CONVERGENCE OF A DAMPED NEWTON'S METHOD FOR DISCRETE MONGE-AMPÈRE FUNCTIONS WITH A PRESCRIBED ASYMPTOTIC CONE

GERARD AWANOU

Abstract. We prove the convergence of a damped Newton's method for the nonlinear system resulting from a discretization of the second boundary value problem for the Monge-Ampère equation. The boundary condition is enforced through the use of the notion of asymptotic cone. The differential operator is discretized based on a partial discrete analogue of the subdifferential.

1. Introduction

In this paper we prove the convergence of a damped Newton's method for a finite difference approximation of the second boundary value problem for Monge-Ampère type equations. The method was introduced in [1] and has the particularity among available discretizations that the discrete problem has a unique solution. The damped Newton's method allows the use of an initial guess which may be far from the solution of the discrete problem. We establish the global convergence of the algorithm for the discretization proposed in [1].

Monge-Ampère type equations with the second boundary value condition arise in geometric optics and optimal transport. In [14] a discretization of the Dirichlet problem which is based on a partial discrete analogue of the subdifferential, for the discretization of the differential operator, was proposed. We consider in this paper the generalization to the second boundary value condition proposed in [1]. The approach in [1] is to interpret the boundary condition as the prescription of the asymptotic cone of the convex solution of the Monge-Ampère equation. The convergence analysis of the damped Newton's method given here generalizes the one given in [14] and is similar in some aspects to the convergence analysis of a damped Newton's method, given in [11], for a dual problem c.f. (1.4) below.

Let Ω and Ω^* be bounded convex polygonal domains of $\mathbb{R}^d$. We recall that for a function u on Ω the subdifferential of u is defined at $x \in \Omega$ by

$$\partial u(x) = \{\, p \in \mathbb{R}^d : u(y) \geq u(x) + p \cdot (y-x), \text{ for all } y \in \Omega \,\}. \tag{1.1}$$

Let R be a locally integrable function on Ω^* such that $R > 0$ on Ω^*. Given points $a_i, i = 1 \ldots, m$ in Ω and positive numbers $\mu_i, i = 1 \ldots, m$ such that $\int_{\Omega^*} R(p) dp = \sum_{i=1}^m \mu_i$, consider the problem of computing a piecewise linear continuous convex function on Ω such that

$$\omega(R, u, a_i) = \mu_i \tag{1.2}$$

$$\overline{\partial u(\Omega)} = \overline{\Omega^*}, \tag{1.3}$$

where for $i = 1, \dots, m$, $\omega(R, u, a_i)$ is the R-Monge-Ampère measure of u at a_i and is defined by

$$\omega(R, u, a_i) = \int_{\partial u(a_i)} R(p) dp.$$

We refer to solutions of (1.2) as Monge-Ampère functions, anticipating applications to a more general setting [9].

Assume that Ω is the convex hull of $v_i, i = 1, \dots, k$. Problem 1.2 with the Dirichlet boundary condition $u(v_i) = g_i$ for $g_i \in \mathbb{R}, i = 1, \dots, k$ with $g_i = g(v_i)$ for a convex function $g \in C(\overline{\Omega})$, has been solved in [17] with what is now known as the Oliker-Prussner method. Whether one considers the Dirichlet boundary condition, or the second boundary condition (1.3), an approach for solving (1.2) is to consider a Legendre dual u^* of the solution u. In the case of (1.3), we define for $y \in \mathbb{R}^d$

$$u^*(y) = \max_{i=1,\dots,m} \{ a_i \cdot y - u(a_i) \}.$$

It can be shown, see for example [5, sextion 5], that for a solution u of (1.2)-(1.3), u^* solves

$$\int_{W_i} R(p) dp = \mu_i, i = 1, \dots, m, \tag{1.4}$$

where $W_i = \{ y \in \Omega^*, a_i \cdot y - u(a_i) \geq a_j \cdot y - u(a_j), j = 1, \dots, m \}$. The sets W_i form a decomposition of Ω^* known as power diagram, and their computations in three dimensions are delicate, see for example [12] where plenty of subtleties are mentioned. For the Dirichlet boundary condition, c.f. [10, Theorem 1.4], the transform takes the form $u^* = \max_{i=1,\dots,m+k} \{ a_i \cdot y - u(a_i) \}$, where for $j = 1, \dots, k$, we put $a_{m+j} = v_j$ and $u(a_{m+j}) = g_j$. Now $W_i = \{ y \in \mathbb{R}^d, a_i \cdot y - u(a_i) \geq a_l \cdot y - u(a_l), l = 1, \dots, m+k \}$.

In order to have an easier approach to the Dirichlet problem for (1.2), Mirebeau in [14] proposed a medius approach, which is between finite difference methods and power diagrams. In [1], this approach was generalized to the second boundary condition (1.3). See also [4] for a related approach. We note that power diagrams were not used explicitly in [17, 20] for the Dirichlet problem for (1.2). The proposed coordinate descent algorithm therein is known not to be efficient. An improvement has been proposed in [15] and has yet to be extended to the second boundary condition (1.3). In fact, it was not even known how to discretize (1.2)-(1.3) without the use of power diagrams (and ensure unicity up to a constant of the discrete solution). We wish to consider such extensions in a separate work and focus in this work on the convergence of a damped Newton's method of a discretization of the Mirebeau type which does not require the use of power diagrams.

The discretization of (1.3) in [1] is based on an observation made in [16] that (1.2)-(1.3) is equivalent to finding a convex function on $\mathbb{R}^d$ with a prescribed asymptotic cone such that (1.2) holds. Before we recall the notion of asymptotic cone, which we detailed in [1], let us mention that the connection between (1.2)-(1.3) and global convex functions with a prescribed asymptotic cone appears to be folklore, as existence results for the second boundary value problem routinely refer to the book [19] which only describes the geometric problem, see the comments in [7, 8].

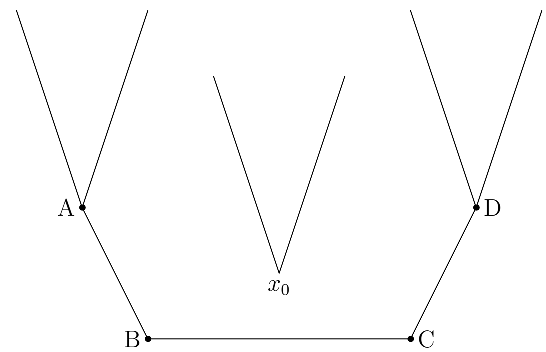

FIGURE 1. The convex hull $\mathcal{M}$ of $\{ A, B, C, D \}$ defines a piecewise linear convex function on a finite interval. The convex hull $\mathcal{M}^*$ of $\mathcal{M}$ and the cone $x_0 + K_{\Omega^*}$ with $\Omega^* = (-3, 3)$ defines a piecewise linear convex function on the real line with asymptotic cone K_{Ω^*}. Reproduced from [2].

