ERRATUM TO "STANDARD FINITE ELEMENTS FOR THE NUMERICAL RESOLUTION OF THE ELLIPTIC MONGE-AMPÈRE EQUATION: CLASSICAL SOLUTIONS "

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ABSTRACT. The proof of Lemma 3.6 in the paper referenced in the title is not correct. Moreover the numerical results were obtained with an iterative method different from the one analyzed in the paper. The purpose of this erratum is to give a correct proof of the main results therein for high order C^1 elements. For high order C^0 elements we make the assumption that at a given point the eigenvalues of the exact solution are the same. The convergence rates are suboptimal for C^0 elements.

1. Introduction

In [3, Lemma 3.6], we claimed a strict contraction property for a mapping T in the H^1 norm. Unfortunately there was a mistake at the end of step 1 of the proof of the lemma. It was stated on [3, p. 12] that "Since $\beta < 1$, for h sufficiently small $a = \beta + Ch^{\frac{1}{2}}(1 + ||u||_1)^{n-1} < 1$". However β, as defined therein, also depends on h, see [3, p. 10]. Moreover $1 - \beta \to 0$ at a rate higher than $h^{1/2}$, and thus the argument as stated is not correct. As a consequence, the strategy which consists in rescaling the equation does not work for C^0 elements and turns out not to be necessary for C^1 elements. We give in section 3 an analysis of the iterative method the numerical results of which were given in [3].

In this erratum we first give corrections for the results claimed in [3] for high order C^1 elements. We then give the iterative method which yield the numerical results presented in [3]. It corresponds to a discretization of the Monge-Ampère equation with interior penalty terms. We prove that the discretization is well-posed and prove the convergence of the iterative method for high order elements under the assumption that at any given point the eigenvalues of the exact solution are the same (3.5).

We use the same notation as in [3]. In addition we extend canonically Sobolev norm and semi norms to matrix fields.

We recall the scale-trace inequality

$$||v||_{0,2,\partial K} \leq Ch^{-\frac{1}{2}}||v||_{0,2,K}, \tag{1.1}$$

when v is a polynomial on K.

2. The argument for C^1 elements

The convergence of the time marching method [3, (3.3)] was given in [1] for cubic and high order C^1 elements. The analysis given in [3] may be viewed as a different approach. The rescaling argument used in [3] turns out not to be necessary. To make this erratum more readable, we have included the parameter α in the corrections given here for C^1 elements. Here α is a constant independent of h. The reader may assume that $\alpha = 1$. In this section, V_h denotes a finite dimensional space of piecewise polynomial C^1 functions with an interpolation operator I_h which satisfies the approximation property [3, (2.3)].

For the statement of [3, Lemma 3.1], instead of "each element T", we meant "each element K".

For the statement of [3, Theorem 3.3] the convergence of the iterative method is in the H^1 semi norm, not the H^1 norm.

We now define

$$B_h(\rho) = \{v_h \in V_h, v_h = g_h \,\text{on}\, \partial\Omega, |v_h - I_h u|_1 \leq \rho\}.$$

In addition, we now require that $\rho < \delta_h/C_p$ where C_p is the constant in the Poincaré's inequality, i.e. for $v_h \in H_0^1(\Omega), ||v_h||_1 \leq C_p|v_h|_1$ and $\delta_h = Ch^{1+n/2}$ for a constant C. We recall from [3, Lemma 3.1] and [3, (3.1)] that for $\rho < \delta_h/C_p$ and $v_h \in B_h(\rho)$, $D^2 v_h$ is piecewise positive definite.

