SMOOTH APPROXIMATIONS OF THE ALEKSANDROV SOLUTION OF THE MONGE-AMPÈRE EQUATION *

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Abstract. We prove the existence of piecewise polynomials strictly convex smooth functions which converge uniformly on compact subsets to the Aleksandrov solution of the Monge-Ampère equation. We extend the Aleksandrov theory to right hand side only locally integrable and on convex bounded domains not necessarily strictly convex. The result suggests that for the numerical resolution of the equation, it is enough to assume that the solution is convex and piecewise smooth.

Key words. Aleksandrov solution, Monge-Ampère, weak convergence of measures, convexity, finite elements

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1. Introduction In a previous work [2], we addressed the numerical approximation of solutions of the Dirichlet problem for the Monge-Ampère equation

$$\det D^2 u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1.1}$$

by elements of a space V_h of piecewise polynomials C^1 functions. The domain $\Omega \subset \mathbb{R}^d, d=2,3$ is assumed to be convex and bounded with boundary $\partial\Omega$. For a smooth function u, $D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,d}$ is the Hessian of u and f, g are given functions with $f \geq 0$ and $g \in C(\partial\Omega)$ with g convex on any line segment contained in $\partial\Omega$.

We considered in [2] the variational problem: find $u_h \in V_h$, $u_h = g_h$ on $\partial\Omega$ and

$$\int_\Omega (\det D^2 u_h - f) v_h \, dx = 0, \forall v_h \in V_h \cap H_0^1(\Omega). \tag{1.2}$$

Here g_h is the canonical interpolant in V_h of a smooth extension of g. Our numerical experiments indicate that problem (1.2) has a solution u_h which is convex and converges to the unique convex solution u of (1.1), *even in situations where the smoothness of u is not guaranteed*.

This points to a theoretical result we establish in this paper: given a quasi-uniform triangulation of a convex bounded domain, there exist piecewise polynomials strictly convex C^1 functions u_h which are Aleksandrov solutions of Monge-Ampère equations $\det D^2 u_h = f_h$ with $f_h > 0$ almost everywhere. Moreover $u_h|_{\partial\Omega}$ converges to g and we have $\int_\Omega f_{h_k} p \, dx \to \int_\Omega f p \, dx$ for all continuous functions p with compact support in Ω and any sequence $h_k \to 0$. The sequence u_{h_k} is shown to converge uniformly to u on compact subsets of Ω. The second contribution of this work is an approximation result of a generalized solution of (1.1) with f only locally integrable by solutions of approximate problems with right hand side integrable. The convergence of solutions of the discretization (1.2) will be addressed in a subsequent work.

By mollification of dilatations of the exact solution, we show that the Aleksandrov solution is the limit of smooth convex functions u_m, which converge uniformly to u and solve Monge-Ampère equations $\det D^2 u_m = f_m$, with f_m converging to f weakly as measures and with boundary data converging to g. We then approximate the

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functions u_m by piecewise polynomials strictly convex C^1 functions which converge uniformly to u_m.

The notion of viscosity solution of (1.1) is probably the best known notion of weak solution of the equation. Its numerical resolution by finite difference methods was considered in [11]. The notion of Aleksandrov solution is equivalent to the notion of viscosity solution for $f \in C(\overline{\Omega})$ and $f > 0$, [13, Proposition 1.7.1]. A numerical method based on Aleksandrov solutions was given in [18] for $d = 2$. Finally, there is the notion of Brenier solution of the equation with a computational fluid dynamics approach taken in [5].

We note that if one approaches the numerical resolution of (1.1) from a viscosity solution theory point of view, one naturally expects a discrete maximum principle which does not necessarily hold for the finite dimensional space V_h. It is the geometric structure of the Monge-Ampère equation, as evidenced by the Aleksandrov theory, which makes the results of this paper possible.

We organize the paper as follows. In section 2 we introduce some notation and review the notion of Aleksandrov solution of the Monge-Ampère equation. In section 3, we present a general result on approximation by smooth functions based on mollification of dilatations of the exact solution. In section 4 we prove our main result which is the existence of C^1 approximations which are piecewise smooth and converge uniformly to the Aleksandrov solution. The results are first presented for the case f bounded. In the last section we extend our results to the more general case of locally integrable right hand side f.

We assume that the reader is familiar with the basic elements of measure theory as given for example in [10].

2. Preliminaries

2.1. Notation We denote by $L^1_{loc}(\mathbb{R}^d)$ the space of locally integrable functions on $\mathbb{R}^d$. We let $C^\infty(S)$ denote the set of infinitely differentiable functions on the domain S and use the notation $\mathcal{D}(\Omega)$ for the space of infinitely differentiable functions with compact support in Ω.

We denote by $d(S,T)$ the distance between two subsets S and T of $\mathbb{R}^d$ and we use the notation $\operatorname{diam} S$ for the diameter of the set S. We use the standard notation $B(x,\rho)$ for the ball of center x and radius ρ in $\mathbb{R}^d$.

For a matrix A, we denote by A_{ij} its entries. The smallest and largest eigenvalue of the symmetric $d \times d$ matrix A are denoted respectively by $\lambda_1(A)$ and $\lambda_d(A)$. We will use the notation C for a generic constant but will index some other constants.

2.2. The Aleksandrov solution The presentation of the Aleksandrov solution of the Monge-Ampère equation given here is essentially taken from [13] to which we refer for further details. Let Ω be an open subset of $\mathbb{R}^d$.

The normal mapping or subdifferential of a real valued function v defined on Ω, is a set-valued mapping ∂v defined from Ω to the set of subsets of $\mathbb{R}^d$ such that for any $x_0 \in \Omega$,

$$\partial v(x_0) = \{ q \in \mathbb{R}^d : v(x) \geq v(x_0) + q \cdot (x - x_0), \text{for all} x \in \Omega \}.$$

For a subset $E \subset \Omega$, we define

$$\partial v(E) = \cup_{x \in E} \partial v(x).$$

Let us denote by $|E|$ the Lebesgue measure of E when E is measurable and let $v\in C(\Omega)$. The class

$$\mathcal{S}=\{E\subset\Omega,\partial v(E)\,\text{is Lebesgue measurable}\},$$

is a Borel σ-algebra and the mapping

$$M[v]:\mathcal{S}\to\overline{\mathbb{R}}, M[v](E)=|\partial v(E)|,$$

is a measure, finite on compact sets, called the Monge-Ampère measure associated with the function v.