We denote by $a_j^*, j = 1, \ldots, N$ the boundary vertices of Ω^*. To the polygonal domain Ω^*, we associate a cone K_{Ω^*} defined as follows. For each $p \in \overline{\Omega^*}$ one associates the half-space $H(p) = \{ (x, z) \in \mathbb{R}^d \times \mathbb{R}, z \geq p \cdot x \}$. The cone K_{Ω^*} is the intersection of the half-spaces $H(p), p \in \overline{\Omega^*}$, i.e.

$$K_{\Omega^*} = \cap_{p \in \overline{\Omega^*}} H(p).$$

A convex function u on $\mathbb{R}^d$ is said to have asymptotic cone K_{Ω^*} if for all $x \in \mathbb{R}^d$, the cone $x + K_{\Omega^*}$ is contained in the epigraph of u. Recall that the epigraph of u is the convex set $\{ (x, z) \in \mathbb{R}^d \times \mathbb{R}, z \geq u(x) \}$. This is illustrated in Figure 1 taken from [2].

It was shown in [16] and reviewed in [1] that (1.2)-(1.3) is equivalent to finding a convex function on $\mathbb{R}^d$ with asymptotic cone K_{Ω^*} such that (1.2) holds. Obviously, computations will not be done on $\mathbb{R}^d$. Let S denotes the convex hull of $\{ a_i, i = 1, \ldots, m \}$ and assume that $a_i, i = 1, \ldots, p$ form the vertices of the boundary of S. It was shown in [1] that for $x \notin S$, we have the extension formula

$$u(x) = \min_{1 \leq i \leq p} \max_{1 \leq j \leq N} (x - a_i) \cdot a_j^* + u(a_i).$$

The above extension formula is the basis of the extension formula (2.3) used below for the discrete problem analyzed in this paper. The extension (2.3) is introduced to make the geometric problem analytically tractable. For the numerical method, we are only working on a bounded domain, yet the geometric problem involves a convex function on the whole space. However the values of the convex function outside of the bounded domain are controlled by its values in the bounded domain and the second boundary condition. The extension (2.3) makes this connection analytically explicit.

We organize the paper as follows. In the next section, we start with some preliminaries and recall the discretization introduced in [1] and its properties. The damped Newton's method is introduced in section 3 in a general setting. In section 4 we give its convergence analysis for our discretization. In the last section we illustrate how to implement the method with a numerical experiment in MATLAB.

2. Preliminaries

Let $f \geq 0$ be an integrable function on Ω. We assume that $R = 0$ on $\mathbb{R}^d \setminus \partial u(\Omega)$ and that the compatibility condition

$$\int_{\Omega} f(x) dx = \int_{\Omega^*} R(p) dp, \tag{2.1}$$

holds. We are interested in approximating a convex function u which solves in the sense of Aleksandrov

$$\begin{aligned} R(Du(x)) \det D^2 u(x) &= f(x) \text{ in } \Omega \\ \overline{\partial u(\Omega)} &= \overline{\Omega^*}. \end{aligned} \tag{2.2}$$

For a Borel set $E \subset \Omega$, it is required that $\int_{\partial u(E)} R(p) dp = \int_E f(x) dx$.

The coordinates of a vector $e \in \mathbb{Z}^d$ are said to be co-prime if their great common divisor is equal to 1. A subset W of $\mathbb{Z}^d$ is symmetric with respect to the origin if $\forall y \in W, -y \in W$.

Let V be a (finite) subset of $\mathbb{Z}^d \setminus \{0\}$ of vectors with co-prime coordinates which span $\mathbb{R}^d$ and which is symmetric with respect to the origin. Furthermore, assume that V contains the elements of the canonical basis of $\mathbb{R}^d$ and that V contains a normal to each side of the target polygonal domain Ω^*.

Let h be a small positive parameter and let $\mathbb{Z}^d_h = a + \{ mh, m \in \mathbb{Z}^d \}$ denote the orthogonal lattice with mesh length h with an offset $a \in \mathbb{Z}^d$. The offset a may make it easier to choose the decomposition of the domain used for the discrete Monge-Ampère equation (2.4) below. Put $\Omega_h = \Omega \cap \mathbb{Z}^d_h$ and denote by $(r_1, \ldots, r_d)$ the canonical basis of $\mathbb{R}^d$. Let

$$\partial \Omega_h = \{ x \in \Omega_h \text{ such that for some } i = 1, \ldots, d, x + h r_i \notin \Omega_h \text{ or } x - h r_i \notin \Omega_h \}.$$

We note that with our notation $\partial \Omega_h \subset \Omega_h$. The unknown in the discrete scheme is a mesh function (not necessarily convex) on Ω_h which is extended to $\mathbb{Z}^d_h$ using the extension formula

$$v_h(x) = \min_{y \in \partial \Omega_h} \max_{1 \leq j \leq N} (x - y) \cdot a_j^* + v_h(y). \tag{2.3}$$

Definition 2.1. *A mesh function v_h on $\mathbb{Z}^d_h$ is said to have asymptotic cone K_{Ω^*} if its values outside Ω_h is given by* (2.3)*.*

We define for a function v_h on $\mathbb{Z}^d_h$, $e \in \mathbb{Z}^d$ and $x \in \Omega_h$

$$\Delta_{he} v_h(x) = v_h(x + he) - 2 v_h(x) + v_h(x - he).$$

We are particularly interested in those mesh functions v_h with asymptotic cone K_{Ω^*} which are discrete convex in the sense that $\Delta_{he} v_h(x) \geq 0$ for all $x \in \Omega_h$ and $e \in V$. We denote by $\mathcal{C}_h$ the cone of discrete convex mesh functions.

For $x \in \Omega_h$ and $e \in V$ such that $x + he \notin \Omega_h$, we define

$$\Gamma(x + he) = \operatorname*{argmin}_{y \in \partial \Omega_h} \max_{1 \leq j \leq N} (x - y) \cdot a_j^* + v_h(y).$$

A priori, $\Gamma(x+he)$ is multi-valued. We assume that for the implementation a unique choice is made for pairs (x,e) such that $x+he \notin \Omega_h$. If $x+he \in \Omega_h$, we put $\Gamma(x+he) = x+he$.

We consider the following analogue of the subdifferential of a function. For $y \in \mathbb{Z}_h^d$ we define

$$\partial_V v_h(y) = \{\, p \in \mathbb{R}^d, p \cdot (he) \geq v_h(y) - v_h(y-he) \,\forall e \in V \,\},$$

and consider the following discrete version of the R-Monge-Ampère measure

$$\omega_a(R, v_h, E) = \int_{\partial_V v_h(E)} R(p) dp.$$

The discretization considered in [14] used a symmetrization of the subdifferential. The subscript a in the notation $\omega_a(R, v_h, E)$ recalls that we consider in this paper an asymmetrical version.

We can now describe our discretization of the second boundary value problem: find $u_h \in \mathcal{C}_h$ with asymptotic cone K_{Ω^*} such that

$$\omega_a(R, u_h, \{\, x \,\}) = \int_{C_x} f(t) dt, x \in \Omega_h, \tag{2.4}$$

where $(C_x)_{x \in \Omega_h}$ form a partition of Ω, i.e. $C_x \cap \Omega_h = \{\, x \,\}, \cup_{x \in \Omega_h} C_x = \Omega$, and $C_x \cap C_y$ is a set of measure 0 for $x \neq y$. In the interior of Ω one may choose as $C_x = x + [-h/2, h/2]^d$ the cube centered at x with $E_x \cap \Omega_h = \{\, x \,\}$. The requirement that the sets C_x form a partition is essential to assure the mass conservation (2.1) at the discrete level, i.e.

$$\sum_{x \in \Omega_h} \omega_a(R, u_h, \{\, x \,\}) = \sum_{x \in \Omega_h} \int_{C_x} f(t) dt = \int_{\Omega} f(t) dt = \int_{\Omega^*} R(p) dp. \tag{2.5}$$

The unknowns in (2.4) are the mesh values $u_h(x), x \in \Omega_h$. For $z \notin \Omega_h$, the value $u_h(z)$ needed for the evaluation of $\partial_V v_h(x)$ is obtained from the extension formula (2.3). Problem (2.4) can be seen as a variant of (1.2)-(1.3).

