APPLICATIONS ORIENTED MATHEMATICS
PRELIMINARY EXAMINATION

Friday, May 12, 2006 1:00-4:00pm

Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best FIVE questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

Potentially useful formulas :

$$Ai(z) \sim \frac{1}{2\sqrt{\pi} z^{1/4}} \exp\left(-\frac{2}{3} z^{3/2}\right), \qquad z \to +\infty$$

$$Bi(z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} \exp\left(\frac{2}{3} z^{3/2}\right), \qquad z \to +\infty$$

$$Ai(z) \sim \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right), \qquad z \to -\infty$$

$$Bi(z) \sim \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right), \qquad z \to -\infty$$

$$Ai(0) = \frac{Bi(0)}{\sqrt{3}} = \frac{1}{3^{2/3}\Gamma(2/3)} = .35502\cdots.$$

Here Ai and Bi are Airy functions, and z is real.

1. Find the leading term in the asymptotic expansion of the following integral, in the indicated limits

$$\int_0^{\pi/2} \sqrt{t}\, e^{x(\sin(t)-t)}\, dt.$$

(a) $x \to +\infty$ (b) $x \to -\infty$.

2. Obtain two-term asymptotic approximations, as $x \to +\infty$, for the following oscillatory integrals

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{t}}\, e^{ixt}\, dt$ (b) $\displaystyle\int_0^1 \sqrt{t}\, e^{ixt}\, dt.$

3. Consider the ODE

$$y''(x) + \left(1 + \frac{2}{x} + \frac{1}{4x^2}\right) y(x) = 0.$$

(a) Classify all points, including $x = \infty$, as ordinary, regular singular or irregular singular.

(b) Find the asymptotic behaviors of all solutions, in the limit $x \to 0$.

(c) Find the asymptotic behaviors of all solutions, in the limit $x \to \infty$.

4. Consider the following eigenvalue problems, for $\lambda \to \infty$,

(a) $y''(x) + \lambda\, e^{x^2/2}\, y(x) = 0, \qquad 0 < x < 1; \qquad y(0) = 0, \qquad y'(1) = 0$

(b) $y''(x) + \lambda\, x\, e^{x^2/2}\, y(x) = 0, \qquad 0 < x < 1; \qquad y(0) = 0, \qquad y(1) = 0.$

Find the asymptotic behavior of the large eigenvalues and the corresponding eigenfunctions. Do not normalize the eigenfunctions. Note that part (b) has a turning point at $x = 0$.

5. Find two-term asymptotic approximations, as $\varepsilon \to 0^+$, for all roots $x = x(\varepsilon)$ of the algebraic equation

$$\varepsilon^2 x^4 + x^3 - 3x^2 + 3x - 1 - \varepsilon = 0.$$

6. Use the two-time method to analyze the following nonlinear oscillator

$$y'' + y + \varepsilon[6(y')^3 - 2y'] = 0,$$

$$y(0) = 2, \qquad y'(0) = 0.$$

How does $y(t)$ behave for large times ?

7. Consider the following singularly perturbed boundary value problems :

(a) $\quad \varepsilon y''(x) \; - \; \sin(2\pi x)\, y'(x) = 0, \quad 0 < x < 1; \quad y(0) = A, \quad y(1) = B$

(b) $\quad \varepsilon y''(x) \; + \; x(1-x)\, y'(x) = 0, \quad 0 < x < 1; \quad y(0) = A, \quad y(1) = B.$

Construct an outer solution, locate any boundary layers and give their thickness, and construct local approximations in these layers.

8. Consider the following singularly perturbed boundary value problem, defined in the triangle $\mathcal{D} = \{(x, y) : 0 < x < 1, 0 < y < x\}$

$$\varepsilon \Delta u - u_x - u_y = \varepsilon[u_{xx} + u_{yy}] - u_x - u_y = 0, \qquad (x, y) \in \mathcal{D}$$

$$u(x, 0) = 1, \quad 0 < x < 1; \qquad u(x, x) = 3, \quad 0 < x < 1; \qquad u(1, y) = 2, \quad 0 < y < 1.$$

Construct an outer solution, locate any boundary/internal layers, and give their thickness. Give the ODE/PDE that applies in each layer, and solve it. Suggestion : first sketch the domain and the subcharacteristics.