Let Ω be a convex domain. We denote by $K(\Omega)$ the set of convex functions on Ω.

DEFINITION 2.1. *Let μ be a Borel measure on Ω. A function $v\in K(\Omega)$ is an Aleksandrov solution of*

$$\det D^2 v=\mu,$$

if and only if $M[v]=\mu$. We recall that a measure μ is absolutely continuous with respect to the Lebesgue measure and with density f if and only if

$$\mu(B)=\int_B f\,dx, \text{for any Borel set}\, B.$$

The measure μ is then identified with f. We have

THEOREM 2.2 ([14] Theorem 1.1). *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Assume $f\in L^1(\Omega)$ and $g\in C(\partial\Omega)$ can be extended to a function $\tilde{g}\in C(\overline{\Omega})$ which is convex in Ω. Then the Monge-Ampère equation* (1.1) *has a unique Aleksandrov solution in $K(\Omega)\cap C(\overline{\Omega})$.* The general case of locally integrable right hand side f is addressed in section 5.

COROLLARY 2.3. *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Assume $f\in C(\overline{\Omega})$ and $g\in C(\partial\Omega)$. Then the Monge-Ampère equation* (1.1) *has a unique Aleksandrov solution in $K(\Omega)\cap C(\overline{\Omega})$.*

Proof. The result follows from the equivalence of viscosity and Aleksandrov solution when $f>0$, [13, Proposition 1.7.1]. In the general case, we note that (1.1) has a unique convex viscosity solution [16]. Thus g extends to a continuous function on Ω, namely the viscosity solution, and we can apply Theorem 2.2. □

Let v in $K(\Omega)\cap C^2(\Omega)$, we have

$$M[v](E)=\int_E \det D^2 v(x)\,dx,$$

for all Borel sets $E\subset\Omega$.

DEFINITION 2.4. *A sequence μ_m of Borel measures converges weakly to a Borel measure μ if and only if*

$$\int_\Omega p(x)\,d\mu_m\to\int_\Omega p(x)\,d\mu,$$

for every continuous function p with compact support in Ω. We also have

DEFINITION 2.5. *Assume that $f_m,f\geq 0$. The sequence f_m converges weakly to f as measures if and only if*

$$\int_\Omega f_m p\,dx\to\int_\Omega fp\,dx,$$

for all continuous functions p with compact support in Ω. We have the following weak continuity result of Monge-Ampère measures with respect to local uniform convergence.

LEMMA 2.1 ([13] Lemma 1.2.3). *Let u_m be a sequence of convex functions in Ω such that $u_m \to u$ uniformly on compact subsets of Ω. Then $M[u_m]$ tend to $M[u]$ weakly.*

REMARK 2.6. *If u_m is a sequence of $C^2(\Omega)$ convex functions such that $u_m \to u$ uniformly on compact subsets of Ω, with u solving* (1.1), *then* $\det D^2 u_m$ *converges weakly to f as measures.*

3. Approximation by mollified functions Let $u \in C(\overline{\Omega})$ be a convex function on the convex bounded domain Ω. Then $\overline{\Omega}$ is convex and u is convex on $\overline{\Omega}$. For $1 < \lambda \leq 2$ and for $x_0 \in \Omega$ define $\Omega^\lambda = \{x' \in \mathbb{R}^n, x' = x_0 + \lambda(x - x_0), \text{for some} \, x \in \Omega\}$ and for $x' \in \Omega^\lambda, x' = x_0 + \lambda(x - x_0)$, we define $u^\lambda(x') = u(x)$. Let

$$d_\lambda \equiv d(\partial\Omega, \partial(\Omega^\lambda)). \tag{3.1}$$

LEMMA 3.1. *We have*

$$d_\lambda > 0, \tag{3.2}$$

for $1 < \lambda \leq 2$.

Proof. Assume that $d_\lambda = 0$ and let $y \in \partial\Omega$. There would exist a sequence $x'_n \in \partial(\Omega^\lambda)$ such that $y = \lim_{n\to\infty} x'_n = \lim_{n\to\infty} x_0 + \lambda(x_n - x_0)$ for $x_n \in \partial\Omega$. Put $x = \lim_{n\to\infty} x_n$ and note that $x \in \partial\Omega$. We have $y = x_0 + \lambda(x - x_0)$ and hence x_0, x and y are on the same line L with $x_0 \in \Omega$ and both x and y in $\partial\Omega$.

A convex bounded domain is Lipschitz continuous [12]. Thus there exists points z_1 and z_2 in $L \cap \Omega$ such that the the rays $\{tz_1 + (1-t)x, 0 < t \leq 1\}$ and $\{ty + (1-t)z_2, 0 \leq t < 1\}$ is entirely contained in Ω. But then the line segment $[z_1, z_2]$ is not entirely contained in Ω. This contradicts the convexity of Ω.

□

LEMMA 3.2. *The set Ω^λ is open, convex and u^λ is convex in Ω^λ. Moreover $\overline{\Omega} \subset \Omega^\lambda$ for $\lambda > 1$, and u^λ converges uniformly to u on $\overline{\Omega}$ as $\lambda \to 1^+$.*

Proof. The proof of the first three statements readily follows from the definitions. We first prove that $\Omega \subset \Omega^\lambda$ and since $d_\lambda > 0$ by Lemma 3.1 for $\lambda > 1$ with both Ω and Ω^λ open, this would prove that $\overline{\Omega} \subset \Omega^\lambda$.

Let $z' \in \partial(\Omega^\lambda)$. We first show that $z' = x_0 + \lambda(z - x_0), z \in \partial\Omega$. There exists a sequence $z'_m \in \Omega^\lambda$ such that $z' = \lim_{m\to\infty} z'_m = \lim_{m\to\infty} x_0 + \lambda(z_m - x_0)$ for $z_m \in \Omega$. We therefore have $z' = x_0 + \lambda(z - x_0) = (1-\lambda)x_0 + \lambda z, z \in \overline{\Omega}$. If $z \in \Omega$ we must have $z' \in \Omega^\lambda$ by definition of Ω^λ. Therefore $z \in \partial\Omega$. It is therefore not restrictive to write $\partial\Omega^\lambda$ or $(\partial\Omega)^\lambda$ for $\partial(\Omega^\lambda)$.