The following theorem is given in [1].

Theorem 2.2. *Problem* (2.4) *has a discrete convex solution* u_h *which is unique up to a constant. Moreover, if* $u_h(x^1) = \alpha$ *for* $x^1 \in \Omega_h$ *for all* $h > 0$ *and* $\alpha \in \mathbb{R}$*, then* u_h *converges uniformly on compact subsets to the unique solution of* (2.2) *which solves* $u(x^1) = \alpha$*.*

We will need below the following results from [1].

Lemma 2.3. *Discrete convex mesh functions with asymptotic cone* K_{Ω^*} *are uniformly bounded on* Ω_h*.*

Lemma 2.4. *Assume that* $f > 0$ *in* Ω *and let* v_h *be a solution of* (2.4)*. Then for* $x \in \Omega_h$*,* $\partial_V v_h(x) \subset \Omega^*$*.*

3. The damped Newton's method

We first give a general convergence result, the assumptions of which are then verified in the next section for our discretization. Let $\mathbb{U}$ denote an open subset of $\mathbb{R}^M$. In section 4 we will identify a mesh function with a vector consisting in its values at grid points. Here, we then denote by v_h a generic element of $\mathbb{U}$.

We are interested in the zeros of a mapping $G : \mathbb{U} \to \mathbb{R}^M$ with $G(v_h) = (G_l(v_h))_{l=1,\ldots,M}$. We let $||.||$ denote a norm in $\mathbb{R}^M$ and put

$$\mathbb{U}_\epsilon = \{ v_h \in \mathbb{U}, G_l(v_h) > -\epsilon, l = 1, \ldots, M \},$$

for a parameter $\epsilon > 0$ to be specified later. We assume that $G \in C^1(\mathbb{U}_\epsilon, \mathbb{R}^M)$ for all $\epsilon > 0$. The current iterate is denoted v_h^k and the following iterate is sought along the path

$$p^k(\tau) = v_h^k - \tau G'(v_h^k)^{-1} G(v_h^k).$$

We make the assumption that there exists $\tau_k \in (0, 1]$ such that $p^k(\tau) \in \mathbb{U}$ for all $\tau \in [0, \tau_k]$. Let $\delta \in (0, 1)$ and choose $\rho \in (0, 1)$, e.g. $\rho = 1/2$.

Algorithm 1 A damped Newton's method

1: Choose $v_h^0 \in \mathbb{U}_\epsilon$ and set $k = 0$
2: **If** $G(v_h^k) = 0$ **stop**
3: Let i_k be the smallest non-negative integer i such that $p^k(\rho^i) \in \mathbb{U}_\epsilon$ and

$$||G(p^k(\rho^i))|| \leq (1 - \delta\rho^i)||G(v_h^k)||.$$

Set $v_h^{k+1} = p^k(\rho^{i_k})$
4: $k \leftarrow k + 1$ and **go to** 2

The general convergence result for damped Newton's methods is analogous to [11, Proposition 6.1] where maps with values probability measures are considered. Therein, the map G is assumed to be in $C^{1,\alpha}, 0 < \alpha \leq 1$. For G to be merely C^1, as in certain geometric optics problems, and with $\det G'(x) \neq 0$ for all $x \in \mathbb{U}_\epsilon$, one has linear convergence [6]. For completeness we adapt the proof of [6] to the case where the domain of the mapping G is an open set of $\mathbb{R}^M$. As with [14, Proposition 2.10] we will assume that the mapping G is proper, i.e. the preimage of any compact set is a compact set.

Theorem 3.1. *Let $G \in C^1(\mathbb{U}_{2\epsilon}, \mathbb{R}^M)$ and assume that G is a proper map with a unique zero u_h in $\overline{\mathbb{U}_\epsilon}$ and $\det G'(x) \neq 0$ for all $x \in \mathbb{U}_{2\epsilon}$. Assume that there exists $\check{\tau}_k$ in $(0, 1]$ such that for all $0 < \tau \leq \check{\tau}_k, p_k(\tau) \in \mathbb{U}$. Then the iterate v_h^k of the damped Newton's method is well defined, converges to u_h and*

$$||v_h^{k+1} - u_h|| \leq C\ o(||v_h^k - u_h||),$$

for $k \geq k_0$ if $i_k = 0$ for $k \geq k_0$ and k_0 sufficiently large. For k sufficiently large, with step 3 replaced with $i_k = 0$, i.e. a full Newton's step, the convergence is guaranteed to be at least linear.

Proof. The proof is divided into three parts. In the first part, we show that given an iterate v_h^k in the admissible set $\mathbb{U}_\epsilon$, one can find a path from v_h^k which is contained in $\mathbb{U}_\epsilon$, provided v_h^k is not a zero of G. We show by a contradiction argument that G decreases in norm along such a path at the rate given in the algorithm. In the second part, we show that the iterate selected along such a path by the damped Newton's algorithm, converges to a zero of G. Finally in the third part, we give the convergence rate.

Since $G \in C^1(\mathbb{U}_{2\epsilon}, \mathbb{R}^M)$, we can find a constant L (depending on ϵ) such that

$$||G'(x)|| \leq L, \ \forall x \in \mathbb{U}_\epsilon.$$

Part 1: The damped Newton's method is well defined. We first note that it follows from the definitions that for all $\tau \in [0, 1]$ we have

$$G(v_h^k) + G'(v_h^k)(p^k(\tau) - v_h^k) = (1 - \tau)G(v_h^k). \tag{3.1}$$

Assume that $G(v_h^k) \neq 0$. We claim that there exists $\tau_k' \in (0, 1]$ such that

$$||G(p^k(\tau))|| \leq (1 - \delta\tau)||G(v_h^k)||, \ \forall \tau \in [0, \tau_k'].$$

If such a τ_k' does not exist, there would exist a sequence τ_l converging to 0 such that

$$||G(p^k(\tau_l))|| > (1 - \delta\tau_l)||G(v_h^k)||, \ \forall k. \tag{3.2}$$

Since G is C^1, c.f. for example [18, \$ 3.2.10], we have

$$G(p^k(\tau_l)) = G(v_h^k) + G'(v_h^k)(p^k(\tau_l) - v_h^k) + o(||p^k(\tau_l) - v_h^k||).$$

Thus, by (3.1), we have

$$G(p^k(\tau_l)) = (1 - \tau_l)G(v_h^k) + o(||p^k(\tau_l) - v_h^k||), \tag{3.3}$$

and thus by (3.2)

$$(1 - \delta\tau_l)||G(v_h^k)|| < ||G(p^k(\tau_l))|| \leq (1 - \tau_l)||G(v_h^k)|| + o(||p^k(\tau_l) - v_h^k||).$$

This implies that $\tau_l(1 - \delta)||G(v_h^k)|| < o(||p^k(\tau_l) - v_h^k||$ which gives using the definition of $p^k(\tau_l)$

$$(1 - \delta)||G(v_h^k)|| < \frac{1}{\tau_l} o\big(\tau_l||G'(v_h^k)^{-1}G(v_h^k)||\big).$$

Taking the limit as $\tau_l \to 0$, we obtain $(1 - \delta)||G(v_h^k)|| \leq 0$, a contradiction as $G(v_h^k) \neq 0$.

