Noname manuscript No.
(will be inserted by the editor)

Convergence of finite difference schemes to the Aleksandrov solution of the Monge-Ampère equation

Gerard Awanou and Romeo Awi

Received: date / Accepted: date

Abstract We present a technique for proving convergence to the Aleksandrov solution of the Monge-Ampère equation of a stable and consistent finite difference scheme. We also require a notion of discrete convexity with a stability property and a local equicontinuity property for bounded sequences.

Keywords discrete Monge-Ampère · Aleksandrov solution · weak convergence of measures

Mathematics Subject Classification (2000) 39A12 · 35J60 · 65N12 · 65M06

1 Introduction

Given an orthogonal lattice with mesh length h on a convex bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$, we are interested in convergent finite difference approximations of the problem: find a convex function $u \in C(\overline{\Omega})$ such that

$$\begin{aligned} \det D^2 u &= \nu \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega, \end{aligned} \tag{1}$$

where ν is a finite Borel measure and $g \in C(\partial\Omega)$ can be extended to a convex function $\tilde{g} \in C(\overline{\Omega})$. When $u \in C^2(\Omega)$, $\det D^2 u$ is the determinant of the

Gerard Awanou
Department of Mathematics, Statistics, and Computer Science (M/C 249), University of Illinois at Chicago, Chicago, IL, 60607-7045
Tel.: +1-312-413-2167
Fax: +1-312-996-1491
E-mail: awanou@uic.edu
Romeo Awi
IMA, University of Minnesota, 207 Church Street SE, 306 Lind Hall, Minneapolis, MN U.S.A. 55455
E-mail: awi@ima.umn.edu

Hessian matrix $D^2 u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,d}$. In the general case, the expression $\det D^2 u$ denotes the Monge-Ampère measure associated with u, c.f. section 2.1.

Let Ω_h denote the computational domain and $\partial \Omega_h$ its boundary. Let $f_h \geq 0$ be a family of mesh functions. In section 2.2 we associate to f_h a Borel measure which is also denoted by f_h, c.f. (3). Assume that f_h converges to ν as measures (c.f. section 2.2). We consider the problem with unknown a mesh convex function u_h

$$\begin{aligned} h^d \mathcal{M}_h[u_h] &= h^d f_h \,\text{in}\, \Omega_h \\ u_h &= g \,\text{on}\, \partial \Omega_h. \end{aligned} \tag{2}$$

Here $\mathcal{M}_h[v_h]$ denotes a stable and consistent discretization of $\det D^2 v$ for a smooth convex function v. There are several notions of discrete convexity. We require that the uniform limit on compact subsets of mesh convex functions is a convex function and that a locally bounded sequence of such functions is locally equicontinuous. Of course we also require (2) to have a solution. A sufficient condition is degenerate ellipticity and Lipschitz continuity as defined by Oberman [17]. We show that a family of solutions u_h of (2) converges uniformly on compact subsets to the unique Aleksandrov solution of (1), c.f. Definition 2 for the notion of Aleksandrov solution.

The Monge-Ampère equation (1) is a fully nonlinear equation which arises in several applications of great importance, e.g. optimal transportation and reflector design. Problems in affine geometry motivated the study of the Dirichlet problem. This paper is therefore relevant to readers interested in numerical analysis, optimal transportation and affine geometry.

The equation $\det D^2 u = \nu$ with ν a sum of Dirac masses, and with Dirichlet boundary condition was solved by Pogorelov [20]. For the so-called second boundary condition we refer to [19, Chapter V section 3]. When the measure ν is absolutely continuous with respect to the Lebesgue measure, with density $f \geq 0$ and $f \in C(\Omega)$, the convergence of a scheme of the type (2) was proved in [11] using the notion of viscosity solution. In this paper, we use the notion of Aleksandrov solution, the consistency of the discretization (2) and approximation by smooth functions to handle the Monge-Ampère measure. In [3] the notion of Aleksandrov solution was also used along with a different procedure for approximation by smooth functions.

We note that the method introduced in [8] is not known to be consistent. Our requirements for convergence are stability, consistency and solvability of (1), as well as stability under uniform convergence on compact subsets of discrete convex mesh functions along with local equicontinuity of locally bounded sequences of such functions. All of these requirements, except stability in the general case where ν is a linear combination of Dirac masses, are met by the finite difference scheme introduced in [11]. Our numerical results indicate that a very good initial guess is required for an iterative method for solving the nonlinear problem (2) if one uses the discretization proposed in [11]. In our numerical experiments, the discrete problem (2) is solved with a time marching method which has also been used in [2]. The difficulty of capturing singular solutions may be related to the choice of the method for solving the nonlinear equation (2). The numerical results obtained may be explained with the approach taken in [3]. For additional numerical evidence of the stability of the discretization proposed in [11] in the general case where ν is a linear combination of Dirac masses, we refer to [6].