Let $y \in \Omega$. Since $\lambda > 1$, $t = 1/\lambda \in [0,1)$ and hence $0 \leq 1 - t \leq 1$. Since Ω is convex, $x = (1-t)x_0 + ty \in \Omega$. But $y = x_0 + \lambda(x - x_0)$ and hence $y \in \Omega^\lambda$. This proves that $\Omega \subset \Omega^\lambda$. We have proved that $\overline{\Omega} \subset \Omega^\lambda$.

We now prove that u^λ converges uniformly to u on $\overline{\Omega}$ as $\lambda \to 1^+$. Since points on $\partial\Omega$ are limits of points in Ω, it is enough to prove the uniform convergence on Ω. Let $\epsilon > 0$. We seek $\lambda_0 \in (1,2)$ such that $1 < \lambda < \lambda_0$ implies $|u^\lambda(x') - u(x')| < \epsilon$ for all $x' \in \Omega$.

Put $l = \max\{|x_0 - x'|, x' \in \Omega\}$. Since u is uniformly continuous on $\overline{\Omega}$, we can choose δ such that $0 < \delta < l$ and $|x' - y'| < \delta$ implies $|u(x') - u(y')| < \epsilon$ for all $x', y' \in \Omega$. We choose $\lambda_0 = l/(l - \delta)$. For $\lambda < \lambda_0$ and $x' \in \Omega$, we have $(1 - 1/\lambda)l < \delta$ and hence

$$\left| \left((1 - \frac{1}{\lambda})x_0 + \frac{1}{\lambda}x' \right) - x' \right| = (1 - \frac{1}{\lambda})|x_0 - x'| \leq (1 - \frac{1}{\lambda})l < \delta.$$

For $x' \in \Omega$, since

$$u^\lambda(x') - u(x') = u\bigg((1-\frac{1}{\lambda})x_0 + \frac{1}{\lambda}x'\bigg) - u(x'),$$

the result follows. □

REMARK 3.1. *The idea to use dilations of the domain for smooth approximations of convex functions in a Sobolev space was first used in [1]. We are interested in this paper in uniform convergence.*

Let $\phi \geq 0$ be infinitely differentiable with compact support in $\{x \in \Omega, |x| < 1\}$ and $\int_{\mathbb{R}^d} \phi \, dx = 1$. For $\epsilon > 0$, let $\phi_\epsilon(x) = 1/\epsilon^d \phi(x/\epsilon)$ and for $v \in L^1(\Omega)$, we define the regularization or mollification of v by

$$v_\epsilon(x) = \phi_\epsilon * v(x) = \int_\Omega \phi_\epsilon(x-y) v(y) \, dy,$$

which is well defined on

$$\Omega_\epsilon = \{ x \in \Omega, d(x, \partial\Omega) > \epsilon \}.$$

For a convex function v, v_ϵ is also convex as a linear combination with positive coefficients of convex functions as

$$v_\epsilon(x) = v * \phi_\epsilon(x) = \int_{B(0,\epsilon)} v(x-y) \phi_\epsilon(y) \, dy.$$

We recall that $v_\epsilon \in C^\infty(\Omega_\epsilon)$ and v_ϵ converges uniformly to $v \in C(\Omega)$ on compact subsets of Ω_ϵ. Properties of mollification are discussed for example in [21]. We state one of the main results of this paper

THEOREM 3.2. *Let u be the convex solution of the Monge-Ampère equation* (1.1). *There exists a sequence $u_\epsilon \in C^\infty(\overline{\Omega})$ of convex functions, obtained by dilatations and convolutions of u, such that $\det D^2 u_\epsilon = f_\epsilon$ converges to f weakly as measures and u_ϵ converges uniformly to u on $\overline{\Omega}$. Explicitly*

$$u_\epsilon = u^\lambda_{\delta_\lambda}, \, with \, \lambda = 1 + \epsilon \, and \lim_{\lambda \to 1^+} \delta_\lambda = 0. \tag{3.3}$$

Proof. Recall from Lemma 3.2 that $d_\lambda = d(\partial\Omega, \partial(\Omega^\lambda)) > 0$ and so $u^\lambda_{\epsilon'} \in C^\infty(\overline{\Omega})$ for $\epsilon' \leq d_\lambda/2$. For $1 < \lambda \leq 2$, since $u^\lambda_{\epsilon'} \to u^\lambda$ uniformly on $\overline{\Omega}$, $\forall \epsilon > 0, \exists \delta_\lambda$ such that $0 < \delta_\lambda \leq \frac{d_\lambda}{2}$ and

$$|u^\lambda_{\delta_\lambda}(x) - u^\lambda(x)| < \epsilon/2, \forall x \in \overline{\Omega}.$$

To alleviate the notation, we do not explicitly write the dependence of δ_λ on ϵ. We define for $\epsilon > 0$

$$u_\epsilon = u^\lambda_{\delta_\lambda}, \text{with} \, \lambda = 1 + \epsilon.$$

Let $\gamma > 0$. By Lemma 3.2, $u^\lambda \to u$ uniformly on $\overline{\Omega}$. Thus, $\exists \epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ we have $|u^{1+\epsilon}(x) - u(x)| < \gamma/2$. For $0 < \epsilon < \min(\epsilon_0, \gamma)$ and $x \in \overline{\Omega}$ we have

$$\begin{aligned} |u_\epsilon(x) - u(x)| &\leq |u_\epsilon(x) - u^{1+\epsilon}(x)| + |u^{1+\epsilon}(x) - u(x)| \\ &\leq \frac{\epsilon}{2} + \frac{\gamma}{2} < \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma. \end{aligned}$$

This proves that $u_\epsilon \to u$ uniformly on $\overline{\Omega}$. By Lemma 3.2, u_ϵ is convex and in $C^\infty(\overline{\Omega})$. By Remark 2.6, $\det D^2 u_\epsilon$ converges to f weakly as measures.
□

We now establish that the sequence u_ϵ from Theorem 3.2 may be assumed to be uniformly strictly convex.