We now show that there exists $\overline{\tau}_k \leq \min\{\, \tau_k', \check{\tau}_k, \tau_k \,\}$, with $\overline{\tau}_k > 0$ such that for all $\tau \in [0, \overline{\tau}_k], p^k(\tau) \in \mathbb{U}_\epsilon$ when $v_h^k \in \mathbb{U}_\epsilon$.

If $p^k(\tau) \notin \mathbb{U}_\epsilon$ for some $\tau \leq \min\{\, \tau_k', \check{\tau}_k, \tau_k \,\}$ it must be that at some time $\overline{\tau}_k \leq \tau, p^k(\overline{\tau}_k) \in \partial\mathbb{U}_\epsilon$. Thus for some $i_k \in \{\, 1, \ldots, M \,\}$ we have $G_{i_k}(p(\overline{\tau}_k)) = -\epsilon$. Let us assume that $\overline{\tau}_k$ is chosen so that $G_i(p^k(\tau)) > -\epsilon$ for all $\tau \in [0, \overline{\tau}_k)$ and all $i \in \{\, 1, \ldots, M \,\}$.

We have

$$||G(p^k(\overline{\tau}_k)) - G(v_h^k)|| \geq |G_{i_k}(p^k(\overline{\tau}_k)) - G_{i_k}(v_h^k)| = | - \epsilon - G_{i_k}(v_h^k)| > 0,$$

since $G_{i_k}(v_h^k) > -\epsilon$ as $v_h^k \in \mathbb{U}_\epsilon$ by assumption. Therefore, by the mean value theorem, c.f. for example [18, \$ 3.2.10],

$$0 < ||G(p^k(\overline{\tau}_k)) - G(v_h^k)|| \leq L||p^k(\overline{\tau}_k) - v_h^k|| \leq L\overline{\tau}_k||G'(v_h^k)^{-1}G(v_h^k)||.$$

We conclude that $\overline{\tau}_k > 0$. And by construction $p^k(\tau) \in \mathbb{U}_\epsilon$ for all $\tau \in [0, \overline{\tau}_k)$.

Part 2: The sequence v_h^k converges to u_h. Let

$$\mathcal{K} = \{ \, v \in \mathbb{U}_\epsilon, ||G(v)|| \leq ||G(v_h^0)|| \, \}.$$

Since G is proper, $\mathcal{K}$ is compact. By construction $||G(v_h^{k+1})|| < ||G(v_h^k)||$ and $v_h^k \in \mathbb{U}_\epsilon$ for all k. Thus up to a subsequence, the sequence v_h^k converges to $v^* \in \overline{\mathbb{U}_\epsilon}$ and since the sequence $||G(v_h^k)||$ is strictly decreasing, there exists $\eta \geq 0$ such that $||G(v_h^k)|| \to \eta$.

If $\eta = 0$, then $G(v^*) = 0$ and since G has a unique zero u_h in $\overline{\mathbb{U}_\epsilon}$, the whole sequence converges to $v^* = u_h$.

Now we show that it is not possible to have $\eta \neq 0$. Assume that $\eta > 0$.

If the sequence i_k of Step 3 of the algorithm is bounded, there would exist a constant $\xi > 0$ such that $\rho^{i_k} \geq \xi$ for all k. This would imply that $1 - \delta \rho^{i_k} \leq 1 - \delta \xi$ and $\eta \leq (1 - \delta \xi) \eta$, i.e. $\eta = 0$. We therefore have $\lim_{k \to \infty} i_k = \infty$ and consequently $\lim_{k \to \infty} \rho^{i_k - 1} = 0$. Put

$$\hat{\tau}_k = \rho^{i_k - 1}.$$

By definition of i_k and (3.3), we have

$$\begin{aligned}
(1 - \delta \hat{\tau}_k)||G(v_h^k)|| &< ||G(p(\hat{\tau}_k))|| \leq (1 - \hat{\tau}_k)||G(v_h^k)|| + ||o\big(\hat{\tau}_k ||G'(v_h^k)^{-1} G(v_h^k)||\big)|| \\
(1 - \delta)||G(v_h^k)|| &< \frac{1}{\hat{\tau}_k} ||o\big(\hat{\tau}_k ||G'(v_h^k)^{-1} G(v_h^k)||\big)||.
\end{aligned}$$

Recall that the sequence $||G(v_h^k)||$ is bounded. Moreover $v^* \in \mathbb{U}_{2\epsilon}$ and thus $\det G'(v^*) \neq 0$. Since $G \in C^1(\mathbb{U}_{2\epsilon}, \mathbb{R}^M)$ it follows that for k sufficiently large $||G'(v_h^k)^{-1}|| \leq C$ for a constant C, c.f. [18, \$ 2.3.3]. We conclude that

$$(1 - \delta)||G(v_h^k)|| < \frac{1}{\hat{\tau}_k} o(\hat{\tau}_k).$$

This implies that $\eta = 0$, a contradiction.

Part 3: We first prove the rate $||v_h^{k+1} - u_h|| \leq C \; o(||v_h^k - u_h||)$, for $k \geq k_0$ if $i_k = 0$ for $k \geq k_0$ and k_0 sufficiently large. We have with $\tilde{\tau}_k = \rho^{i_k}$

$$\begin{aligned}
G'(v_h^k)(v_h^{k+1} - u_h) &= G'(v_h^k)\big(v_h^k - u_h - \tilde{\tau}_k G'(v_h^k)^{-1} G(v_h^k)\big) \\
&= G'(v_h^k)(v_h^k - u_h) - \tilde{\tau}_k G(v_h^k).
\end{aligned}$$

Moreover $0 = G(u_h) = G(v_h^k) - G'(v_h^k)(v_h^k - u_h) + o(||v_h^k - u_h||)$. Thus $G'(v_h^k)(v_h^{k+1} - u_h) = (1 - \tilde{\tau}_k) G(v_h^k) + o(||v_h^k - u_h||)$. Since $G'(v_h^k)^{-1}$ is uniformly bounded for k sufficiently large, the claim follows when $\tilde{\tau}_k = 1$.

The linear convergence of Newton's method for C^1 mappings is classical but is nethertheless reviewed here. Let now $v_h^{k+1} = v_h^k - G'(v_h^k)^{-1} G(v_h^k)$ and put $e_h^k = v_h^k - u_h$. Let $\beta > 0$ such that for $||v_h^k - u_h|| \leq \beta$ we have $||G'(v_h^k)^{-1}|| \leq C_1$. If necessary by taking β smaller, we may assume that for $||v - u_h|| \leq \beta$ and $||w - u_h|| \leq \beta$ we have $||G'(v) - G'(w)|| \leq 1/C_1$ using the local uniform continuity of G' on $\mathbb{U}_\epsilon$. We have

$$e_h^{k+1} = e_h^k - G'(v_h^k)^{-1} G(v_h^k) = G'(v_h^k)^{-1} \int_0^1 \big(G'(v_h^k) - G'(u_h + t e_h^k)\big) e_h^k dt,$$

from which we get $||e_h^{k+1}|| \leq ||e_h^k||$. This shows that when $||v_h^k - u_h|| \leq \beta$ we also have $||v_h^{k+1} - u_h|| \leq \beta$ and the linear convergence rate. □

4. Convergence of the damped Newton's method for the discretization

Let $M+1$ denote the cardinality of Ω_h and denote the points of Ω_h by $x^i, i = 1, \ldots, M+1$.

