10/10 group discussion problems

(1) Find $D_{\mathbf{u}}f$ at the point $(2,1)$ where $f(x,y) = \ln(x^2+y^2)$ and $\mathbf{u} = < -1, 2 >$.

Solution: The calculation of the directional derivative is routine. The answer is 0. Since $\nabla f \neq \mathbf{0}$, this actually implies that $\mathbf{u}$ is tangent to the level surface passing though $(2,1)$.

(2) Find the points on the hyperboloid $x^2 - y^2 + 2z^2 = 1$ where the normal line is parallel to the line that joins the points $(3,-1,0)$ and $(5,3,6)$.

Solution: In the 1:00 class, I left the square off the z term, so your answer will differ slightly.

The direction vector of the line is $< 2,4,6 >$. The hyperboloid is a level set of the function $f(x,y,z) = x^2 - y^2 + 2z^2$, so ∇f at each point is orthogonal to the surface. Therefore, we need to find the points (x,y,z) on the hyperboloid so that ∇f is parallel to $< 2,4,6 >$, that is points (x,y,z) so that

$$< 2x, -2y, 4z >= a < 2, 4, 6 >,$$

for some scalar a and so that the point (x,y,z) satisfies

$$x^2 - y^2 + 2z^2 = 1.$$

The equation relating the two vectors quickly gives $x = a$, $y = -2a$ and $z = \frac{3a}{2}$. Plugging these values into the defining equation of the hyperboloid and solving for a, one obtains $a = \pm\sqrt{\frac{2}{3}}$, yielding the two desired points.

(3) Show that every line normal to a sphere passes through its center.

Solution: A sphere centered at (a,b,c) of radius r is described by the level set at level r^2 of a the function $f(x,y,z) = (x-a)^2 + (y-b)^2 + (z-c)^2$. At each point, the gradient vector gives the direction vector of the normal line to the level set passing through that point. We calculate

$$\nabla_{(x,y,z)} f =< 2(x-a), 2(y-b), 2(z-c) >,$$

which is twice the position vector of (x,y,z) relative to the center of the sphere.

(4) Suppose that you know $D_{\mathbf{u}}f$ and $D_{\mathbf{v}}f$ at a point where $\mathbf{u}$ and $\mathbf{v}$ are non-parallel unit vectors. Can you find ∇f at this point? How?

Solution: Some students did this problem by introducing coordinates and explicitly calculating. That approach has the disadvantage of not working for any dimension. Here's another way to look at it:

Since $|\mathbf{u}| = |\mathbf{v}|$, one can check that $\mathbf{u}+\mathbf{v}$ is perpendicular to $\mathbf{u}-\mathbf{v}$. Also, one can check that $D_{\mathbf{u}}f + D_{\mathbf{v}}f = D_{\mathbf{u}+\mathbf{v}}f$ and $D_{\mathbf{u}}f - D_{\mathbf{v}}f = D_{\mathbf{u}-\mathbf{v}}f$. Since $\mathbf{u}+\mathbf{v}$ is perpendicular to $\mathbf{u}-\mathbf{v}$, we can write any vector as the sum of its projections onto these two vectors. In particular,

$$\nabla f = \mathbf{proj}_{\mathbf{u}+\mathbf{v}}\nabla f + \mathbf{proj}_{\mathbf{u}-\mathbf{v}}\nabla f.$$

Then,

$$\mathbf{proj}_{\mathbf{u}+\mathbf{v}}\nabla f = \frac{\nabla f\cdot(\mathbf{u}+\mathbf{v})}{|\mathbf{u}+\mathbf{v}|^2}(\mathbf{u}+\mathbf{v}) = \frac{D_{\mathbf{u}+\mathbf{v}}f}{|\mathbf{u}+\mathbf{v}|^2}(\mathbf{u}+\mathbf{v})$$

and

$$\mathbf{proj}_{\mathbf{u}-\mathbf{v}}\nabla f = \frac{\nabla f\cdot(\mathbf{u}-\mathbf{v})}{|\mathbf{u}-\mathbf{v}|^2}(\mathbf{u}-\mathbf{v}) = \frac{D_{\mathbf{u}-\mathbf{v}}f}{|\mathbf{u}-\mathbf{v}|^2}(\mathbf{u}-\mathbf{v}).$$

Therefore,

$$\nabla f = \frac{D_{\mathbf{u}+\mathbf{v}}f}{|\mathbf{u}+\mathbf{v}|^2}(\mathbf{u}+\mathbf{v}) + \frac{D_{\mathbf{u}-\mathbf{v}}f}{|\mathbf{u}-\mathbf{v}|^2}(\mathbf{u}-\mathbf{v}),$$

so the answer is yes.