We strengthen the result of [3, Lemma 3.5] as

$$|I_h u - T(I_h u)|_1 \leq C_1 h^d. \tag{2.1}$$

With $w_h = I_h u - T(I_h u)$, we have using [3, (3.8)] and [3, Lemma 2.1]

$$|w_h|_1^2 = \frac{\alpha^n}{\nu} \sum_{K \in \mathcal{T}_h} \int_K (\text{cof}\, r_h) : D^2(I_h u - u) w_h \, dx,$$

where $r_h = tD^2 I_h u + (1-t)D^2 u, t \in [0,1]$. Using the divergence free row property of the cofactor matrix

$$\begin{aligned}
\frac{\nu}{\alpha^n}|w_h|_1^2 &= \sum_{K \in \mathcal{T}_h} \int_K \text{div}\, \Big((\text{cof}\, r_h) D(I_h u - u) \Big) w_h \, dx \\
&= - \sum_{K \in \mathcal{T}_h} \int_K \big((\text{cof}\, r_h) D(I_h u - u) \big) \cdot D w_h \, dx \\
&\quad + \sum_{K \in \mathcal{T}_h} \int_{\partial K} ((\text{cof}\, r_h) D(I_h u - u)\big) \cdot (w_h n) \, ds \\
&\equiv R_1 + R_2.
\end{aligned}$$

Since $|r_h|_{0,\infty} \leq C$, $|R_1| \leq Ch^d |w_h|_1$. In addition, since $I_h u$ is C^1 and u is smooth, we have

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} ((\text{cof}\, D^2 u) D(I_h u - u)\big) \cdot (w_h n) \, ds = 0.$$

Thus

$$R_2 = \sum_{K \in \mathcal{T}_h} \int_{\partial K} ((\text{cof}\, r_h - \text{cof}\, D^2 u) D(I_h u - u)\big) \cdot (w_h n) \, ds.$$

As in the proof of [6, Lemma 2.6], we have for two matrix fields η and τ

$$|\operatorname{cof}\eta - \operatorname{cof}\tau|_{0,\infty} \leq C|t\eta + (1-t)\tau|_{0,\infty}^{n-2}|\eta - \tau|_{0,\infty}.$$

Moreover, as $|I_h u - u|_{2,\infty} \leq Ch^{d-1}$ and $|D(I_h u - u)|_{1,\partial K} \leq Ch^{d-1/2}|u|_{d+1,K}$, we get

$$|R_2| \leq Ch^{d-1}h^{d-\frac{1}{2}} \sum_{K\in\mathcal{T}_h} \int_{\partial K} |w_h|\, ds \leq Ch^{2d-\frac{3}{2}} \sum_{K\in\mathcal{T}_h} h_K^{\frac{n-1}{2}} ||w_h||_{0,2,\partial K}.$$

By the the trace, Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$|R_2| \leq Ch^{2d-\frac{3}{2}} \bigg(\sum_{K\in\mathcal{T}_h} h_K^{n-1} \bigg)^{\frac{1}{2}} \bigg(\sum_{K\in\mathcal{T}_h} ||w_h||_{1,2,K}^2 \bigg)^{\frac{1}{2}}.$$

$$|R_2| \leq Ch^{2d-1}||w_h||_1 \leq Ch^{2d-1}|w_h|_1 \leq Ch^d|w_h|_1,$$

where we used $w_h = 0$ on $\partial\Omega$. This completes the proof of (2.1).

For the statement of [3, Lemma 3.6], the strict contraction property of the mapping T should be in the H^1 semi norm, i.e.

$$|T(\alpha v_h) - T(\alpha w_h)|_1 \leq a|\alpha v_h - \alpha w_h|_1, 0 < a < 1. \tag{2.2}$$

For [3, Lemma 3.7] which states that T maps $\alpha B_h(\rho)$ into itself, we now take

$$\rho = \frac{C_1}{1-a}h^d, 2 + \frac{n}{2} \leq d, \tag{2.3}$$

where a is a constant defined below (2.12). This assures that for h sufficiently small $\rho \leq \delta_h/(2C_p) = \delta/(4C_{inv}C_p)h^{1+n/2}$. We also note that the constant C_{inv} appearing in the inverse estimates is still generic as it depends on the norms used.