The set of discrete convex mesh functions on Ω_h which satisfy (2.3), and with $v_h(x^1) = \alpha$ for an arbitrary number α and $x^1 \in \Omega_h$, is identified with a subset $\mathbb{U}$ of $\mathbb{R}^M$ by considering a mapping $\mathcal{A}$

$$\begin{aligned} \mathcal{C}_h &\longrightarrow \mathbb{U} \\ v_h &\longmapsto (v_h(x_i))_{i=2,\ldots,M+1}. \end{aligned}$$

We consider a map $G : \mathbb{U} \to \mathbb{R}^M$ defined by

$$G_i(v_h) = \omega_a(R, v_h, \{\, x^i \,\}) - \int_{C_{x^i}} f(t)dt \text{ for } i = 2, \ldots, M+1. \tag{4.1}$$

Note that for $v_h \in \mathbb{U}$, we have $v_h(x^1) = \alpha$ and this value is used for computing $\omega_a(R, v_h, \{\, x^i \,\})$ in the expression of $G_i(v_h)$. Also, by (2.5) we have

$$\omega_a(R, v_h, \{\, x^1 \,\}) = \int_{\Omega^*} R(p)dp - \sum_{i=2}^{M+1} \omega_a(R, v_h, \{\, x^i \,\}).$$

Hence

$$\omega_a(R, v_h, \{\, x^1 \,\}) = \int_{C_{x^1}} f(t)dt,$$

when $G_i(v_h) = 0, i = 2, \ldots, M+1$. In other words, the above equation is automatically satisfied when $G_i(v_h) = 0, i = 2, \ldots, M+1$.

We consider the set $\mathbb{V}$ which is the image by the identification mapping $\mathcal{A}$, of mesh functions v_h with $v_h(x^1) = \alpha$, v_h extended to $\mathbb{Z}_h^d$ using (2.3) and which satisfy $\Delta_{he} v_h(x^i) \geq 0, i = 2, \ldots, M+1$ and for all $e \in V$. We have $\mathbb{U} \subset \mathbb{V}$. Elements of $\mathbb{U}$ (identified with mesh functions) are not required to satisfy $\Delta_{he} v_h(x^1) \geq 0$.

For the choice of ϵ, we now make the assumption that

$$\text{there exists a constant } \epsilon > 0 \text{ such that } f > 2\epsilon/h^d \text{ on } \Omega.$$

Define as in the previous section $\mathbb{V}_\epsilon = \{\, v_h \in \mathbb{V}, G_i(v_h) > -\epsilon, i = 2, \ldots, M+1 \,\}$. The set $\mathbb{V}_\epsilon$ is non empty since the solution u_h of (2.4) with $u_h(x^1) = \alpha$ solves $G_i(u_h) = 0 > -\epsilon$ for all i. We note that for $v_h \in \mathbb{V}_\epsilon$ we have $\omega_a(R, v_h, \{\, x^i \,\}) > \epsilon > 0$ for all $i = 2, \ldots, M+1$. Hence $\Delta_{he} v_h(x^i) > 0$ for all $i = 2, \ldots, M+1$ and $e \in V$.

The goal of this section is to verify the assumptions of Theorem 3.1 for the equation $G(v_h) = 0$ with G given by (4.1). We will prove the following theorem

Theorem 4.1. *Given an initial guess $v_h^0 \in \mathbb{V}_\epsilon$, the iterate v_h^k of the damped Newton's method, with $\mathbb{U}_\epsilon$ replaced by $\mathbb{V}_\epsilon$, is well defined, converges to the unique solution u_h of (2.4) which satisfies $u_h(x^1) = \alpha$, and*

$$||v_h^{k+1} - u_h|| \leq C\ o(||v_h^k - u_h||),$$

for $k \geq k_0$ if $i_k = 0$ for $k \geq k_0$ and k_0 sufficiently large. For k sufficiently large, with step 3 replaced with $i_k = 0$, i.e. a full Newton's step, the convergence is guaranteed to be at least linear.

The proof of Theorem 4.1 proceeds in several steps. In the next theorem, we show that the mapping G has a unique zero in $\overline{\mathbb{V}_\epsilon}$, the solution u_h of (2.4) which satisfies $u_h(x^1) = \alpha$. The path condition in Theorem 3.1 is proved in Theorem 4.3 under the assumption that the mapping G is differentiable with an invertible Jacobian. The C^1 continuity of G is established in Theorem 4.5 and the invertibility of the Jacobian in Theorem 4.6. Finally, in Theorem 4.7, we prove that the mapping G is proper. We are then in a position to give the proof of Theorem 4.1.

Theorem 4.2. *The mapping G has a unique zero in $\overline{\mathbb{V}_\epsilon}$.*

Proof. Since the solution u_h of (2.4) which satisfies $u_h(x^1) = \alpha$ is in $\overline{\mathbb{V}_\epsilon}$, a zero exists. Let $v_h \in \overline{\mathbb{V}_\epsilon}$ such that $G(v_h) = 0$. As discussed at the beginning of this section, the compatibility condition (2.5) implies that $G_1(v_h) = 0$.

We then have $\omega_a(R, v_h, \{ x^1 \}) > \epsilon > 0$. Therefore $\partial_V v_h(x^1)$ is a set of non zero measure. In particular, it is non empty. There exists p such that for each direction e, as V is symmetric with respect to the origin, $v_h(x^1) - v_h(x^1 - e) \leq p \cdot e \leq v_h(x^1 + e) - v_h(x^1)$. Thus $\Delta_{he} v_h(x^1) \geq 0$ as well and v_h is the unique solution of (2.4). □

Theorem 4.3. *Assume that G is differentiable at $v \in \mathbb{V}_\epsilon$ and* $\det G'(v) \neq 0$. *There exists $\check{\tau}$ in $(0, 1]$ such that for all $0 < \tau \leq \check{\tau}, p(\tau) = v - \tau G'(v)^{-1} G(v) \in \mathbb{V}$.*

Proof. Since $v \in \mathbb{V}_\epsilon$, $\Delta_{he} v(x^i) > 0$ for all $i = 2, \ldots, M+1$ and $e \in V$. Let $c_0 = \min\{ \Delta_{he} v_h(x^i), i = 2, \ldots, M+1, e \in V \}$ and let $\zeta = c_0/8$. We note that if $|w-v|_\infty := \max\{ |w(x)-v(x)|, x \in \Omega_h \} \leq \zeta$, then $\Delta_{he} w(x^i) \geq 0, i = 2, \ldots, M+1, e \in V$. Indeed, for $x \in \Omega_h, x \neq x^1$, when $x \pm e \in \Omega_h$

$$\begin{aligned}\Delta_{he} w(x) &= w(x+he) - 2w(x) + w(x-he) \\ &= \Delta_{he} v(x) + \big((w-v)(x+he) - 2(w-v)(x) + (w-v)(x-he)\big) \\ &\geq \Delta_{he} v(x) - 4\zeta \geq c_0 - 4\zeta = \frac{c_0}{2} > 0.\end{aligned}$$

If $x + e \notin \Omega_h$, we have $w(x+e) = w(y) + (x+e-y) \cdot a_l^*$ for some integer $l, 1 \leq l \leq N$ and $y \in \partial\Omega_h$. We have $v(x+e) \leq v(y) + (x+e-y) \cdot a_l^*$ and thus

$$w(x+e) - v(x+e) \geq w(y) - v(y) \geq -\zeta.$$

Similarly, when $x - e \notin \Omega_h$, we have $w(x-e) - v(x-e) \geq -\zeta$.