THEOREM 3.3. *There exists a sequence u_ϵ which satisfies the conditions of Theorem 3.2 and such that $\det D^2 u_\epsilon \geq \epsilon^d$ and $|(D^2 u_\epsilon)_{i,j}| \leq C_0(\epsilon), i,j=1,\ldots,d$ for a constant $C_0>0$ which depends on ϵ and $\lim_{\epsilon\to 0} C_0(\epsilon)=\infty$. Moreover the smallest eigenvalue of $\operatorname{cof} D^2 u_\epsilon$ is uniformly in x bounded below by $C_1(\epsilon)=\epsilon^d/(dC_0(\epsilon))$.*

Proof. Let us denote by $\hat{u}_\epsilon$ the sequence given by Theorem 3.2. We choose $y_0 \in \Omega$ and define

$$w_\epsilon = \frac{\epsilon}{2}|x-y_0|^2.$$

We have $w_\epsilon \in C^\infty(\overline{\Omega})$ and $D^2 w_\epsilon = \epsilon^d I$ where I is the $d\times d$ identity matrix. Put

$$u_\epsilon(x) = \hat{u}_\epsilon(x) + w_\epsilon(x).$$

Then $u_\epsilon \in C^\infty(\overline{\Omega})$, is convex and converges uniformly to u on $\overline{\Omega}$. It then follows by Lemma 2.1 that $\det D^2 u_\epsilon$ converges to f weakly as measures.

Next, using for example Proposition 3.3 of [19], we have

$$\det D^2 u_\epsilon \geq \det D^2 \hat{u}_\epsilon + \det D^2 w_\epsilon \geq \epsilon^d.$$

Let us denote by $\lambda_i, i=1,\ldots,d$, the eigenvalues of $D^2 u_\epsilon$ with $0<\lambda_1 \leq \ldots \leq \lambda_d$. Since $D^2 u_\epsilon$ is symmetric and invertible, $\operatorname{cof} D^2 u_\epsilon = (\det D^2 u_\epsilon)(D^2 u_\epsilon)^{-1}$ has smallest eigenvalue $(\det D^2 u_\epsilon)/\lambda_d$.

On the other hand, recall from (3.3), that $\hat{u}_\epsilon$ is obtained from u by dilatation and convolution, i.e. $\hat{u}_\epsilon = u_\delta^\lambda$ for some λ and δ which depends on ϵ. For $i,j=1,\ldots,d$

$$\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j} u_\epsilon(x) &= \frac{1}{\delta^d}\int_{\mathbb{R}^n} \frac{\partial^2}{\partial x_i \partial x_j}\left(\phi\left(\frac{x-y}{\delta}\right)\right) u^\lambda(y)\,dy + \epsilon\kappa_i^j \\
&= \frac{1}{\delta^{d+2}}\int_{\mathbb{R}^n} \frac{\partial^2 \phi}{\partial x_i \partial x_j}\left(\frac{x-y}{\delta}\right) u^\lambda(y)\,dy + \epsilon\kappa_i^j \\
&= \frac{1}{\delta^{d+2}}\int_{B(x,\delta)} \frac{\partial^2 \phi}{\partial x_i \partial x_j}\left(\frac{x-y}{\delta}\right) u^\lambda(y)\,dy + \epsilon\kappa_i^j,
\end{aligned}$$

where $\kappa_i^j=1$ if $i=j$ and $\kappa_i^j=0$ otherwise. Since ϕ and its derivatives have compact support, and u is bounded on $\overline{\Omega}$, we conclude that

$$|\partial^2 u_\epsilon/(\partial x_i \partial x_j)| \leq C(\epsilon) = \frac{C}{\delta^2} + \epsilon, \tag{3.4}$$

for a constant $C>0$ which is independent of ϵ and x. It follows that $|\lambda_i| \leq dC(\epsilon), i=1,\ldots,d$, by using for example the Gershgorin circle theorem. We then get that the smallest eigenvalue of $\operatorname{cof} D^2 u_\epsilon$ is bounded below by $\epsilon^d/(dC(\epsilon))$.

Explicitly $\hat{u}_\epsilon = u_\delta^\lambda$ with $\lambda = 1+\epsilon$, $0<\delta\leq d_\lambda/2$ and $\lim_{\lambda\to 1} d_\lambda = 0$. Using (3.4), we conclude that $\lim_{\epsilon\to 0} C_0(\epsilon)=\infty$.
□

4. Approximation by piecewise polynomials C^1 functions In this section we assume that $0 \leq f \leq M$ for a constant M. For simplicity, for the construction of the finite dimensional spaces, we now assume that the domain Ω has a polygonal boundary. We establish our main result on approximation of the Monge-Ampère equation (1.1) by piecewise polynomial C^1 functions.

4.1. Additional notation and approximation results

We recall the standard notation $W^{k,p}(\Omega)$ for the Sobolev spaces and use $||.||_{k,p}$ and $|.|_{k,p}$ respectively for their norms and semi-norms. We will also use the notation $H^k(\Omega)$ for $W^{k,2}(\Omega)$ and in this case, the norm and semi-norms are denoted respectively by $||.||_k$ and $|.|_k$.

Let $\mathcal{T}$ denote a triangulation of Ω into simplices K which is conforming in the sense that the intersection of any two simplices is either empty or is a vertex, an edge or more generally a common subface. We denote by h_K the diameter of K and by ρ_K the radius of the largest ball contained in K. We assume that the triangulation is shape regular, i.e. there exists a constant $C > 0$ such that for any triangle K, $h_K/\rho_K \leq C$. We also assume that the triangulation is quasi-uniform, i.e. $h/h_{min} \leq C$ where h and h_{min} are the maximum and minimum respectively of $\{h_K, K \in \mathcal{T}_h\}$.