The Banach fixed point theorem still applies. Put $\hat{B}_h(\rho) = B_h(\rho) - I_h u$ and note that $I_h u = g_h$ on $\partial\Omega$. Thus for $w_h \in \hat{B}_h(\rho), w_h = 0$ on $\partial\Omega$ and by Poincaré's inequality we can define a norm on $\hat{B}_h(\rho)$ by $||w_h|| = |w_h|_1$. We define on $\hat{B}_h(\rho)$ a mapping $\hat{T}$ by $\hat{T}(w_h) = T(w_h + I_h u) - I_h u$. If w_h is a fixed point of $\hat{T}$, $w_h + I_h u$ is a fixed point of T and conversely if v_h is a fixed point of T, $v_h - I_h u$ is a fixed point of $\hat{T}$. It can be readily checked that $\hat{T}$ satisfies the assumptions of the Banach fixed point theorem on $\alpha\hat{B}_h(\rho)$ endowed with the norm $||.||$, when T does on $\alpha B_h(\rho)$.

Note that by Poincaré's inequality, for $u_h \in B_h(\rho)$, $||I_h u - u_h||_1 \leq C|I_h u - u_h|_1 \leq C\rho$, a property which is used in the proof of [3, Theorem 3.3].

The first displayed equation in the proof of [3, Lemma 3.2] should be

$$m||z||^2 \leq [(\operatorname{cof}\, D^2 v_h(x))z] \cdot z \leq M||z||^2, z \in \mathbb{R}^n,$$

and the correct values of m and M are $m = (m')^n/M' \times 1/(2^{n-1}3)$ and $M = (M')^n/m' \times 3^n/2^{n-1}$.

We first give a quantitative estimate of the constant δ introduced in [3, Lemma 3.1]. The outline of the proof of [3, Lemma 3.1] is as follows.

For all $\epsilon > 0$, there exists $\delta > 0$ such that for $v \in W^{2,\infty}(\Omega)$, $|v - u|_{2,\infty} \leq \delta$ implies $|\lambda_i(D^2v(x)) - \lambda_i(D^2u(x))| < \epsilon$ a.e. in Ω, for $i = 1$ and $i = n$. Then ϵ is taken successively as $m'/2$ and $M'/2$. It follows from [8, Theorem 1 and Remark 2 p. 39] that for two symmetric $n \times n$ matrices A and B,

$$|\lambda_i(A) - \lambda_i(B)| \leq n \max_{r,s} |A_{rs} - B_{rs}|, i = 1, n. \tag{2.4}$$

Thus we can take $\delta \leq m'/(2n)$. Since $m' \leq M'$, we have $\delta \leq M'/2$. We make the assumption that

$$\delta \leq \min\left(\frac{m'}{2n}, \frac{m}{2C_p^2 C_2}\right), \tag{2.5}$$

where C_2 is a constant introduced in (2.6) below.

The mapping T'_K can be shown to be continuous on $\alpha B_h(\rho)$ by using the expression of $\langle T'_K(\alpha v_h)(\alpha w_h), z_h\rangle$ given as [3, (3.9)], trace inequalities, inverse estimates and a mean value theorem for the cofactor matrix [6, Lemma 2.6].

We modify step 1 of the proof of [3, Lemma 3.6] to prove that for $v_h, w_h \in B_h(\rho)$

$$|\langle T'_K(\alpha v_h)(\alpha w_h), z_h\rangle| \leq \beta |\alpha w_h|_{1,K} |z_h|_{1,K} + C_2 \delta \frac{\alpha^{n-1}}{\nu} ||\alpha w_h||_{1,K} ||z_h||_{1,K}, \tag{2.6}$$

for a constant C_2 independent of h.

We note from [3, p. 10-11] that

$$0 \leq \beta \leq 1 - \frac{\alpha^{n-1} m}{\nu}.$$

Without loss of generality, we can thus take as the value of β an upper bound, i.e.

$$\beta = 1 - \frac{\alpha^{n-1} m}{\nu}. \tag{2.7}$$

The displayed equation just above [3, (3.11)] says that

$$\left| \int_K [(I - \frac{1}{\nu} \operatorname{cof} D^2 \alpha v_h) D w_h] \cdot D z_h \, dx \right| \leq \beta |w_h|_{1,K} |z_h|_{1,K}. \tag{2.8}$$