If ν has density $f \geq 0$ and $f \in C(\overline{\Omega})$, under our assumptions, u_h converges uniformly on compact subsets to the unique viscosity solution of (1) if the latter is known to have a unique viscosity solution. This follows from the equivalence of the notion of viscosity and Aleksandrov solutions, the proof of which we outline. If $f > 0$ and $f \in C(\overline{\Omega})$, a continuous viscosity solution of (1) is also an Aleksandrov solution of (1) [13, Proposition 1.7.1]. The result is also valid for $f \geq 0$ and $f \in C(\overline{\Omega})$. Indeed, consider the problems $\det D^2 u_\epsilon = f + \epsilon, \epsilon > 0$. By [15, Lemma 5.1], u_ϵ converges uniformly on compact subsets to u. One then uses the equivalence of the notion of viscosity and Aleksandrov solutions in the non degenerate case [13, Propositions 1.7.1 and 1.3.4] and the stability of the notion of Aleksandrov and viscosity solutions under uniform convergence on compact subsets [13, Lemma 1.2.3] and [7, Theorem 2.3].

In this paper we provide the convergence proof of a time marching method for solving the nonlinear problem (2). The main contribution of this paper is the method of proof for convergence of finite difference schemes satisfying our assumptions. The numerical results indicate that such schemes may not lead to an effective numerical algorithm. Our results clarify the nature of efficient discretizations for (1). Another consequence of our results is the equivalence of the notions of viscosity and Aleksandrov solutions for $f \geq 0$ and $f \in C(\Omega) \cap L^\infty(\Omega)$. Indeed as we show in section 4.2, u_h obtained through (2) and the discretization proposed in [11] converges to the Aleksandrov solution. It is also known to converge to the unique viscosity solution of (1) when the latter exists. Hence the result.

The paper is organized as follows: in the next section we give some notations, recall key results on the Aleksandrov solution and finite difference schemes. In section 3 we prove the claimed convergence result. We conclude with a proof of convergence of the time marching method and numerical experiments.

2 Preliminaries

In this section, we recall key results on the Aleksandrov solution of the Monge-Ampère equation. We then associate discrete measures to mesh functions. For a smooth solution of (1) we immediately get a discretization of the Monge-Ampère measure. Finally, we introduce finite difference schemes.

2.1 The Monge-Ampère measure

In this paper, we take the analytic approach to the Monge-Ampère measure [21]. Let $K(\Omega)$ denote the cone of convex functions on Ω and let $M(\Omega)$ denote the set of Borel measures on Ω. We will consider only values of measures on the Borel sets, i.e. for a Borel measure ν, the ν-measurable sets are the Borel sets.

For $v \in C^2(\Omega) \cap K(\Omega)$, we define a Borel measure $\mathcal{M}[v]$ by

$$\mathcal{M}[v](B) = \int_B \det D^2 v(x)\, dx,$$

where B is a Borel set.

The topology on $M(\Omega)$ is induced by the weak convergence of measures.

Definition 1 A sequence $\nu_n \in M(\Omega)$ converges to $\nu \in M(\Omega)$ if and only if for all continuous functions ϕ with compact support in Ω,

$$\int_\Omega \phi\, d\nu_n \to \int_\Omega \phi\, d\nu.$$

The above definition of convergence of a sequence of measures extends immediately to a family of measures, i.e. a family $\nu_h \in M(\Omega)$ converges to ν if and only if for all sequences $h_n \to 0$, ν_{h_n} converges to ν.

We note that there are several equivalent definitions of weak convergence of measures which can be found for example in [9, Theorem 1, section 1.9] for measures on $\mathbb{R}^d$ or [1, Theorem 4.5.1] for a weaker notion of convergence. We have

Lemma 1 *Let $\nu_n, \nu \in M(\Omega)$ and $\nu_n(B) \to \nu(B)$ for any Borel set $B \subset \Omega$ with $\nu(\partial B) = 0$ and $\overline{B} \subset \Omega$. Then ν_n converges to ν.*

Proof The proof is analogous to the one in [1, Theorem 4.5.1], so we omit it.

Proposition 1 *[21, Proposition 3.1] The mapping $\mathcal{M}$ maps $C(\Omega)$-bounded subsets of $C^2(\Omega) \cap K(\Omega)$ into bounded subsets of $M(\Omega)$. Moroever $\mathcal{M}$ has a unique extension to a continuous operator on $K(\Omega)$.*

For a convex function v, we will refer to $\mathcal{M}[v]$ as the Monge-Ampère measure associated with v. It can be shown that it coincides with the notion of Monge-Ampère measure obtained through the normal mapping [21, Proposition 3.4].

