

Discussion problems for Friday, November 2

Evaluate the following:

(1) $\int_{-\infty}^{\infty} e^{-x^2} dx$.

Solution: This was done in lecture. It is based on the following (tricky) observation:

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA.$$

The double integral can be evaluated by changing to polar coordinates:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dA = \int_0^{\infty} \int_0^{2\pi} r e^{-r^2} d\theta dr = 2\pi \int_0^{\infty} r e^{-r^2} dr = \pi,$$

so taking the square root, we find that the original integral evaluates to $\sqrt{\pi}$.

(2) $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoid with vertices $(1, 0)$, $(2, 0)$, $(0, 1)$ and $(0, 2)$.

Solution: This integral can be evaluated by substituting $u = y-x$ and $v = y+x$. To use this to integrate, we need to compute the Jacobian of the transformation and to figure out what the region R is transformed to under this change of variables.

To calculate the Jacobian, we need to solve for x and y in the two substitution equations. This results in $x = \frac{v-u}{2}$ and $y = \frac{v+u}{2}$. Therefore the Jacobian is equal to $|(-1/2)(1/2) - (1/2)(1/2)| = 1/2$.

The region R is bounded, in the xy -plane by the four lines $y = 0$, $x = 0$, $y = 1 - x$ and $y = 2 - x$. To find out what happens after our substitution, plug the expressions for x and y in terms of u and v into these four equations to get $u = -v$, $u = v$, $u = 1$ and $u = 2$. It is now routine to integrate $\cos\left(\frac{u}{v}\right)$ over the region bounded by these four lines.