Since $|w-v|_\infty \leq C||w-v||$ and

$$||p(\tau) - v|| \leq \tau ||G'(v)^{-1}||\, ||G(v)||,$$

we have

$$|p(\tau) - v|_\infty \leq C||p(\tau) - v|| \leq C\tau||G'(v)^{-1}||\,||G(v)|| \leq \zeta,$$

for τ sufficiently small. Here, since v is fixed, $||G'(v)^{-1}||\,||G(v)||$ is a constant independent of τ. The proof is complete.

□

We fix i in $\{ 2, \ldots, M+1 \}$ and consider the mapping

$$v \mapsto G_i(v) = \int_{\partial_V v(x^i)} R(p) dp.$$

Recall from (2.5) that

$$G_1(v) = \int_{\Omega^*} R(p)dp - \sum_{i=2}^{M+1} G_i(v).$$

We make the abuse of notation of indexing elements of $\mathbb{R}^{\#V}$, where $\#V$ denotes the cardinality of the set V, with elements of V. Thus for $\lambda \in \mathbb{R}^{\#V}$, $\lambda = (\lambda_a)_{a \in V}$. Let

$$Q(\lambda) = \{ p \in \mathbb{R}^d, p \cdot e \leq \lambda_e, \ \forall e \in V \},$$

and consider the mapping $\hat{G}$ defined by $\hat{G}(\lambda) := \int_{Q(\lambda)} R(p)dp$. For a given index $j > 1$ we are interested in the variations of G_i with respect to $v(x^j)$, i.e. the derivative at $v(x^j)$ of the application which is the composite of $\hat{G}$ and the mapping

$$r \mapsto \lambda = \left(\frac{w(x^i + ha) - w(x^i)}{h} \right)_{a \in V},$$

where w is the mesh function defined by

$$w(x) = v(x), x \neq x^j, \quad w(x^j) = r.$$

We will need the following slight generalization of [13, Lemma 16].

Lemma 4.4. *Let V be a set of non zero vectors which span $\mathbb{R}^d$ with the property that for e and f in V, $e = rf$ for a scalar r if and only if $r = -1$. Assume furthermore that V is symmetric with respect to the origin. Let R be continuous on Ω^* and for $\lambda \in \mathbb{R}^{\#V}, Q(\lambda) \subset \Omega^*$. Then the mapping*

$$\lambda \mapsto \hat{G}(\lambda) := \int_{Q(\lambda)} R(p)dp,$$

is C^1 continuous on $\{ \lambda \in \mathbb{R}^{\#V}, \lambda_{-a} + \lambda_a > 0, \forall a \in V \}$ with

$$\frac{\partial \hat{G}}{\partial \lambda_a} = \frac{1}{||a||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot a = \lambda_a \}} R(p)dp.$$

Proof. It is proven in [13, Lemma 16] that the mapping $\hat{G}$ is continuous thanks to the assumption that the vectors e in V are non zero. The proof of the existence of $\partial \hat{G}/(\partial \lambda_a)$ for a fixed, and its continuity is therein reduced to the continuity of a mapping $\tilde{G}(\lambda) := \int_{\tilde{Q}(\lambda)} R(p)dp$ where $\tilde{Q}(\lambda) = \{ p \in \mathbb{R}^d, p \cdot \tilde{e} \leq \lambda_{\tilde{e}}, \ \forall e \in V \}$, $\tilde{e}$ is the orthogonal projection of e onto a plane orthogonal to a and $\lambda_{\tilde{e}} = \lambda_e - \lambda_a \, e \cdot a/||a||^2$.

Either the vectors e and a are linearly independent, in which case $\tilde{e} \neq 0$ or $e = -a$. In the latter case $\tilde{e} = 0$ but then $\lambda_{\tilde{e}} = \lambda_{-a} - \lambda_a (-a) \cdot a/||a||^2 = \lambda_{-a} + \lambda_a > 0$, and the corresponding constraint in the definition of $\tilde{Q}(\lambda)$ is vacuous.

We conclude for all $e \in V$, $\lambda_{\tilde{e}} > 0$ giving the continuity of $\tilde{G}$ and hence the C^1 continuity of $\hat{G}$. □

Theorem 4.5. *Under the assumptions of Lemma 4.4, the mapping G is C^1 continuous on $\mathbb{V}_\epsilon$ for R continuous on Ω^*.*

Proof. For $v \in \mathbb{V}_\epsilon$, $\partial_V v(x) \subset \Omega^*$ for all $x \in \Omega_h, x \neq x^1$. The proof is similar to the proof of Lemma 2.4. Define for $v \in \mathbb{V}_\epsilon$

$$\lambda_a(v)(x) = \frac{v(x+ha) - v(x)}{h}.$$

We recall that as a consequence of the assumption $f > 0$ on Ω, for $v \in \mathbb{V}_\epsilon$ $\lambda_a(v)(x) + \lambda_{-a}(v)(x) > 0$ for all $x \in \Omega_h, x \neq x^1$. By Lemma 4.4 the functional G is C^1 continuous on $\mathbb{V}_\epsilon$. □

Given $i > 1$, $\partial_V v(x^i)$ and hence $\omega_a(R, v, \{ x^i \}$ depends on $x^i, x^i + he, e \in V$ when $x^i + he \in \Omega_h$ and on $\Gamma(x^i + he)$ when $x^i + he \notin \Omega_h$. Note that when $x^i \notin \Omega_h$, it is not possible to have $\Gamma(x^i + he') = x^i$ for some $e' \in V$. Therefore for $i, j > 1$

$$\text{if } x^j \notin \{x^i + he, e \in V\}, x^j \notin \partial\Omega_h \text{ and } j \neq i, \partial G_i(v)/(\partial v(x^j)) = 0. \tag{4.2}$$

(4.3) If $x^j \notin \{x^i + he, e \in V\}, x^j \in \partial\Omega_h$ and $j \neq i$,

$$\frac{\partial G_i(v)}{\partial v(x^j)} = \sum_{\substack{e' \in V \\ x_i + he' \notin \Omega_h, \Gamma(x^i + he') = x^j}} \frac{1}{h||e'||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e' = \lambda_{e'} \}} R(p) dp.$$

For $e \in V$ and $x^i + he \in \Omega_h \setminus \{ x^1 \}$

$$\frac{\partial G_i(v)}{\partial v(x^i + he)} = \frac{1}{h||e||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e = \lambda_e \}} R(p) dp, \tag{4.4}$$

and when $x^i \notin \partial\Omega_h$

$$\frac{\partial G_i(v)}{\partial v(x^i)} = -\sum_{e \in V} \frac{1}{h||e||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e = \lambda_e \}} R(p) dp. \tag{4.5}$$

Finally when $x^i \in \partial\Omega_h$

$$\begin{aligned} \frac{\partial G_i(v)}{\partial v(x^i)} = &-\sum_{e \in V} \frac{1}{h||e||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e = \lambda_e \}} R(p) dp \\ &+ \sum_{\substack{e' \in V \\ x_i + he' \notin \Omega_h, \Gamma(x^i + he') = x^i}} \frac{1}{h||e'||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e' = \lambda_{e'} \}} R(p) dp. \end{aligned} \tag{4.6}$$

We will prove that the Jacobian matrix of G is invertible as a weakly chained diagonally dominant matrix, a notion we now recall.