It remains to estimate the term

$$R \equiv \int_{\partial K} z_h [(\operatorname{cof} D^2 v_h) D w_h] \cdot n_K \, ds.$$

Since u is smooth, z_h is continuous on Ω and Dw_h is also continuous on Ω by the assumptions in this section, we have

$$\int_{\partial K} z_h [(\operatorname{cof} D^2 u) D w_h] \cdot n_K \, ds = 0.$$

We therefore have

$$\begin{aligned} R &= \int_{\partial K} z_h [(\operatorname{cof} D^2 v_h) D w_h] \cdot n_K \, ds - \int_{\partial K} z_h [(\operatorname{cof} D^2 u) D w_h] \cdot n_K \, ds \\ &= \int_{\partial K} z_h [(\operatorname{cof} D^2 v_h - \operatorname{cof} D^2 u) D w_h] \cdot n_K \, ds. \end{aligned}$$

We have by the trace and scaled trace inequalities

$$\begin{aligned} |R| &\leq C || \operatorname{cof} D^2 v_h - \operatorname{cof} D^2 u ||_{0,\infty} ||z_h||_{0,\partial K} ||D w_h||_{0,\partial K} \\ &\leq C h^{-\frac{1}{2}} || \operatorname{cof} D^2 v_h - \operatorname{cof} D^2 u ||_{0,\infty} ||z_h||_{1,K} ||w_h||_{1,K}. \end{aligned}$$

As with [6, Lemma 2.6], we have for some $t \in [0,1]$

$$|| \operatorname{cof} D^2 v_h - \operatorname{cof} D^2 u||_{0,\infty} \leq C||tD^2 v_h + (1-t)D^2 u||_{0,\infty}^{n-2} |v_h - u|_{2,\infty}.$$

Since $|u - I_h u|_{2,\infty} \leq Ch^{d-1}$ and by an inverse estimate and Poincaré's inequality $|v_h - I_h u|_{2,\infty} \leq Ch^{-1-n/2}||v_h - I_h u||_1 \leq Ch^{-1-n/2}|v_h - I_h u|_1$, we conclude using $\rho \leq C\delta h^d$ that $|v_h - u|_{2,\infty} \leq C\delta h^{d-1-n/2}$. Thus for $d \geq 3$

$$|R| \leq C\delta ||z_h||_{1,K} ||w_h||_{1,K}. \tag{2.9}$$

Combining (2.8)–(2.9), we obtain (2.6).

As in the first part of Step 3 of [3, Lemma 3.6], we have

$$|T(\alpha v_h) - T(\alpha w_h)|_1^2 = \sum_{K \in \mathcal{T}_h} \langle T_K(\alpha v_h) - T_K(\alpha w_h), \psi_h \rangle, \tag{2.10}$$

where $\psi_h = T(\alpha v_h) - T(\alpha w_h)$. By the mean value theorem, as T_K' is continuous, we have

$$\begin{aligned} |\langle T_K(\alpha v_h) - T_K(\alpha w_h), \psi_h \rangle| &= \Big| \langle \int_0^1 T_K'(\alpha v_h + t(\alpha w_h - \alpha v_h))(\alpha w_h - \alpha v_h) \\ &\qquad dt, \psi_h \rangle \Big| \\ &= \Big| \int_0^1 \langle T_K'(\alpha v_h + t(\alpha w_h - \alpha v_h))(\alpha w_h - \alpha v_h), \\ &\qquad \psi_h \rangle dt \Big| \\ &\leq \int_0^1 |\langle T_K'(\alpha v_h + t(\alpha w_h - \alpha v_h))(\alpha w_h - \alpha v_h) \\ &\qquad , \psi_h \rangle| \, dt, \end{aligned} \tag{2.11}$$

where we use the following property of the Riemann integral of Banach valued functions. If $A \in L(V_K', \mathbb{R})$ is continuous, then $A \int_0^1 F(t) dt = \int_0^1 A(F(t)) dt$, for $F(t) \in V_K'$ [7, Proposition 3.1.3]. For $\psi_h \in V_K$, we then take $A(F) = \langle F, \psi_h \rangle$ for $F \in V_K'$.