We let V_h denote a finite dimensional space of piecewise polynomial C^1 functions of local degree $r \geq 3$, i.e., V_h is a subspace of

$$\{s \in C^1(\Omega),\ s|_t \in \mathcal{P}_r,\ \forall t \in \mathcal{T}\},$$

where $\mathcal{P}_r$ denotes the space of polynomials of degree less than or equal to r. Such spaces can be realized as finite element spaces [7] or more generally as spline spaces [17]. We make the assumption that the degree r is sufficiently high so that the following approximation properties hold:

$$||v - I_h v||_{k,p} \leq C_{ap} h^{l+1-k} |v|_{l+1,p}, \tag{4.1}$$

where I_h is an interpolation operator mapping the Sobolev space $W^{l+1,p}(\Omega)$ into V_h, $1 \leq p \leq \infty$ and $0 \leq k \leq l \leq r$. Unless the mesh has a special form, in general one needs $r \geq 5$ for $d = 2$ and $r \geq 9$ for $d = 3$. The constant C_{ap} depends only on r, l, the domain Ω and is independent of h. We also make the assumption that the following inverse inequality holds

$$||v||_{s,p} \leq C_{inv} h^{l-s+\min(0, \frac{d}{p} - \frac{d}{q})} ||v||_{l,q}, \forall v \in V^h, \tag{4.2}$$

and for $0 \leq l \leq s, 1 \leq p, q \leq \infty$. The constant C_{inv} is independent of h. The above assumptions hold for finite element spaces [7].

4.2. Discrete convexity

Lemma 4.1 ([15]). *For two symmetric $d \times d$ matrices A and B, we have*

$$|\lambda_1(A) - \lambda_1(B)| \leq c_d \max_{i,j} |A_{ij} - B_{ij}|,$$

where c_d is a constant which depends only on d.

Remark 4.1. *It follows from Lemma 4.1 that for $\max_{i,j} |A_{ij} - B_{ij}| < \lambda_1(B)/(2c_d)$, and B positive definite, A is also positive definite with $\lambda_1(A) \geq \lambda_1(B)/2$.*

The proof of the following result can be found in section 5 of [8].

Lemma 4.2. *Let v_h be a function which is a polynomial of degree r and convex on each element K. Assume that $v_h \in C^1(\Omega)$. Then v_h is convex.*

4.3. Approximation results In [2] and also [6], the following result was proven

THEOREM 4.2. *Let u_ϵ and g_ϵ be $C^\infty(\overline{\Omega})$ functions such that $\det D^2 u_\epsilon = f_\epsilon > c_\epsilon > 0$ with $u_\epsilon = g_\epsilon$ on $\partial\Omega$. Let $g_{\epsilon,h} = I_h g_\epsilon$. Then the problem: find $u_{\epsilon,h} \in V_h$, $u_{\epsilon,h} = g_{\epsilon,h}$ on $\partial\Omega$ and*

$$\int_\Omega (\det D^2 u_{\epsilon,h} - f_\epsilon) v_h \, dx = 0, \forall v_h \in V_h \cap H_0^1(\Omega),$$

has a unique solution $u_{\epsilon,h}$ in a sufficiently small neighborhood of $I_h u_\epsilon$ and we have

$$||u_\epsilon - u_{\epsilon,h}||_1 \leq C h^l ||u_\epsilon||_{l+1}, 5 \leq l \leq r.$$

From the above result, one derives easily an estimate in the $H^2(\Omega)$ Sobolev norm. Recall that $V_h \subset H^2(\Omega)$. We have by (4.1) and (4.2)

$$\begin{aligned}
||u_\epsilon - u_{\epsilon,h}||_2 &\leq ||u_\epsilon - I_h u_\epsilon||_2 + ||I_h u_\epsilon - u_{\epsilon,h}||_2 \\
&\leq C h^{l-1} ||u_\epsilon||_{l+1} + C h^{-1} ||I_h u_\epsilon - u_{\epsilon,h}||_1 \\
&\leq C h^{l-1} ||u_\epsilon||_{l+1} + C h^{-1} ||I_h u_\epsilon - u_\epsilon||_1 + C h^{-1} ||u_\epsilon - u_{\epsilon,h}||_1 \\
&\leq C h^{l-1} ||u_\epsilon||_{l+1}.
\end{aligned}$$

By the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$, we have

$$||u_\epsilon - u_{\epsilon,h}||_{0,\infty} \leq C ||u_\epsilon - u_{\epsilon,h}||_2 \leq C h^{l-1} ||u_\epsilon||_{l+1}, 5 \leq l \leq r.$$

It follows that $u_{\epsilon,h}$ converges uniformly to u_ϵ on compact subsets of Ω.

Moreover, again by (4.1) and (4.2), using $d = 2, 3$

$$\begin{aligned}
||u_\epsilon - u_{\epsilon,h}||_{2,\infty} &\leq ||u_\epsilon - I_h u_\epsilon||_{2,\infty} + ||I_h u_\epsilon - u_{\epsilon,h}||_{2,\infty} \\
&\leq C h^{l-1} ||u_\epsilon||_{l+1,\infty} + h^{-\frac{d}{2}} ||I_h u_\epsilon - u_{\epsilon,h}||_2 \\
&\leq C h^{l-1} ||u_\epsilon||_{l+1,\infty} + C h^{l-1-\frac{d}{2}} ||u_\epsilon||_{l+1} \\
&\leq C h^{l-4} ||u_\epsilon||_{l+1,\infty}.
\end{aligned}$$

Since $5 \leq l \leq r$, for $h \leq h_\epsilon$ for some h_ϵ, we have $C h^{l-4} ||u_\epsilon||_{l+1,\infty} \leq \lambda_1(D^2 u_\epsilon)/(2c_d)$. Thus $\max_{ij} |(D^2 u_\epsilon)_{ij} - (D^2 u_{\epsilon,h})_{ij}| \leq ||u_\epsilon - u_{\epsilon,h}||_{2,\infty} \leq \lambda_1(D^2 u_\epsilon)/(2c_d)$. Since $f_\epsilon > c_\epsilon > 0$, there exists a constant which we also denote by c_ϵ such that $\lambda_1(D^2 u_\epsilon) \geq c_\epsilon$. Hence by Remark 4.1 and Lemma 4.2, $u_{\epsilon,h}$ is piecewise strictly convex and convex.

We now consider the family $\{u_{\epsilon,h}, h \leq h_\epsilon\}$ of convex functions and can now prove one of the main results of this paper.

