Discussion problem for Monday, November 5

The task today was to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The suggested method was to show that both sides of the equation are equal to the double integral

$$\iint_{I^2} \frac{dA}{1-xy},$$

where I^2 denotes the unit square. To show that the double integral is equal to the left hand side, rewrite

$$\frac{1}{1-xy}$$

as

$$\sum_{m=0}^{\infty} (xy)^m,$$

which can be done because $|xy| < 1$ on the unit square. I gave you a black box theorem saying that the following equality is true:

$$\iint_{I^2} \sum_{m=0}^{\infty} (xy)^m \, dA = \sum_{m=0}^{\infty} \iint_{I^2} (xy)^m \, dA.$$

Evaluating the double integral and setting $n = m + 1$, you are finished with this part.

To show that the double integral evaluates to $\frac{\pi^2}{6}$, it is helpful to make the substitution

$$x = \frac{u-v}{\sqrt{2}}$$
$$y = \frac{u+v}{\sqrt{2}}.$$

Calculate the Jacobian, getting 1. The region of integration is transformed from I^2 to the region in the uv-plane bounded by the four lines

$$u = v,$$
$$u = -v,$$
$$u = v + \sqrt{2}, \text{ and}$$
$$u = -v + \sqrt{2}.$$

The integrand becomes

$$\frac{2dA}{2 - u^2 + v^2}.$$

This is symmetric about the u axis since changing v to $-v$ leaves the expression unchanged, so we have

$$\iint_{I^2} \frac{dA}{1-xy} = 2\int_0^{\sqrt{2}/2}\int_0^{u} \frac{2dvdu}{2-u^2+v^2} + 2\int_{\sqrt{2}/2}^{\sqrt{2}}\int_0^{-u+\sqrt{2}} \frac{2dvdu}{2-u^2+v^2}.$$

This is where pretty much everyone got stuck in class (I think), so I'll try to go into some detail in evaluating

$$\int \frac{2dv}{2-u^2+v^2} = 2\int \frac{dv}{2-u^2+v^2}.$$

This task is much more heroic than I anticipated. Recall (or rediscover or relearn!) that in this situation you should make the substitution $v = \sqrt{2-u^2}\tan\theta$. Then we have (what we now know to be the Jacobian) $dv = \sqrt{2-u^2}\sec^2\theta d\theta$. Substituting all of this in, we have

$$2\int \frac{dv}{2-u^2+v^2} = 2\int \frac{\sqrt{2-u^2}\sec^2\theta\ d\theta}{2-u^2+(2-u^2)\tan^2\theta}.$$

One of the Pythagorean identities says that the denominator is equal to $(2-u^2)\sec^2\theta$, so that the whole integral is just equal to

$$\frac{\theta}{\sqrt{2-u^2}} + C$$

for some constant C. From our substituting equation, we see that

$$\theta = \arctan\frac{v}{\sqrt{2-u^2}}$$

so we now have

$$\int \frac{2dv}{2-u^2+v^2} = \frac{1}{\sqrt{2-u^2}}\arctan\frac{v}{\sqrt{2-u^2}} + C.$$

Evaluating this from 0 to u and from 0 to $-u+\sqrt{2}$ we have (look at the top of the page)

$$\iint_{I^2}\frac{dA}{1-xy} =$$

$$2\int_0^{\sqrt{2}/2}\frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}} + 2\int_{\sqrt{2}/2}^{\sqrt{2}}\frac{1}{\sqrt{2-u^2}}\arctan\frac{-u+\sqrt{2}}{\sqrt{2-u^2}}$$

Notice that

$$\arctan\frac{u}{\sqrt{2-u^2}} = \arcsin\frac{u}{\sqrt{2}}$$

by drawing the right triangle with legs of lengths u and $\sqrt{2-u^2}$.

Using the same trick for the second summand, set

$$\theta = \arctan \frac{\sqrt{2}-u}{\sqrt{2-u^2}}.$$

Then using a particular right triangle, we get

$$\cos\theta = \frac{\sqrt{2-u^2}}{\sqrt{4-2\sqrt{2}u}}.$$

Square both sides and rearrange to get

$$\frac{u}{\sqrt{2}} = 2\cos^2\theta - 1 = \cos 2\theta = \sin\frac{\pi}{2} - 2\theta,$$

where the last two equalities are simply trig identities. Solve for θ to get

$$\theta = \frac{\pi}{4} - \arcsin\frac{u}{\sqrt{2}}.$$

We can now finish off the integral by rewriting

$$2\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}} \arctan\frac{u}{\sqrt{2-u^2}} + 2\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \arctan\frac{-u+\sqrt{2}}{\sqrt{2-u^2}}$$

as

$$2\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}} \arcsin\frac{u}{\sqrt{2}} du + 2\int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}} \left(\frac{\pi}{4} - \arcsin\frac{u}{\sqrt{2}}\right) du.$$

Now you can finish this on your own by integrating these. Make a substitution for the arcsin parts...