Alternatively, one can establish (2.11) by direct calculation without using a functional notation.

Since $v_h + t(w_h - v_h) \in B_h(\rho), t \in [0,1]$, we get from (2.11) and (2.6)

$$\begin{aligned} |\langle T_K(\alpha v_h) - T_K(\alpha w_h), \psi_h \rangle| &\leq \beta |\alpha w_h - \alpha v_h|_{1,K} |\psi_h|_{1,K} \\ &\quad + C_2 \delta \frac{\alpha^{n-1}}{\nu} ||\alpha w_h - \alpha v_h||_{1,K} ||\psi_h||_{1,K}. \end{aligned}$$

By Cauchy-Schwarz inequality, we obtain from (2.10)

$$\begin{aligned} |T(\alpha v_h) - T(\alpha w_h)|_1^2 &\leq \beta |\alpha w_h - \alpha v_h|_1 |\psi_h|_1 \\ &\quad + C_2 \delta \frac{\alpha^{n-1}}{\nu} ||\alpha w_h - \alpha v_h||_1 ||\psi_h||_1. \end{aligned}$$

But $\psi_h = 0$ on $\partial\Omega$ and thus by Poincaré's inequality,

$$|T(\alpha v_h) - T(\alpha w_h)|_1^2 \leq \left(\beta + C_p^2 C_2 \delta \frac{\alpha^{n-1}}{\nu}\right)|\alpha w_h - \alpha v_h|_1 |\psi_h|_1.$$

Therefore

$$|T(\alpha v_h) - T(\alpha w_h)|_1 \leq \left(\beta + C_p^2 C_2 \delta \frac{\alpha^{n-1}}{\nu}\right)|\alpha w_h - \alpha v_h|_1.$$

Using the value of β given by (2.7), the contraction constant in the above equation is given by

$$a = 1 - \frac{\alpha^{n-1} m}{\nu} + C_p^2 C_2 \delta \frac{\alpha^{n-1}}{\nu} = 1 - \frac{\alpha^{n-1}}{\nu}\left(m - C_p^2 C_2 \delta\right). \tag{2.12}$$

Note that $a \geq 0$ since $\beta \geq 0$. By (2.5), $C_p^2 C_2 \delta \leq m$ and thus $a < 1$. We have proved (2.2).

The proof that T maps $\alpha B_h(\rho)$ into itself is essentially the same as the one given in [3, Lemma 3.7]. We have

$$\begin{aligned} |I_h u - T(v_h)|_1 \leq |I_h u - T(I_h u)|_1 + |T(I_h u) - T(v_h)|_1 &\leq C_1 h^d + a|v_h - I_h u|_1 \\ &\leq (1-a)\rho + a\rho \leq \rho. \end{aligned} \tag{2.13}$$

For [3, Remark 3.8], the convergence of the iterative method should be with the H^1 semi norm.

A few references given in [3], [4, 5, 1, 2] have been updated.

3. Analysis of the discretization corresponding with the numerical results given in [3]

We use the same notation as in [3] and the previous section. In this section, V_h denotes a finite dimensional space of piecewise polynomial C^1 functions with an interpolation operator I_h which satisfies the approximation property [3, (2.3)]. The numerical results given in [3] were actually obtained with the iterative method

$$\begin{aligned} \nu \int_\Omega D u_h^{k+1} \cdot D v_h \, dx = -\nu \sum_{K \in \mathcal{T}_h} \int_K (\Delta u_h^k) v_h \, dx - \int_\Omega f v_h \, dx \\ + \sum_{K \in \mathcal{T}_h} \int_K (\det D^2 u_h^k) v_h \, dx, \end{aligned} \tag{3.1}$$