THEOREM 4.3. *The family $\{u_{\epsilon,h}, h \leq h_\epsilon\}$ has a subsequence which converges uniformly on compact subsets of Ω to the solution u of (1.1).*

Proof. Let K be a compact subset of Ω. Since $u_\epsilon \to u$ uniformly on K, for $k \geq 1$, $\exists \epsilon_k > 0$ such that for $0 < \epsilon \leq \epsilon_k$, and $x \in K$, $|u(x) - u_\epsilon(x)| < 1/(2k)$. Since $u_{\epsilon_k,h} \to u_{\epsilon_k}$ uniformly on K, we can choose $h_k < h_{\epsilon_k}$ such that $|u_{\epsilon_k}(x) - u_{\epsilon_k,h_k}(x)| < 1/(2k)$ for all $x \in K$. The result then follows from the triangular inequality. □

As a consequence of Lemma 2.1, $\det D^2 u_{\epsilon_k,h_k}$ converges weakly to f as measures.

REMARK 4.4. *Although Theorem 4.3 asserts the existence of a discrete strictly convex function, the latter may have a discrete Hessian with very small eigenvalues, in absolute value, prohibiting the direct use of Newton's method. The use of fixed point iterative methods, as in [2], which enforce locally convexity are the best options available.*

For simplicity, we now refer to $u_{\epsilon,h}$ as u_h for an arbitrary family $\epsilon \to 0$, e.g. $\epsilon = 1/m, m = 1, 2 \ldots$

5. The general case of locally integrable right hand side f In this part, we show how our results can be extended to the general case where $0 \leq f$ but with f locally integrable. Bakelman [3, 4] introduced a notion of weak solution of the problem

$$\begin{aligned} \det D^2 u &= \mu \text{ in } \Omega \\ u &= g \, \text{on} \, \partial\Omega, \end{aligned} \tag{5.1}$$

where μ is not necessarily a finite Borel measure and with the boundary condition satisfied in a generalized sense. We first recall what he called the border of a convex function and the notion of solution in the sense of Bakelman. We extend his existence and uniqueness results when μ has density f to the case where the domain is only assumed to be convex. We do it by considering the problems

$$\begin{aligned} \det D^2 u &= f_M \text{ in } \Omega \\ u &= g \, \text{on} \, \partial\Omega, \end{aligned} \tag{5.2}$$

where $f_M(x) = f(x)$ for $f(x) \leq M$ and $f_M(x) = 0$ otherwise. We show that the solutions u_M of (5.2) converge uniformly on compact subsets of Ω to a generalized solution in the sense of Bakelman of (1.1). And so do the approximate solutions $u_{M,h}$.

We now view $\mathbb{R}^d$ as a hyperplane of $\mathbb{R}^{d+1}$. Let v be a bounded convex function on Ω. We denote by S_v the graph of v and by $\overline{Co}(S_v)$ the closed convex hull of the graph of v. Let Z be the cylinder with base $\partial\Omega$ and generators parallel to the x_{d+1} axis. It can be shown that the closed set

$$H = Z \cap \overline{Co}(S_v),$$

is a union of sets $l(x), x \in \partial\Omega$ and $l(x)$ is either a point $\{(x, z_0)\}$, a closed segment $\{(x,z), z_0 \leq z \leq z_1\}$ or a closed ray $\{(x,z) z_0 \leq z < \infty\}$. See [3] and [4, p. 128]. The function defined on $\partial\Omega$ by

$$g_v(x) = z_0, (x, z_0) \in l(x),$$

is called the *border of the convex function* $v(x)$.

Remark 5.1. *If $v \in K(\Omega) \cap C(\overline{\Omega})$, then $g_v = v$ on $\partial\Omega$.* We have

Lemma 5.1. *Assume that v and w are convex functions on Ω such that $v(x) \leq w(x), x \in \Omega$. Then $g_v \leq g_w$ on $\partial\Omega$.*

Proof. Since $v \leq w$ in Ω, for $x \in \partial\Omega$, points (x,z) in $Z \cap \overline{Co}(S_v)$ are below the corresponding points (x,z) in $Z \cap \overline{Co}(S_w)$. By definition of border as an infimum, we obtain the result. □

Definition 5.2. *We assume that the measure μ is finite on compact subsets of Ω. Let $V(\mu, g)$ denote the set of convex functions v on Ω such that $M[v] = \mu$ and $g_v(x) \leq g(x)$ for all $x \in \partial\Omega$. A convex function $u \in V(\mu, g)$ is a solution of* (5.1) *in the sense of Bakelman if*

$$g_u(x) \geq g_v(x), x \in \partial\Omega,$$

for all $v \in V(\mu, g)$.

Remark 5.3. *Assume that Ω is a bounded convex domain of $\mathbb{R}^d$ and g can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. Then if μ is a finite Borel measure,* (5.1) *has a unique convex Aleksandrov solution in $C(\overline{\Omega})$ which assumes the boundary condition in the classical sense. (c.f. [14, Theorem 1.1]).*

As a consequence of [4, Theorem 10.4] we have

THEOREM 5.4. *Let Ω be a bounded convex domain of $\mathbb{R}^d$ and let μ be a Borel measure on Ω. Assume that z_ϵ is a family of convex functions such that*

$$-\infty < r_1 \leq z_\epsilon|_{\partial\Omega} \leq r_2 < \infty$$
$$M[z_\epsilon] \leq \mu.$$

Then we have

$$r_1' \leq z_\epsilon \leq r_2, x \in \Omega,$$

where r_1' depends only on r_1, μ and Ω.

In [3] a comparison principle is proved for convex solutions of (1.1) which are not necessarily in $C(\overline{\Omega})$. We give a new proof based on the proof for $C(\overline{\Omega})$ solutions given in [13].

THEOREM 5.5. *Let Ω be a bounded convex domain and let z_1 and z_2 be convex functions in Ω such that $r \geq z_1 \geq z_2$ on $\partial\Omega$, $r \in \mathbb{R}$. Assume that*

$$M[z_1] \leq M[z_2].$$

Then $z_1 \geq z_2$ in Ω.