Definition 2 Given a Borel measure ν on Ω, a convex function $u \in C(\Omega)$ is an Aleksandrov solution of

$$\det D^2 u = \nu,$$

if and only if $\mathcal{M}[u] = \nu$.

We recall an existence and uniqueness result for the solution of (1).

Proposition 2 ([15] Theorem 1.1) *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Assume ν is a finite Borel measure and $g \in C(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. Then the Monge-Ampère equation* (1) *has a unique Aleksandrov solution in $K(\Omega) \cap C(\overline{\Omega})$.*

Throughout this paper, we will follow the convention of denoting by p a measure ν which is absolutely continuous with respect to the Lebesgue measure and with density p.

2.2 Discrete measures associated with mesh functions

Let h be a small positive parameter and let

$$\mathbb{Z}_h^d = \{\, mh, m \in \mathbb{Z}^d \,\},$$

denote the regular uniform grid of $\mathbb{R}^d$. By a mesh function we mean a real-valued function defined on $\mathbb{Z}_h^d$.

The computational domain is defined as $\Omega_h = \Omega \cap \mathbb{Z}_h^d$ and its boundary is simply $\partial\Omega_h = \{\, x \in \overline{\Omega} \cap \mathbb{Z}_h^d, x \notin \Omega_h \,\}$. We denote by $\mathcal{C}_h$ the cone of discrete convex mesh functions. We recall that there are several possibilities for defining a discrete convex function. For example, we may require that a mesh function v_h is discrete convex if and only if $v_h(x+e) - 2v_h(x) + v_h(x-e) \geq 0$ for all $x \in \Omega_h$ and $e \in \mathbb{Z}_h^d$ for which $x \pm e \in \overline{\Omega}_h$. We prove in section 4.2 that for that notion of discrete convex mesh functions, the assumptions of this paper hold.

Let v_h be a mesh function such that $v_h \geq 0$ on Ω_h. We associate to v_h a Borel measure which we denote here by v_h and defined by

$$v_h(B) = h^d \sum_{x \in \Omega_h \cap B} v_h(x), \tag{3}$$

for any Borel set B.

Given a continuous function v on Ω, we use the notation $r_h(v)$ to denote its restriction to Ω_h. If $v \geq 0$ is a continuous function on Ω, we have

$$\lim_{h \to 0} r_h(v)(B) = \int_B v(x)\, dx.$$

for any $B \subset \Omega$ satisfying $|\partial B| = 0$. In other words, the measures v_h converge weakly to v.

Definition 3 Let $v_h \in \mathcal{C}_h$ for each $h > 0$. We say that v_h converges to a convex function $v \in C(\Omega)$ uniformly on compact subsets of Ω if and only if for each compact set $K \subset \Omega$, each sequence $h_k \to 0$ and for all $\epsilon > 0$, there exists $h_{-1} > 0$ such that for all h_k, $0 < h_k < h_{-1}$, we have

$$\max_{x \in K \cap \mathbb{Z}_h^d} |v_{h_k}(x) - v(x)| < \epsilon.$$

2.3 Finite difference schemes

Let $f_h \geq 0$ be a family of mesh functions which converge to ν as measures and let $\mathcal{M}_h[r_h v]$ denotes a discretization of $\det D^2 v$ for a smooth convex function v. We are interested in the discrete Monge-Ampère equation (2).

The discretization $\mathcal{M}_h[r_h v]$ is said to be consistent if for all C^2 convex functions v, and a sequence $x_h \in \Omega_h$ such that $x_h \to x \in \Omega$, $\lim_{h\to 0} \mathcal{M}_h[r_h v](x_h) = \det D^2 v(x)$.

We denote by $\mathcal{M}_h[v_h]$ the discretization of the determinant which reduces to $\mathcal{M}_h[r_h v]$ for a smooth function v. The discretization is said to be stable if the problem $\mathcal{M}_h[v_h] = f_h$ has a solution v_h which is bounded independently of h. We give in section 4.2 an example of a discretization $\mathcal{M}_h[v_h]$ which is stable when the functions f_h are uniformly bounded. The example of discretization $\mathcal{M}_h[v_h]$ analyzed in [4] is stable under the assumption of this paper that f_h converges to ν as measures. However, the one analyzed there is not known to be consistent.

3 Convergence

We recall that our assumptions are that the nonlinear equation (2) is solvable with the discretization stable and consistent. We also require that the uniform limit on compact subsets of mesh convex functions is a convex function and that a locally bounded sequence of such functions is locally equicontinuous.

