Discussion problems for November 19

(1) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ if
  (a) $\mathbf{F}(x,y) = (4x^3y^2 - 2xy^3)\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\mathbf{j}$ and C is given by $\mathbf{r}(t) = (t + \sin(\pi t))\mathbf{i} + (2t + \cos(\pi t))\mathbf{j}$.
  (b) $\mathbf{F}(x,y,z) = e^y\mathbf{i} + (xe^y + e^z)\mathbf{j} + ye^z\mathbf{k}$ and C is the directed line segment from the point $(0,2,0)$ to $(4,0,3)$.

(2) A 160 lb man carries a 25 lb can of paint at a constant speed up a helical staircase that encircles a 90 ft high silo with radius 20 ft three times. Suppose that there is a hole in the can and 9 lb of paint leaks out at a constant rate over the course of the ascent. How much work is done?

Solutions:

(1) Both of these (especially part (a)) would be tedious to evaluate directly. You should appeal to the fundamental theorem for line integrals which states that if C is a smooth curve given by a vector function $\mathbf{r}(t)$ for $a \le t \le b$, and if ∇f is continuous, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

We'll use this theorem by finding a function f so that $\mathbf{F} = \nabla f$. Such a function f is called the *potential function* for $\mathbf{F}$.

  (a) To find the potential function, we'll first integrate the $\mathbf{i}$ component of $\mathbf{F}$ with respect to x to get $f(x,y) = x^4y^2 - x^2y^3 + c(y)$ for some function $c(y)$. Now take the partial derivative of f with respect to y to figure out what $c(y)$ should be:

$$f_y(x,y) = 2x^4y - 3x^2y^2 + c'(y).$$

  Comparing this to the $\mathbf{j}$ component of $\mathbf{F}$ we see that $c'(y) = 4y^3$, so $c(y) = y^4 + K$ for some constant K. Therefore $f(x,y) = x^4y^2 - x^2y^3 + y^4 + K$. Now use the theorem to evaluate and get an answer of 0.

  (b) We'll use the same procedure for this part. First integrate the $\mathbf{i}$ component with respect to x to get $f(x,y,z) = xe^y + c(y,z)$ for some function $c(y,z)$. Now differentiate with respect to y: $f_y(x,y,z) = xe^y + c_y(y,z)$. Comparing with the $\mathbf{j}$ component, we see that $c_y(y,z) = e^z$. Integrating with respect to y, we see that $c(y,z) = ye^z + d(z)$, for some function $d(z)$. So now we have that

$$f(x,y,z) = xe^y + ye^z + d(z).$$

  We can find what $d(z)$ is by differentiating with respect to z: $f_z(x,y,z) = ye^z + d'(z)$. Comparing with the $\mathbf{k}$ component of $\mathbf{F}$, we see that $d'(z) = 0$. Therefore $f(x, y, z) = xe^y + ye^z + K$ for some constant K. Now apply the theorem to get an answer of 2.

(2) One approach to this problem is first to parameterize the helix by, for example, $\mathbf{r}(t) =< 20\cos(6\pi), 20\sin(6\pi), 90t >$ for $0 \leq t \leq 1$. Depending on how you interpreted the question, the force is either acting in the upward direction, or in the downward direction, depending on whether you think the man or the gravity is doing the work. This isn't a philosophy course, so let's just take the approach of the man doing the work and forget about this issue. Then the force field in question is $\mathbf{F} =< 0, 0, 185 - 9t >$. To find the work done, just evaluate $\int_0^1 \mathbf{F} \cdot \mathbf{r}'(t)dt$ to get 16245.

In this case, thinking before working doesn't save you much work, but it often does. You could have noticed at the beginning of the problem that $\mathbf{F}$ is a conservative vector field, so is independent of path under integration. Then the work done can be found by just assuming the man travels straight up and ignoring the staircase altogether.