Proof. The theorem without the requirement of an upper bound for z_1 and z_2 on $\partial\Omega$ is given as Theorem 10.1 of [3]. If z_1 and z_2 are in $C(\overline{\Omega})$, the proof can be found in [13] Theorem 1.4.6. We use a contradiction argument. Let

$$U_1 = \{x \in \Omega, z_1(x) < z_2(x)\}.$$

Assume that U_1 is non-empty. Since z_1 and z_2 are convex, they are uniformly bounded above by r and are therefore continuous. Thus U_1 is open. Let U be a non-empty open subset of U_1 such that $\overline{U} \subset U_1$. Let $a = \min_{x \in \overline{U}}\{z_1(x) - z_2(x)\}$. Put $b = \inf_{x \in \partial\Omega}\{z_1(x) - z_2(x)\}$. By assumption $a \leq 0 \leq b$. There exists $x_0 \in U_1$ such that $a = z_1(x_0) - z_2(x_0)$. It follows that $a < 0 \leq b$. We choose $\delta > 0$ such that $\delta(\operatorname{diam}\Omega)^2 < (b-a)/2$. We define

$$w(x) = z_2(x) + \delta|x - x_0|^2 + \frac{b+a}{2}.$$

Let $G = \{x \in \overline{U}, z_1(x) < w(x)\}$. We have $x_0 \in G$ and as in [13] Theorem 1.4.6, one shows that $G \cap \partial\Omega = \emptyset$ with $M[z_1](G) > M[z_2](G)$ contradicting the assumption of the theorem. We conclude that U_1 must be empty and hence $z_1 \geq z_2$ in Ω. □

We will also need

LEMMA 5.2 ([14] Lemma 5.1). *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain, μ_j, μ be Borel measures in Ω with μ finite, $u_j \in C(\overline{\Omega})$ convex in Ω and g can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω with*

1. *$u_j = g$ on $\partial\Omega$,*
2. *$M[u_j] = \mu_j$ in Ω,*
3. *$\mu_j \to \mu$ weakly in Ω, and*
4. *$\mu_j(\Omega) \leq A$ for all j for a constant A.*

Then there exists $u \in C(\overline{\Omega})$ convex in Ω such that u_j converges, up to a subsequence, to u uniformly on compact subsets of Ω, and $M[u] = \mu$ with $u = g$ on $\partial\Omega$.

We can now prove the following theorem.

THEOREM 5.6. *Assume that Ω is a bounded convex domain of $\mathbb{R}^d$ and g can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. If $f \geq 0$ is locally integrable, the problem* (1.1) *has a unique convex solution in the sense of Bakelman. Moreover the solution u is the uniform limit on compact subsets of Ω of the solutions u_M of the problems* (5.2) *with truncated right hand sides as $M \to \infty$.*

Proof. For $\epsilon > 0$ sufficiently small, we recall that

$$\Omega_\epsilon = \{x \in \Omega, d(x, \partial\Omega) > \epsilon\}.$$

We define a finite Borel measure μ_ϵ by

$$\mu_\epsilon(B) = \int_{B \cap \Omega_\epsilon} f(x)\,dx.$$

Given $M > 0$, we define finite Borel measures $\mu_{\epsilon,M}$ by

$$\mu_{\epsilon,M}(B) = \int_{B \cap \Omega_\epsilon} f_M(x)\,dx.$$

Part 1: Limits as $M \to \infty$.

We claim that up to a subsequence $\mu_{\epsilon,M}$ converges weakly to μ_ϵ as $M \to \infty$. By [9, Theorem 1, section 1.9], it is enough to show that $\mu_{\epsilon,M}(B) \to \mu_\epsilon(B)$ for any bounded Borel set B with $\mu_\epsilon(\partial B) = 0$. Since $f_M \to f$ a.e. as $M \to \infty$, by Fatou's lemma

$$\mu_\epsilon(B) \leq \liminf_{M \to \infty} \mu_{\epsilon,M}(B).$$

Moreover, $0 \leq f_M(x) \leq f(x)$ for all $x \in \Omega$. Thus $\mu_{\epsilon,M}(B) \leq \mu_\epsilon(B)$ and this proves the claim.

Note also that $\mu_{\epsilon,M}(\Omega) \leq \mu_\epsilon(\Omega) < \infty$ and $\mu_\epsilon(\Omega)$ is independent of M.

By Remark 5.3, there exists $u_{\epsilon,M} \in C(\overline{\Omega})$ such that

$$\begin{aligned} \det D^2 u_{\epsilon,M} &= \mu_{\epsilon,M} \text{ in } \Omega \\ u_{\epsilon,M} &= g \,\text{on}\, \partial\Omega. \end{aligned}$$

By Lemma 5.2, as $M \to \infty$, $u_{\epsilon,M}$ converges uniformly on compact subsets of Ω, up to a subsequence, to the convex solution u_ϵ of the problem

$$\begin{aligned} \det D^2 u_\epsilon &= \mu_\epsilon \text{ in } \Omega \\ u_\epsilon &= g \,\text{on}\, \partial\Omega. \end{aligned}$$

Part 2: Existence and uniqueness of the Bakelman solution.

We denote by μ the measure with density f, i.e.

$$\mu(B) = \int_B f(x)\,dx,$$

for any Borel set B. Since $\mu_\epsilon \leq \mu$ and $u_\epsilon = g \,\text{on}\, \partial\Omega$, applying Theorem 5.4 to the family u_ϵ, we obtain the uniform boundedness in x of u_ϵ. We conclude the existence of a function $u(x)$ and a subsequence u_{ϵ_m} such that $u_{\epsilon_m}(x)$ converges pointwise to $u(x)$. As a limit of convex functions, u is necessarily convex. Moreover by [20, Theorem 10.8], the convergence is uniform on compact subsets of Ω. By Lemma 2.1, $M[u_{\epsilon_m}] \to M[u]$

weakly. Again by [9, Theorem 1, section 1.9], $M[u_{\epsilon_m}](B)\to M[u](B)$ for any bounded Borel set B with $M[u](\partial B)=0$. But

$$M[u_{\epsilon_m}](B)=\int_{B\cap\Omega_{\epsilon_m}} f(x)\,dx\to\int_B f(x)\,dx, \text{as} \epsilon_m\to 0.$$

We conclude that $M[u](B)=\int_B f(x)\,dx$ for any closed ball B contained in Ω and hence $M[u]$ has density f.

Next, for $\epsilon'<\epsilon$, we have $\mu_{\epsilon'}\geq\mu_\epsilon$ with $u_{\epsilon'}=u_\epsilon$ on $\partial\Omega$. Since $g\in C(\partial\Omega)$, u_ϵ is uniformly bounded above on $\partial\Omega$. By the comparison principle given in Theorem 5.5 we have $u_{\epsilon'}\leq u_\epsilon$ in Ω and hence $u\leq u_\epsilon$ for all ϵ. By Lemma 5.1, we have $g_u\leq g_{u_\epsilon}=g$ on $\partial\Omega$.