Lemma 2 *There exists a constant C_0 such that for all Borel sets $B \subset \Omega$*

$$|\mathcal{M}_h[w_h](B) - \mathcal{M}_h[v_h](B)| \leq C_0 \max_{x \in \overline{B} \cap \mathbb{Z}^d_h} |w_h(x) - v_h(x)|.$$

Proof We have by (3)

$$\begin{aligned}
|\mathcal{M}_h[w_h](B) - \mathcal{M}_h[v_h](B)| &= h^d \left| \sum_{x \in B \cap \Omega_h} w_h(x) - \sum_{x \in B \cap \Omega_h} v_h(x) \right| \\
&\leq \sum_{x \in \overline{B} \cap \Omega_h} h^d |w_h(x) - v_h(x)| \\
&\leq \max_{x \in \overline{B} \cap \mathbb{Z}^d_h} |w_h(x) - v_h(x)| \sum_{x \in \overline{B} \cap \Omega_h} h^d \\
&\leq C \max_{x \in \overline{B} \cap \mathbb{Z}^d_h} |w_h(x) - v_h(x)|,
\end{aligned}$$

with a constant C bounded by a multiple of the measure of Ω.

The constant C_0 in Lemma 2 satisfies

$$\sum_{x \in \Omega_h} h^d \leq C_0. \tag{4}$$

Lemma 3 *For $v \in K(\Omega) \cap C^2(\Omega)$, $h^d \mathcal{M}_h[r_h v]$ converges weakly to* $\det D^2 v$.

Proof By definition of the Monge-Ampère measure and of discretization of the integral, we have for a Borel set B with $\mathcal{M}[v](\partial B) = 0$ and $\overline{B} \subset \Omega$

$$\int_B \det D^2 v(x)\, dx = \lim_{h \to 0} h^d \sum_{x \in B \cap \Omega_h} \det D^2 v(x).$$

Using (4)

$$\Big| h^d \sum_{x \in B \cap \Omega_h} \det D^2 v(x) - h^d \mathcal{M}_h[r_h v](B) \Big| = \Big| \sum_{x \in B \cap \Omega_h} h^d \left(\det D^2 v(x) - \mathcal{M}_h[r_h v](x) \right) \Big|$$
$$\leq C_0 \max_{x \in \overline{B} \cap \Omega_h} | \det D^2 v(x) - \mathcal{M}_h[r_h v](x)|,$$

We claim that by the consistency assumption, as $h \to 0$, $\max_{x \in \overline{B} \cap \Omega_h} | \det D^2 v(x) - \mathcal{M}_h[r_h v](x)| \to 0$. Suppose

$$\lim_{h \to 0} \max_{x \in B \cap \Omega_h} | \det D^2 v(x) - M_h[r_h v](x)| \neq 0.$$

One can then find $\epsilon > 0$ and a sequence $\{h_k\}_{k=1}^{\infty}$ converging to 0 such that for all k, one has

$$\max_{x \in B \cap \Omega_{h_k}} | \det D^2 v(x) - M_{h_k}[r_{h_k} v](x)| \geq \epsilon.$$

That is for all k, there exists $x_k \in B \cap \Omega_{h_k}$ such that

$$| \det D^2 v(x_k) - M_{h_k}[r_{h_k} v](x_k)| \geq \epsilon. \tag{5}$$

Using a subsequence if necessary, $\{x_k\}_{k=1}^{\infty}$ converges to some $\bar{x} \in \bar{B} \subset \Omega$. Then $\det D^2 v(x_k) \to \det D^2 v(\bar{x})$ by continuity of $\det D^2 v$ and by the consistency assumption

$$\det D^2 v(\bar{x}) - M_{h_k}[r_{h_k} v](x_k) \to 0.$$

A contradiction. Therefore

$$\lim_{h \to 0} \max_{x \in B \cap \Omega_h} | \det D^2 v(x) - M_h[r_h v](x)| = 0.$$

By Lemma 1 the result follows.

We have the following weak convergence result for discrete Monge-Ampère measures

Theorem 1 *Let $v_h \in \mathcal{C}_h$ converge uniformly on compact subsets to $v \in K(\Omega) \cap C(\overline{\Omega})$ in the sense of Definition 3. Then $h^d \mathcal{M}_h[v_h]$ converges weakly to $\mathcal{M}[v]$.*

Proof Let $v_\epsilon \in K(\Omega) \cap C^\infty(\Omega)$ converge uniformly to v on Ω. The existence of v_ϵ may be proven as in [5] by mollifications of dilations of v. Let B be a Borel set with $\mathcal{M}[v](\partial B) = 0$ and $\overline{B} \subset \Omega$. Given $\delta > 0$, we seek $h_0 > 0$ such that $|h^d \mathcal{M}_h[v_h](B) - \mathcal{M}[v](B)| < \delta$ for all $0 < h < h_0$.