$\forall v_h \in V_h \cap H_0^1(\Omega)$, given a sufficiently close initial guess u_h^0 and with $u_h^0 = u_h^{k+1} = g_h$ on $\partial\Omega$. By integration by parts, we obtain

$$\begin{aligned} \nu \int_\Omega D u_h^{k+1} \cdot D v_h \, dx = \nu \int_\Omega D u_h^k \cdot D v_h \, dx - \nu \sum_{K \in \mathcal{T}_h} \int_{\partial K} v_h (D u_h^k) \cdot n ds - \int_\Omega f v_h \, dx \\ + \sum_{K \in \mathcal{T}_h} \int_K (\det D^2 u_h^k) v_h \, dx. \end{aligned}$$

Formally, the iterative method converges to a solution of the following discretization of the Monge-Ampère equation: find $u_h \in V_h$ such that $u_h = g_h$ on $\partial\Omega$ and

$$\sum_{K\in\mathcal{T}_h} \int_K (\det D^2 u_h) v_h \, dx - \nu \sum_{K\in\mathcal{T}_h} \int_{\partial K} v_h (D u_h) \cdot n \, ds = \int_\Omega f v_h \, dx, \forall v_h \in V_h \cap H_0^1(\Omega). \quad (3.2)$$

To prove the existence of a local solution to (3.2) and the convergence of the time marching method (3.1), we will use a fixed point argument. Error estimates follow as in the proof of [3, Theorem 3.3].

For a given $v_h \in V_h$, $v_h = g_h$ on $\partial\Omega$, define $S(v_h) \in V_h$ as the solution of

$$\nu \int_\Omega D S(v_h) \cdot D w_h \, dx = \nu \int_\Omega D(v_h) \cdot D w_h \, dx + \sum_{K\in\mathcal{T}_h} \int_K (\det D^2 v_h) w_h \, dx - \nu \sum_{K\in\mathcal{T}_h} \int_{\partial K} w_h (D v_h) \cdot n \, ds - \int_\Omega f w_h \, dx, \forall w_h \in V_h \cap H_0^1(\Omega), \quad (3.3)$$

with $v_h - S(v_h) = 0$ on $\partial\Omega$.

Lemma 3.1. *We have*

$$|I_h u - S(I_h u)|_1 \leq C_2 h^{d-1}. \quad (3.4)$$

Proof. Note that u is smooth and $w_h \in V_h \cap H_0^1(\Omega) \subset C^0(\Omega)$. We then have $\sum_{K\in\mathcal{T}_h} \int_{\partial K} w_h (Du) \cdot n \, ds = 0$. On the other hand $f = \det D^2 u$. Thus, with $w_h = S(I_h u) - I_h u$, and $v_h = I_h u$, we have

$$\nu \int_\Omega D[S(v_h) - v_h] \cdot D w_h \, dx = \sum_{K\in\mathcal{T}_h} \int_K (\det D^2 I_h u - \det D^2 u) w_h \, dx - \nu \sum_{K\in\mathcal{T}_h} \int_{\partial K} w_h [D(I_h u - u)] \cdot n \, ds.$$

By approximation properties, the trace inequality, Cauchy-Schwarz inequality and Poincaré's inequality

$$\left| \sum_{K\in\mathcal{T}_h} \int_{\partial K} w_h [D(I_h u - u)] \cdot n \, ds \right| \leq C h^{d-\frac{1}{2}} \sum_{K\in\mathcal{T}_h} ||w_h||_{1,K} ||u||_{k+1,K} \leq C h^{d-\frac{1}{2}} |w_h|_1.$$

On the other hand, with $z_h = \det D^2 I_h u - \det D^2 u$ and arguing as in the proof of [3, Lemma 3.5], we have

$$||z_h||_{0,K} \leq C h^{d-1}.$$

We conclude using Cauchy-Schwarz inequality that

$$\nu |w_h|_1 \leq C h^{d-1}.$$

This concludes the proof. ∎

We next make the assumption that the eigenvalues of D^2u are the same at a given point, i.e.

$$M' = m'. \tag{3.5}$$

We then take

$$\nu = \frac{m}{1+h}. \tag{3.6}$$

For $v_h \in B_h(\rho), \rho < \delta_h$, we define

$$\gamma = \sup_{||y||=1} \bigg| \bigg[\Big(I - \frac{1}{\nu} \operatorname{cof} D^2 v_h(x) \Big) y \bigg] \cdot y \bigg|.$$