Finally, let $v\in V(\mu,g)$. By definition $M[v]=\mu$ and $g_v\leq g$ on $\partial\Omega$. We show that $g_u\geq g_v$. Note that $u_\epsilon=g\geq g_v$ on $\partial\Omega$ with $M[u_\epsilon]=\mu_\epsilon\leq\mu=M[v]$. By Theorem 5.5, we conclude that $u_\epsilon\geq v$ in Ω and hence $u\geq v$ in Ω. By Lemma 5.1, we obtain $g_u\geq g_v$. This proves that u is a Bakelman solution of (1.1).

The uniqueness of the Bakelman solution is an immediate consequence of Theorem 5.5.

Part 3: Limits as $\epsilon\to 0$.

Let the finite Borel measure μ_M be given by

$$\mu_M(B)=\int_B f_M(x)\,dx,$$

and let 1_D denote the indicator function of the set D. Then $\mu_{\epsilon,M}(B)=\int_B 1_{\Omega_\epsilon} f_M(x)\,dx$ and $\mu_{\epsilon,M}(B)\leq\mu_\epsilon(B)$ for all $\epsilon>0$. Moreover $1_{\Omega_\epsilon} f_M(x)\to f_M(x)$ a.e as $\epsilon\to 0$. With the same arguments as above, one shows that $\mu_{\epsilon,M}\to\mu_M$ weakly as $\epsilon\to 0$ with $\mu_{\epsilon,M}(\Omega)\leq\mu_M(\Omega)<\infty$ and $\mu_M(\Omega)$ is independent of ϵ.

Thus again by Lemma 5.2, as $\epsilon\to 0$, $u_{\epsilon,M}$ converges uniformly on compact subsets of Ω, up to a subsequence, to the solution u_M of the problem

$$\begin{aligned}\det D^2 u_M &= \mu_M \text{ in } \Omega\\ u_M &= g \,\text{on}\, \partial\Omega.\end{aligned}$$

Part 4. Finishing up.

We show that the Bakelman solution u is the uniform limit on compact subsets of Ω of u_M as $M\to\infty$. Note that we can not use the approach in part I as f is assumed to be only locally integrable. However the approach taken in part II can be adapted. For the convenience of the reader we explicitly repeat the argument.

Since $\mu_M\leq\mu$ and $u_M=g\,\text{on}\,\partial\Omega$, applying Theorem 5.4 to the family u_M, we obtain the uniform boundedness in $x\in\Omega$ of $u_M(x)$. We conclude the existence of a function $w(x)$ and a subsequence u_{M_k} such that $u_{M_k}(x)$ converges pointwise to $w(x)$. As a limit of convex functions, w is necessarily convex. Moreover by [20, Theorem 10.8], the convergence is uniform on compact subsets of Ω. By Lemma 2.1, $M[u_{M_k}]\to M[w]$ weakly. Again by [9, Theorem 1, section 1.9], $M[u_{M_k}](B)\to M[w](B)$ for any bounded Borel set B with $M[w](\partial B)=0$. But

$$M[u_{M_k}](B)=\int_B f_{M_k}(x)\,dx\to\int_B f(x)\,dx, \text{as} k\to\infty.$$

We conclude that $M[w](B)=\int_B f(x)\,dx$ for any closed ball B contained in Ω and hence $M[w]$ has density f.

Next, for $M' > M$, we have $\mu_{M'} \geq \mu_M$ with $u_{M'} = u_M$ on $\partial\Omega$. Since $g \in C(\partial\Omega)$, u_M is uniformly bounded above on $\partial\Omega$. By the comparison principle given in Theorem 5.5 we have $u_{M'} \leq u_M$ in Ω and hence $w \leq u_{M_k}$ for all k. By Lemma 5.1, we have $g_w \leq g_{u_{M_k}} = g$ on $\partial\Omega$.

Finally, let $v \in V(\mu, g)$. By definition $M[v] = \mu$ and $g_v \leq g$ on $\partial\Omega$. We show that $g_w \geq g_v$. Note that $u_{M_k} = g \geq g_v$ on $\partial\Omega$ with $M[u_{M_k}] = \mu_{M_k} \leq \mu = M[v]$. By Theorem 5.5, we conclude that $u_{M_k} \geq v$ in Ω and hence $w \geq v$ in Ω. By Lemma 5.1, we obtain $g_w \geq g_v$. This proves that w is a Bakelman solution of (1.1).

By uniqueness of the Bakelman solution $w = u$. The limit u being unique, the whole sequence u_M must converge to u. This concludes the proof.
□

Theorem 5.6 suggests that, for the numerical resolution of (1.1), one should solve the problems (5.2) with truncated right hand sides for increasing values of M. We now prove that this process gives a convergent scheme.

THEOREM 5.7. *Assume that u_M converges uniformly to u on compact subsets of Ω as $M \to \infty$ and $u_{M,h}$ converges uniformly to u_M on compact subsets of Ω as $h \to 0$. Then $u_{M,h}$ has a subsequence which converges uniformly to u on compact subsets of Ω as $h \to 0$ and $M \to \infty$.*

Proof. Let T be a compact subset of Ω and let $k > 1$. Since u_M converges uniformly to u on T, there exists $M_k > 0$ such that for $M \geq M_k$, $|u_M(x) - u(x)| < 1/(2k)$ for all $x \in T$. Next, $u_{M_k,h}$ converges uniformly to u_{M_k} on T. Thus, there exists $h_k > 0$ such that for $0 < h \leq h_k$, $|u_{M_k,h_k}(x) - u_{M_k}(x)| < 1/(2k)$. The result then follows from the triangular inequality. □

REMARK 5.8. *By the results of section 4, $u_{M,h}$ may be assumed to be convex and thus $\det D^2 u_{M_k,h_k} \to f$ weakly as measures for appropriate sequences $h_k \to 0$ and $M_k \to \infty$.*

REMARK 5.9. *Throughout this paper, we used the well known fact that a uniformly bounded sequence of convex functions on a convex domain Ω is locally uniformly equicontinuous and hence has a pointwise convergent subsequence. The result is a consequence of [13, Lemma 3.2.1].*

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