By Proposition 1, $\exists\, \epsilon_0 > 0$ such that $|\mathcal{M}[v_{\epsilon_0}](B) - \mathcal{M}[v](B)| < \delta/3$. By Lemma 3, $\exists h_0 > 0$ such that for all $0 < h < h_0$, $|h^d \mathcal{M}_h[(r_h(v_{\epsilon_0})](B) - \mathcal{M}[v_{\epsilon_0}](B)| < \delta/3$.

We may assume that $\max_{x \in \overline{B}} |v_{\epsilon_0}(x) - v(x)| < \delta/(6C_0)$ and since v_h converges to v on $\overline{B}$, we may assume that for $h < h_0$, $\max_{x \in \overline{B} \cap \mathbb{Z}^d_h} |v(x) - v_h(x)| < \delta/(6C_0)$. Thus we have $\max_{x \in \overline{B} \cap \mathbb{Z}^d_h} |v_{\epsilon_0}(x) - v_h(x)| < \delta/(3C_0)$. By Lemma 2, $|\mathcal{M}_h[r_h(v_{\epsilon_0})](B) - \mathcal{M}_h[v_h](B)| < \delta/3$. This concludes the proof by Lemma 1.

We can now prove the main result of this paper.

Theorem 2 *The mesh function u_h defined by* (2) *converges uniformly on compact subsets to the Aleksandrov solution u of* (1).

Proof By the stability assumption, the family u_h is uniformly bounded and by our assumption on the discretization, locally equicontinuous. By the Arzela-Ascoli theorem, there exists a subsequence u_{h_k} which converges uniformly on compact subsets to a function v. Since $u_h \in \mathcal{C}_h$ the function v is convex by our assumptions on discrete convex functions. Since u_h is uniformly bounded, v is convex and bounded on Ω, hence continuous on Ω. Arguing as in the proof of [4, Theorem 4.3] one proves that $v \in C(\overline{\Omega})$. Since $h^d \mathcal{M}_h[v_h]$ converges weakly to $\mathcal{M}[v]$ and $u_h = g$ on $\partial\Omega$, the function v is an Aleksandrov solution of (1). By uniqueness, $v = u$ and hence the whole family u_h converges uniformly on compact subsets to u.

4 Convergence of a time marching iterative method

Let us denote by $\mathcal{M}(\Omega^h)$ the set of mesh functions, i.e. the set of real valued functions defined on Ω^h. Since Ω_h is a finite set, there is a canonical identification of $\mathcal{M}(\Omega^h)$ with $\mathbb{R}^N$ for some integer N. We will now also use the restriction operator r_h for vector and matrix fields. For $x \in \mathbb{R}^N$, $|x| = (\sum_{i=1}^N x_i^2)^{\frac{1}{2}}$ denotes the Euclidean norm of x and $|x|_\infty = \max_{i=1,\ldots,N} |x_i|$ denotes its maximum norm.

We make the assumption that the mapping $\mathcal{M}_h$ is Lipschitz continuous with Lipschitz constant $K > 0$ i.e.

$$|\mathcal{M}_h[v_h] - \mathcal{M}_h[w_h]|_\infty \leq K|v_h - w_h|_\infty, v_h, w_h \in \mathbb{R}^N.$$

Here we make the abuse of notation of identifying a mesh function with its vector representation. We also make the assumption that problem (2) has a unique solution u_h which can be computed by a time marching method

$$\begin{aligned} u_h^{k+1} &= u_h^k + \frac{1}{\mu}\mathcal{M}_h[u_h^k] \,\text{in}\, \Omega_h \\ u_h &= r_h(g) \,\text{on}\, \partial\Omega_h, \end{aligned} \tag{6}$$

for $\mu \geq \mu_0$ where $\mu_0 > 0$ and u_h^0 is a suitable initial guess. Such assumptions are satisfied by proper Lipschitz continuous degenerate elliptic schemes as defined by Oberman [17]. We were not able to get numerical evidence of convergence for the above iterative method for the discretization proposed in [11] even if we use the exact solution as initial guess. Similar results for Newton's method were reported in [8].