Let $A = (a_{ij})_{i,j=2,\dots,M+1}$ be a $M \times M$ matrix. A row i of A is diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ and strictly diagonally dominant if the strict inequality holds. A matrix A is said to be weakly diagonally dominant if all its rows are diagonally dominant. Recall that the directed graph of A is the graph with vertices $\{ 2, \dots, M+1 \}$ and edges defined as follows: there exists an edge from i to j if and only $a_{ij} \neq 0$.

The matrix A is said to be weakly chained diagonally dominant if it is weakly diagonally dominant and for each row r there is a path in the graph of A to a row p which is strictly diagonally dominant. By path, we mean a finite sequence of joined edges.

It is known, c.f. for example [3], [21] or [14, Lemma 2.5], that a weakly chained diagonally dominant matrix is invertible.

Theorem 4.6. *At each v in $\mathbb{V}_\epsilon$ $\det G'(v) \neq 0$.*

Proof. Put $A = G'(v)$ and recall that the entries of A are given by (4.2)–(4.6). For $e \in V$, either $x_i + he \in \Omega_h$ or $x_i + he \notin \Omega_h$. When $x_i + he \notin \Omega_h, \Gamma(x_i + he) = x_j$ for some $x_j \in \partial\Omega_h$.

Using (4.5), we have for $x^i \notin \partial\Omega_h$

$$\left| \frac{\partial G_i(v)}{\partial v(x^i)} \right| = \left| \sum_{e \in V} \frac{1}{h||e||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e = \lambda_e \}} R(p) dp \right|.$$

The non zero elements of row i correspond to those directions e' for which $x^i + he' \in \Omega_h$ and the directions e'' for which $x^i + he'' \notin \Omega_h$ but for which $\Gamma(x^i + he'') = x, x \in \partial\Omega_h, x \neq x^1$. Note that when $\Gamma(x^i + he') = x_j$, the expression of $\partial G_i(v)/\partial x^j$ takes into account all directions e' for which $\Gamma(x^i + he') = x_j$.

Therefore, if $x^1 \notin \partial\Omega_h$ and $x^i \notin \partial\Omega_h$

$$\left| \frac{\partial G_i(v)}{\partial v(x^i)} \right| = \sum_{j \neq i, j > 1} \left| \frac{\partial G_i(v)}{\partial v(x^j)} \right|.$$

On the other hand, if $x^1 \in \partial\Omega_h$ and $x^i \notin \partial\Omega_h$, we may have some directions e for which $x^i + he' \notin \Omega_h$ and $\Gamma(x^i + he') = x^1$. In the latter case we only have

$$\left| \frac{\partial G_i(v)}{\partial v(x^i)} \right| \geq \sum_{j \neq i, j > 1} \left| \frac{\partial G_i(v)}{\partial v(x^j)} \right|.$$

Let I^1 denote the set of indices $i \in \{ 2, \dots, M+1 \}$ such that $e_{i1} = (x^1 - x^i)/h \in V$, i.e. $x^i + he_{i1} = x^1, e_{i1} \in V$, and put

$$L_e = \frac{1}{h||e||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e = \lambda_e \}} R(p) dp.$$

Since $i > 1$, when $x^i \in \partial\Omega_h, x^i \neq x^1$. Recall that when $x^i + he' \in \Omega_h$, we have $\Gamma(x^i + he') = x^i + he'$. Using (4.6), we have in this case

$$\left| \frac{\partial G_i(v)}{\partial v(x^i)} \right| = \sum_{\substack{e' \in V \\ \Gamma(x^i + he') \neq x^i}} \frac{1}{h||e'||} \int_{Q(\lambda) \cap \{ p \in \mathbb{R}^d, p \cdot e' = \lambda_{e'} \}} R(p) dp.$$

The sum on the right includes directions e' for which $\Gamma(x^i + he') = x^1$. There may be no such directions.

If $x^i \in \partial \Omega_h$ and $i \notin I^1$ we have

$$\left| \frac{\partial G_i(v)}{\partial v(x^i)} \right| = \sum_{\substack{e \in V \\ x^i + he \in \Omega_h \setminus \{ x^1 \}}} L_e + \sum_{\substack{e \in V \\ x^i + he \notin \Omega_h, \Gamma(x^i+he) \neq x^i}} L_e = \sum_{j \neq i, j > 1} \left| \frac{\partial G_i(v)}{\partial v(x^j)} \right| + \sum_{\substack{e \in V \\ x^i + he \notin \Omega_h, \Gamma(x^i+he) = x^1}} L_e.$$

Finally if $x^i \in \partial \Omega_h$ and $i \in I^1$ we have

$$\left| \frac{\partial G_i(v)}{\partial v(x^i)} \right| = \sum_{j \neq i, j > 1} \left| \frac{\partial G_i(v)}{\partial v(x^j)} \right| + \sum_{\substack{e \in V \\ x^i + he \notin \Omega_h, \Gamma(x^i+he) = x^1}} L_e + L_{e_{i1}},$$

and the corresponding rows are strictly diagonally dominant. Note that I^1 is non empty since V is assumed to contain the coordinate unit vectors.

The proof of the path property is now similar to the one given in [14, Corollary 2.6] for the Dirichlet case. Recall that $\partial_V v(x_i)$ is bounded since V contains elements of the canonical basis of $\mathbb{R}^d$. Let V_i denote the set of directions $e \in V$ for which the facet $\{ p \in \mathbb{R}^d, p \cdot e = (v(x+he) - v(x))/h \}$ of $\partial_V v(x_i)$ has non zero measure. Since $\partial_V v(x_i)$ is bounded, the set V_i spans $\mathbb{R}^d$. If $x_j = x_i + he, e \in V_i$, we have $A_{ij} \neq 0$ and $A_{ii} \neq 0$ from (4.2)–(4.6). Among the points $x_i + he, e \in V_i$ one is closer to an element $x^k \in \Omega_h$ with $k \in I^1$. By induction, we can find a sequence of indices $k_1, \ldots, k_m$ such that $A_{ik_1} \neq 0, A_{k_l k_{l+1}} \neq 0$ and $k_m \in I^1$. This completes the proof.

□

Theorem 4.7. *The mapping $G : \mathbb{V}_\epsilon \to \mathbb{R}$ is proper.*

Proof. Let K be a compact subset of $\mathbb{R}$. Since G is continuous by Lemma 4.5, $G^{-1}(K)$ is closed. Let us define $\Omega' = \Omega \setminus \{ x^1 \}$ and $\Omega'_h = \Omega' \cap \mathbb{Z}^d_h$. Elements of $\mathbb{V}$ are discrete convex on Ω'_h with asymptotic cone K_{Ω^*}. By Lemma 2.3 they are uniformly bounded on $\overline{\Omega'} \cap \mathbb{Z}^d_h$. Thus $\mathbb{V}_\epsilon \subset \mathbb{V}$ is bounded and hence $G^{-1}(K)$ is compact. □

We can now give the proof of Theorem 4.1.