Since

$$m||y||^2 \leq [(\operatorname{cof}\ D^2 v_h(x))y] \cdot y \leq M||y||^2, y \in \mathbb{R}^n,$$

we get

$$\gamma \leq \max \bigg\{ \bigg|1 - \frac{m}{\nu}\bigg|, \bigg|1 - \frac{M}{\nu}\bigg| \bigg\} = h.$$

Arguing as in the proof of [3, (3.11)] we obtain

$$\bigg| \Big(I - \frac{1}{\nu} \operatorname{cof} D^2 v_h(x) \Big) p \bigg] \cdot q \bigg| \leq \gamma ||p||\,||q||, p, q \in \mathbb{R}^n. \tag{3.7}$$

We can now state

Theorem 3.2. *For $\rho = 2C_2 h^{d-1}, d \geq 3 + n/2$, the mapping S is a strict contraction in $B_h(\rho)$ which maps $B_h(\rho)$ into itself.*

Proof. We first prove that for v_h and w_h in $B_h(\rho)$, $|S(v_h) - S(w_h)|_1 \leq 1/2 |v_h - w_h|_1$. Put $z_h = S(v_h) - S(w_h)$. Using (3.3) and an integration by parts, we have

$$|z_h|_1^2 = - \sum_{K \in \mathcal{T}_h} \int_K (\Delta v_h - \Delta w_h) z_h \, dx + \frac{1}{\nu} \sum_{K \in \mathcal{T}_h} \int_K (\det D^2 v_h - \det D^2 w_h) z_h \, dx.$$

Using [3, Lemma 2.1], there exists $t \in [0,1]$ such that $\det D^2 v_h - \det D^2 w_h = \operatorname{cof}(r_h) : D^2(v_h - w_h)$ with $r_h = tD^2 v_h + (1-t) D^2 w_h$. Thus, using the divergence row free property of the cofactor matrix

$$\begin{aligned}
|z_h|_1^2 &= \sum_{K \in \mathcal{T}_h} \int_K \Big(\frac{1}{\nu} \operatorname{cof} r_h - I \Big) : D^2(v_h - w_h) z_h \, dx \\
&= \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div} \bigg(\Big(\frac{1}{\nu} \operatorname{cof} r_h - I \Big) D(v_h - w_h) \bigg) z_h \, dx \\
&= \int_\Omega \bigg[\Big(I - \frac{1}{\nu} \operatorname{cof} r_h \Big) D(v_h - w_h) \bigg] \cdot D z_h \, dx \\
&\qquad - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \bigg[\Big(I - \frac{1}{\nu} \operatorname{cof} r_h \Big) D(v_h - w_h) \bigg] \cdot (z_h n) \, ds \\
&\equiv U_1 + U_2.
\end{aligned}$$

Since $r_h \in B_h(\rho)$, as for [3, (3.11)], we have $\beta = \gamma = h$ and

$$|U_1| \leq h |v_h - w_h|_1 |z_h|_1. \tag{3.8}$$

On the other hand by the trace and scale trace inequalities, Cauchy-Schwarz and Poincaré's inequalities

$$\begin{aligned} |U_2| &\leq \gamma \sum_{K \in \mathcal{T}_h} |v_h - w_h|_{1,\partial K} |z_h|_{0,\partial K} \\ &\leq Ch \sum_{K \in \mathcal{T}_h} h^{-\frac{1}{2}} |v_h - w_h|_{1,K} ||z_h||_{1,K} \\ &\leq Ch^{\frac{1}{2}} |v_h - w_h|_1 |z_h|_1, \end{aligned}$$

where we used $z_h = 0$ on $\partial\Omega$. We therefore have $|z_h|_1^2 \leq (h + Ch^{\frac{1}{2}})|v_h - w_h|_1 |z_h|_1$, from which the contraction property follows.

The proof that S maps $B_h(\rho)$ into itself is similar to (2.13).

□

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