Let us denote by Δ_h the standard finite difference discretization of the Laplace operator and let e_i denote the i^{th} vector of the canonical basis of $\mathbb{R}^d$. For $x \in \Omega_h$ and $v_h \in \mathcal{M}(\Omega^h)$, we have

$$\Delta_h v_h(x) = \sum_{i=1}^{d} \frac{v_h(x + he_i) - 2v_h(x) + v_h(x - he_i)}{h^2}.$$

When the measure ν is a combination of Dirac masses we obtained better numerical results with the preconditioned iterative method

$$\begin{aligned} -\Delta_h u_h^{k+1} &= -\Delta_h u_h^k + \frac{1}{\mu}\mathcal{M}_h[u_h^k] \, \text{in} \, \Omega_h \\ u_h &= r_h(g) \, \text{on} \, \partial\Omega_h, \end{aligned} \tag{7}$$

for $\mu \geq \mu_1$ for a positive real number μ_1 under the above assumptions. Moreover numerical experiments indicate that the method (7) converges faster than (6). The idea to use the Laplacian for faster iterative methods has a long story in various contexts [10, p. 58], and a remark in that direction for proper Lipschitz continuous degenerate elliptic schemes was made in [12]. See also [18]. We use the terminology preconditioned iterative method for (7) by analogy with preconditioned techniques for linear equations. An advantage of the preconditioned iterative method (7) is that fast Poisson solvers and standard multigrid methods can be used at each step.

The proof of convergence of the iterative method (7) does not follow the approach in [17] for proving convergence of the basic iterative method (6). The proof of the latter does not seem to extend to the preconditioned version (7). We take a different approach which consists in using the fact that (6) converges to the discrete solution of (2) and properties of the inverse of the operator Δ_h.

4.1 Convergence of the preconditioned iterative method

It can be shown [14, Theorem 4.4.1], that for $f \in C(\overline{\Omega})$ the problem

$$\begin{aligned} \Delta_h[z_h] &= r_h(f) \, \text{in} \, \Omega_h \\ z_h &= 0 \, \text{on} \, \partial\Omega_h, \end{aligned}$$

has a unique solution. We denote by Δ_h^{-1} the inverse of the operator Δ_h with homogeneous boundary conditions. Let $||\Delta_h^{-1}||$ denote the operator norm of

Δ_h^{-1}, i.e.

$$||\Delta_h^{-1}|| = \sup_{|v_h|_\infty \neq 0} \frac{|\Delta_h^{-1} v_h|_\infty}{|v_h|_\infty}.$$

By [14, Theorem 4.4.1], $||\Delta_h^{-1}||$ is bounded independently of h. We note that [14, Theorem 4.4.1] is proven for dimension $n = 2$ but the proof extends immediately to arbitrary dimension.

The main result of this section is the following theorem.

Theorem 3 *Let $\mathcal{M}_h$ denote a Lipschitz continuous finite difference scheme such that the mapping $T_1 : \mathcal{M}(\Omega^h) \to \mathcal{M}(\Omega^h)$ defined by*

$$T_1[v_h] = v_h + \frac{1}{\mu}\mathcal{M}_h[v_h],$$

is a strict contraction for $\mu \geq \mu_0 > 0$. Then for some $\mu_1 > 0$, the mapping $T_2 : \mathcal{M}(\Omega^h) \to \mathcal{M}(\Omega^h)$ defined by

$$T_2[v_h] = v_h - \frac{1}{\mu}\Delta_h^{-1}\mathcal{M}_h[v_h],$$

is also a strict contraction for $\mu \geq \mu_1$.

Proof By assumption, there exists a constant C_1 such that $0 < C_1 < 1$ and

$$|T_1[v_h] - T_1[w_h]|_\infty \leq C_1 |v_h - w_h|_\infty,$$

for all $v_h, w_h \in \mathcal{M}(\Omega^h)$. One may decompose $T_2[v_h] - T_2[w_h]$ as

$$\begin{aligned} T_2[v_h] - T_2[w_h] &= T_2[v_h] - T_1[v_h] + T_1[v_h] - T_1[w_h] + T_1[w_h] - T_2[w_h] \\ &= (T_1[v_h] - T_1[w_h]) + (T_2[v_h] - T_1[v_h]) - (T_2[w_h] - T_1[w_h]). \end{aligned}$$

Moreover

$$T_1[v_h] - T_2[v_h] = \frac{1}{\mu}\left(\mathcal{M}_h[v_h] + \Delta_h^{-1}\mathcal{M}_h[v_h]\right) = \frac{1}{\mu}\left(I + \Delta_h^{-1}\right)\mathcal{M}_h[v_h],$$

where I denotes the identity operator on $\mathcal{M}(\Omega^h)$. We then get

$$(T_2[v_h] - T_1[v_h]) - (T_2[w_h] - T_1[w_h]) = -\frac{1}{\mu}\left(I + \Delta_h^{-1}\right)(\mathcal{M}_h[v_h] - \mathcal{M}_h[w_h]).$$

We recall that $\mathcal{M}_h$ is Lipchitz continuous, i.e.