Proof of Theorem 4.1. The result follows immediately from Theorem 3.1 and Theorems 4.2-4.7. □

Remark 4.8. *It is clearly possible to prove better convergence rates for the damped Newton's method with further regularity assumptions on the density R as in* [11]. *We wish to do that in the more general setting of generated Jacobians.*

Remark 4.9. *Theorem 3.1 does not guarantee a fast decrease of the error. For a fixed δ, it may be possible to have i_k large. In fact, i_k may depend on the iterate v^k_h. Since $G'(x)$ is invertible for all $x \in \Omega_h$, it should be possible as in the proof of* [13, Proposition 24, step 2] *to prove that i_k depends continuously on v^k_h and use a compactness argument to get a uniform bound on i_k.*

	h				
	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$	$1/2^9$
Error	$2.92\ 10^{+1}$	$2.54\ 10^{-2}$	$1.31\ 10^{-2}$	$6.68\ 10^{-3}$	$3.38\ 10^{-3}$
Rate		10	0.95	0.97	0.98

5. A numerical experiment

For the implementation of the numerical method (2.4), note that the set $\partial_V v_h(x)$ for a mesh point x is a polyhedron defined by a finite number of inequalities. There are programs available on MATLAB Central which allow to compute the vertices of a polyhedron from the defining inequalities. Numerical integration over a triangulation of the polyhedron can then be used to compute $\omega_a(R, v_h, \{ x \})$ for $x \in \Omega_h$. Formulas for the Jacobian matrix are given in the proof of Theorem 4.5. In our MATLAB implementation, we found the vertices of $\partial_V v_h(x)$ by parameterizing its edges using the linear inequalities.

Take $d = 2$, $\Omega = (0,1)^2$ and the exact solution $u(x,y) = x^2/2 + xy + y^2$. In this case Ω^* is the polygon of area 1 with vertices $(0,0),(1,1),(1,2)$ and $(1,3)$. Here, $\Omega_h = \Omega \cap \mathbb{Z}^2_h$. We take $R(x,y) = e^{-0.5(x^2+y^2)}$ with corresponding right hand side $f(x,y)$. As an initial guess we consider a quadratic v_h^0 with $\partial v_h(\Omega) \subset \Omega^*$.

For integration over edges, for the entries of the Jacobian matrix, we used a Gaussian quadrature rule with degree of precision 7. For the right hand side a three point quadrature rule with degree of precision 2 was used. The stencil V was taken as $V = -V_1 \cup V_1$ where V_1 consists of the vectors $(1,0),(0,1),(1,1),(1,-1),(2,1),(-1,2),(1,2)$ and $(-2,1)$.

For the imposition of the constraint $v_h(x^1) = 0$, we approximate the solution of the equation $R(Du) \det D^2 u = f + u(x^1)$. The compatibility condition (2.1) implies that $u(x^1) = 0$. In our experiment we used $x^1 = (h,h)$. As for the parameters of the damped Newton's method, we simply took $\delta = 0$ and $\rho = 1/2$. In the table, we show the maximum errors over $[0,1]^2 \cap \mathbb{Z}^2_h$.

Acknowledgments

The author would like to thank the referees for their comments which lead to a better presentation. The author was partially supported by NSF grant DMS-1720276. The author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme "Geometry, compatibility and structure preservation in computational differential equations" where a significant part of this work was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

References

[1] G. Awanou. The second boundary value problem for a discrete Monge-Ampère equation. https://arxiv.org/abs/1910.14376.

[2] G. Awanou. Computational geometric optics: Monge-Ampère. To appear in Notices of the American Mathematical Society, February 2021.

[3] P. Azimzadeh and P. A. Forsyth. Weakly chained matrices, policy iteration, and impulse control. *SIAM Journal on Numerical Analysis*, 54(3):1341–1364, 2016.

[4] J.-D. Benamou and V. Duval. Minimal convex extensions and finite difference discretisation of the quadratic Monge–Kantorovich problem. *European J. Appl. Math.*, 30(6):1041–1078, 2019.

[5] R. J. Berman. Convergence rates for discretized Monge-Ampère equations and quantitative stability of optimal transport. arXiv preprint arXiv:1803.00785, 2020.

[6] S. Bütikofer. *Generalized Newton-type methods for nonsmooth equations in optimization and complementarity problems*. PhD thesis, ETH Zurich, 2008.

[7] L. A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.*, 5(1):99–104, 1992.

[8] P. Delanoë. Gradient rearrangement for diffeomorphisms of a compact manifold. *Differential Geom. Appl.*, 20(2):145–165, 2004.

[9] J. H. G. Fu. Monge-Ampère functions. I, II. *Indiana Univ. Math. J.*, 38(3):745–771, 773–789, 1989.

[10] X. Gu, F. Luo, J. Sun, and S.-T. Yau. Variational principles for Minkowski type problems, discrete optimal transport, and discrete Monge-Ampere equations. *Asian J. Math.*, 20(2):383–398, 2016.

[11] J. Kitagawa, Q. Mérigot, and B. Thibert. Convergence of a Newton algorithm for semi-discrete optimal transport. *J. Eur. Math. Soc. (JEMS)*, 21(9):2603–2651, 2019.

[12] B. Lévy. A numerical algorithm for L_2 semi-discrete optimal transport in 3D. *ESAIM Math. Model. Numer. Anal.*, 49(6):1693–1715, 2015.

[13] Q. Mérigot, J. Meyron, and B. Thibert. An algorithm for optimal transport between a simplex soup and a point cloud. *SIAM J. Imaging Sci.*, 11(2):1363–1389, 2018.

[14] J.-M. Mirebeau. Discretization of the 3D Monge-Ampère operator, between wide stencils and power diagrams. *ESAIM Math. Model. Numer. Anal.*, 49(5):1511–1523, 2015.

[15] R. H. Nochetto and W. Zhang. Pointwise rates of convergence for the Oliker-Prussner method for the Monge-Ampère equation. *Numer. Math.*, 141(1):253–288, 2019.

[16] V. Oliker. Mathematical aspects of design of beam shaping surfaces in geometrical optics. In *Trends in nonlinear analysis*, pages 193–224. Springer, Berlin, 2003.

[17] V. I. Oliker and L. D. Prussner. On the numerical solution of the equation $(\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2) - ((\partial^2 z/\partial x \partial y))^2 = f$ and its discretizations. I. *Numer. Math.*, 54(3):271–293, 1988.

[18] J. M. Ortega and W. C. Rheinboldt. *Iterative solution of nonlinear equations in several variables*, volume 30 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. Reprint of the 1970 original.

[19] A. V. Pogorelov. *Monge-Ampère equations of elliptic type*. Translated from the first Russian edition by Leo F. Boron with the assistance of Albert L. Rabenstein and Richard C. Bollinger. P. Noordhoff, Ltd., Groningen, 1964.

[20] W. Qiu and L. Tang. A note on the Monge–Ampère type equations with general source terms. *Mathematics of Computation*, 2020.

[21] P. Shivakumar and K. H. Chew. A sufficient condition for nonvanishing of determinants. *Proceedings of the American mathematical society*, pages 63–66, 1974.

Department of Mathematics, Statistics, and Computer Science, M/C 249. University of Illinois at Chicago, Chicago, IL 60607-7045, USA

Email address: awanou@uic.edu

URL: http://www.math.uic.edu/~awanou