Solutions to
So you think you know calculus?

In order to understand differential equations, it is vital that you understand calculus. Work on these problems to refresh your memory. Be sure to review any of the parts which you do not understand after class. Please do not use your calculators or any integral tables. If you need help, ask your neighbor or Chris.

(1) Calculate

$$\frac{d}{dx}\left(\frac{e^{3x}\cdot\cos(x^2+1)}{\ln(5x)}\right).$$

- First use the quotient rule. When differentiating the numerator, you must use the product rule and the chain rule. You should get

$$\frac{(3e^{3x}\cos(x^2+1)-2xe^{3x}\sin(x^2+1))\ln(5x)-1/x(e^{3x}\cos(x^2+1))}{(\ln(5x))^2}.$$

(2) If $x^5+y^3+\cos(xy)-xy^2=4$, calculate $\frac{dy}{dx}$.

- You must use implicit differentiation on this one because the equation cannot be solved for y. First apply $\frac{d}{dx}$ to both sides, remembering that y is a function of x:

$$\frac{d}{dx}\left(x^5+y^3+\cos(xy)-xy^2\right)=\frac{d}{dx}(4)$$

$$5x^4+3y^2\frac{dy}{dx}-\sin(xy)(y-x\frac{dy}{dx})-(y^2+2xy\frac{dy}{dx})=0$$

Now this is just an algebra problem. The goal is to solve for $\frac{dy}{dx}$. Do this by collecting all the terms with a $\frac{dy}{dx}$ on the left and moving all of the other terms to the right. Then factor out $\frac{dy}{dx}$ and divide to obtain:

$$\frac{dy}{dx}=\frac{-5x^4+y\sin(xy)+y^2}{3y^2+x\sin(xy)-2xy}.$$

(3) Calculate the following integrals:

(a)

$$\int\frac{x}{x^2+1}\,dx$$

- This is basic 'u substitution. Let $u = x^2 + 1$ so that $du = 2xdx$, or $dx = du/2x$. Therefore,

$$\int \frac{x}{x^2+1}\,dx = \int \frac{x}{u}\frac{du}{2x} = 1/2 \int \frac{du}{u} = 1/2 \ln|u| + C.$$

Then plug $u = x^2 + 1$ back in to obtain

$$1/2 \ln|x^2 + 1| + C.$$

(b)

$$\int te^{t+1}\,dt$$

- This one is integration by parts. Let $u = t$ and $dv = e^{t+1}$. Then $du = 1dt$ and $v = e^{t+1}$. Now

$$\int te^{t+1}\,dt = uv - \int v\,du = te^{t+1} - \int e^{t+1}\,dt = te^{t+1} - e^{t+1} + C.$$

(c)

$$\int \frac{5}{\sqrt{64-\theta^2}}\,d\theta$$

- This one is sine substitution. Recall that whenever you see

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx,$$

you are supposed to make the substitution $x = a\sin(z)$. In our case, we will make the substitution $\theta = 8\sin z$. Then $d\theta = 8\cos z\,dz$, so

$$\int \frac{5}{\sqrt{64-\theta^2}}\,d\theta = 5\int \frac{1}{\sqrt{8^2 - 8^2\sin^2 z}} 8\cos z\,dz = 5\int \frac{8\cos z\,dz}{8\sqrt{1-\sin^2 z}}.$$

Recall the pythagorean identity for cosine and sine: $1 - \sin^2 z = \cos^2 z$. Therefore,

$$5\int \frac{8\cos z\,dz}{8\sqrt{1-\sin^2 z}} = 5\int \frac{\cos z\,dz}{\sqrt{\cos^2 z}} = 5\int dz = 5z + C.$$

From our original assumption, we see that $z = \arcsin(\theta/8)$, so the solution is $5\arcsin(\theta/8) + C$.

(d)

$$\int \frac{2y-7}{y^2-10y+25}\,dy$$

- For this integral, first notice that the denominator of the integrand factors as $(y-5)^2$. This means that we should try to break up the integral into a sum of two integrals using partial fractions:

$$\frac{2y-7}{y^2-10y+25} = \frac{A}{(y-5)^2} + \frac{B}{y-5}.$$

Now, we need to figure out what A and B must be for this equation to be true. To do this, first multiply through by $(y-5)^2$:

$$2y-7 = A + B(y-5).$$

Now simplify the right hand side so that we can match coefficients:

$$2y-7 = By + (A-5B).$$

The constant term on each side must be the same and the coefficient of y on each side must be the same, so $B = 2$ and $A - 5B = 7$. Therefore A=3. Now,

$$\int \frac{2y-7}{y^2-10y+25}\,dy = \int \frac{3}{(y-5)^2}\,dy + \int \frac{2}{y-5}\,dy$$

Now it is basic integration:

$$\frac{-3}{(y-5)} + 2\ln|y-5| + C.$$

(e)

$$\int \frac{1}{x^2-8x+19}\,dx$$

- The denominator of the integrand does not factor, so we are supposed to complete the square and then use a tangent substitution. First complete the square:

$$x^2-8x+19 = x^2-8x+\left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 19$$

$$= x^2-8x+16-16+19 = (x-4)^2+3.$$

So we have

$$\int \frac{1}{x^2-8x+19}\,dx = \int \frac{1}{(x-4)^2+3}\,dx = \int \frac{1}{(x-4)^2+(\sqrt{3})^2}\,dx.$$

Recall that when you see $\int \frac{1}{x^2+a^2}\,dx$, you are supposed to make the substitution $x = a\tan\theta$. So in our case, we make the substitution $x-4 = \sqrt{3}\tan\theta$, and differentiate both sides to get $dx = \sqrt{3}\sec^2\theta d\theta$. This will now work like the sine substitution above. Plug everything in:

$$\int \frac{1}{(x-4)^2+(\sqrt{3})^2}\,dx = \int \frac{\sqrt{3}\sec^2\theta}{3\tan^2\theta+3}\,d\theta.$$

We can cancel a factor of $\sqrt{3}$ from the numerator and the denominator. Now use the fact that $\tan^2\theta + 1 = \sec^2\theta$:

$$\frac{1}{\sqrt{3}}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta = \frac{1}{\sqrt{3}}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta = \frac{1}{\sqrt{3}}\theta + C.$$

Now solve for θ from our original substitution to get the answer in terms of x: $\theta = \arctan(\frac{x-4}{\sqrt{3}})$, so the solution is $\frac{1}{\sqrt{3}}\theta = \frac{1}{\sqrt{3}}\arctan(\frac{x-4}{\sqrt{3}}) + C$.