$$|\mathcal{M}_h[v_h] - \mathcal{M}_h[w_h]|_\infty \leq K|v_h - w_h|_\infty, \forall v_h, w_h \in \mathcal{M}(\Omega^h).$$

One deduces that

$$\begin{aligned} |T_2[v_h] - T_2[w_h]|_\infty &\leq |T_1[v_h] - T_1[w_h]|_\infty + |\frac{1}{\mu}\left(I + \Delta_h^{-1}\right)(\mathcal{M}_h[v_h] - \mathcal{M}_h[w_h])|_\infty \\ &\leq C_1|v_h - w_h|_\infty + \frac{K}{\mu}||I + \Delta_h^{-1}||\,|v_h - w_h|_\infty \\ &\leq \left(C_1 + \frac{K}{\mu}|I + \Delta_h^{-1}\|\right)|v_h - w_h|_\infty. \end{aligned}$$

Since $\|I+\Delta_d^{-1}\| \leq \|I\|+\|\Delta_h^{-1}\|$ is bounded independently of the discretization step h and $0 < C_1 < 1$, one may choose μ big enough such that

$$C_1 + \frac{K}{\mu}\|I + \Delta_h^{-1}\| < 1,$$

making T_2 a strict contraction mapping. This concludes the proof.

Under the assumption of the above theorem, both the iterative methods (6) and (7) converge linearly to the unique solution u_h of (2).

4.2 A numerical example

In this section we consider a particular notion of discrete convexity. A mesh function v_h is *discrete convex* if and only if $\Delta_e v_h(x) = v_h(x+e) - 2v_h(x) + v_h(x-e) \geq 0$ for all $x \in \Omega_h$ and $e \in \mathbb{Z}_h^d$ for which $\Delta_e v_h(x)$ is defined. Then the uniform limit of discrete convex mesh functions is convex [4, Lemma 2.11]. Moreover a bounded sequence of such functions is locally equicontinuous [4].

Following [11], we define

$$M_h[v_h](x) = \inf_{(\alpha_1,\ldots,\alpha_n)\in W_h(x)} \prod_{i=1}^{d} \max\left(\frac{v_h(x+\alpha_i) - 2v_h(x) + v_h(x-\alpha_i)}{|\alpha_i|^2}, 0\right).$$

where for $x \in \Omega_h$, $W_h(x)$ denotes the set of orthogonal bases of $\mathbb{R}^d$ such that for $(\alpha_1, \ldots, \alpha_n) \in W_h(x)$, $x \pm \alpha_i \in \Omega_h$, for all i.

Note that $M_h[v_h] \geq 0$ implies that v_h is discrete convex. Hence the discrete convexity assumption is enforced in the discretization. Moreover, as pointed out in [4], if one considers $\mathcal{M}_h[v_h] = M_h[v_h](x) + \epsilon(h)v_h(x)$ where $\epsilon(h)$ is taken close to machine precision with $\epsilon(h) \to 0$ as $h \to 0$, the discretization is proper and hence uniqueness holds.

It is known that $M_h[v_h]$ satisfies the assumptions of degenerate ellipticity and Lipschitz continuity as defined by Oberman [17]. The consistency of the scheme was proved in [11] while for $f \in C(\overline{\Omega})$, and hence uniformly bounded, a proof of stability can be found in [2]. The proof uses the strict contraction property of the mapping T_1 which holds for a proper, degenerate elliptic and Lipschitz continuous scheme [17, Theorem 7]. Let u_h^0 be a fixed mesh function and recall that we have

$$\begin{aligned}
|u_h|_\infty = |T_1[u_h]|_\infty &\leq |T_1[u_h] - T_1[u_h^0]|_\infty + |T_1[u_h^0]|_\infty \\
&\leq a|u_h - u_h^0|_\infty + |T_1[u_h^0]|_\infty \\
&\leq a|u_h|_\infty + a|u_h^0|_\infty + |T_1[u_h^0]|_\infty,
\end{aligned}$$

where we denote by a the strict contraction constant of the mapping T_1, $0 < a < 1$. The result follows.

As pointed out in the introduction, we do not know how to prove stability of the discretization when the right hand side is a finite Borel measure. The

	h					
μ	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$
50	4.71 10^{-1}	2.86 10^{-1}	1.69 10^{-1}	9.77 10^{-2}	5.50 10^{-2}	3.02 10^{-2}

Table 1

approach taken in [3] suggests that it may be possible to prove stability in that case as well.

For the numerical experiments, the space dimension d is taken as 2 and the computational domain is the unit square $(0,1)^2$. Numerical experiments with ν a Dirac mass was reported in earlier papers, e.g. [11]. Here we consider the example of [8] where ν is the sum of two Dirac masses, i.e. we take

$$u(x,y) = \begin{cases} |y-\frac{1}{2}| & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ \min\left\{ \sqrt{(x-\frac{1}{4})^2 + (y-\frac{1}{2})^2}, \sqrt{(x-\frac{3}{4})^2 + (y-\frac{1}{2})^2} \right\} & \text{otherwise,} \end{cases}$$

and $\nu = \pi/2\, \delta_{(1/4,1/2)} + \pi/2\, \delta_{(3/4,1/2)}$. For simplicity, we only use a 17 point stencil. The initial guess is taken as the exact solution and the nonlinear equations solved with (7). Errors are given in the maximum norm and reported on Table 1.

Acknowledgements We would like to thank the referees for a careful reading of the paper. Gerard Awanou was partially supported by a Division of Mathematical Sciences of the US National Science Foundation grant No 1319640.

References

1. Ash, R.B.: Measure, integration, and functional analysis. Academic Press, New York-London (1972)
2. Awanou, G.: Convergence of a hybrid scheme for the elliptic Monge-Ampère equation. http://homepages.math.uic.edu/ awanou/up.html
3. Awanou, G.: On standard finite difference discretizations of the elliptic Monge-Ampère equation (2014). http://arxiv.org/abs/1311.2812.
4. Awanou, G.: Discrete Aleksandrov solutions of the Monge-Ampère equation (2015). http://arxiv.org/abs/1408.1729
5. Awanou, G.: Smooth approximations of the Aleksandrov solution of the Monge-Ampère equation. Commun. Math. Sci. **13**(2), 427–441 (2015)
6. G. Awanou and L. Matamba Messi. A variational method for computing numerical solutions of the Monge-Ampère equation (2015). http://arxiv.org/abs/1510.00453.
7. Barles, G.: Solutions de viscosité des équations de Hamilton-Jacobi, *Mathématiques & Applications (Berlin) [Mathematics & Applications]*, vol. 17. Springer-Verlag, Paris (1994)
8. Benamou, J.D., Froese, B.D.: A viscosity framework for computing Pogorelov solutions of the Monge-Ampère equation (2014). http://arxiv.org/pdf/1407.1300v2.pdf
9. Evans, L.C., Gariepy, R.F.: Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1992)
10. Faragó, I., Karátson, J.: Numerical solution of nonlinear elliptic problems via preconditioning operators: theory and applications, *Advances in Computation: Theory and Practice*, vol. 11. Nova Science Publishers Inc., Hauppauge, NY (2002)

11. Froese, B., Oberman, A.: Convergent finite difference solvers for viscosity solutions of the elliptic Monge-Ampère equation in dimensions two and higher. SIAM J. Numer. Anal. **49**(4), 1692–1714 (2011)
12. Glowinski, R.: Numerical methods for fully nonlinear elliptic equations. In: ICIAM 07—6th International Congress on Industrial and Applied Mathematics, pp. 155–192. Eur. Math. Soc., Zürich (2009)
13. Gutiérrez, C.E.: The Monge-Ampère equation. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA (2001)
14. Hackbusch, W.: Elliptic differential equations, *Springer Series in Computational Mathematics*, vol. 18, english edn. Springer-Verlag, Berlin (2010). Theory and numerical treatment, Translated from the 1986 corrected German edition by Regine Fadiman and Patrick D. F. Ion
15. Hartenstine, D.: The Dirichlet problem for the Monge-Ampère equation in convex (but not strictly convex) domains. Electron. J. Differential Equations pp. No. 138, 9 pp. (electronic) (2006)
16. Mohammadi, B.: Optimal transport, shape optimization and global minimization. C. R. Math. Acad. Sci. Paris **344**(9), 591–596 (2007)
17. Oberman, A.M.: Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. SIAM J. Numer. Anal. **44**(2), 879–895 (electronic) (2006)
18. Oberman, A.M.: Finite difference methods for the infinity Laplace and p-Laplace equations. J. Comput. Appl. Math. **254**, 65–80 (2013)
19. Pogorelov, A.V.: Monge-Ampère equations of elliptic type. Translated from the first Russian edition by Leo F. Boron with the assistance of Albert L. Rabenstein and Richard C. Bollinger. P. Noordhoff, Ltd., Groningen (1964)
20. Pogorelov, A.V.: Extrinsic geometry of convex surfaces. American Mathematical Society, Providence, R.I. (1973). Translated from the Russian by Israel Program for Scientific Translations, Translations of Mathematical Monographs, Vol. 35
21. Rauch, J., Taylor, B.A.: The Dirichlet problem for the multidimensional Monge-Ampère equation. Rocky Mountain J. Math. **7**(2), 345–364 (1977)