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On standard finite difference discretizations of the elliptic Monge-Ampère equation

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Abstract Given an orthogonal lattice with mesh length h on a bounded two-dimensional convex domain Ω, we propose to approximate the Aleksandrov solution of the Monge-Ampère equation by regularizing the data and discretizing the equation in a subdomain using the standard finite difference method. The Dirichlet data is used to approximate the solution in the remaining part of the domain. We prove the uniform convergence on compact subsets of the solution of the discrete problems to an approximate problem on the subdomain. The result explains the behavior of methods based on the standard finite difference method and designed to numerically converge to non-smooth solutions.

Keywords Monge-Ampère · standard finite difference · Aleksandrov solution · approximation by smooth functions

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1 Introduction

Let Ω be a bounded convex domain of $\mathbb{R}^2$ and let $g \in C(\partial\Omega)$, $f \in C(\Omega)$ with $0 < c_0 \leq f \leq c_1$ for constants $c_0, c_1 \in \mathbb{R}$. We assume that $g \in C(\partial\Omega)$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. We are interested in the finite difference approximation of the Aleksandrov solution of the Monge-Ampère equation

$$\det D^2 u = f \,\text{in}\, \Omega, u = g \,\text{on}\, \partial\Omega. \tag{1}$$

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To a convex function u, one associates a measure $M[u]$ and (1) is said to have an Aleksandrov solution if the density of $M[u]$ with respect to the Lebesgue measure is f. If $u \in C^2(\Omega)$, $M[u]$ is a measure with density $\det D^2 u$, where $D^2 u = \bigg((\partial^2 u)/(\partial x_i \partial x_j) \bigg)_{i,j=1,2}$ is the Hessian of u. There are several equivalent definitions of the Monge-Ampère measure in the general case and the simplest approach is to use an analytic definition based on approximation by smooth functions. See section 3.1 and [32] for the equivalent definitions.

We propose to approximate the Aleksandrov solution of (1) by regularizing the data and discretizing the equation in a subdomain. We use a standard finite difference method in the sense that the scheme may not satisfy a discrete maximum principle. The Dirichlet data is used to approximate the solution in the remaining part of the domain. We prove the uniform convergence on compact subsets of the solution of the discrete problems to an approximate problem on the subdomain.

We introduce a compatible discretization in the sense that the divergence of the transpose of the cofactor matrix of the discrete Hessian vanishes, a feature of the continuous problem. Existence, convergence rate and local uniqueness of a solution can be proven using Schauder estimates as for the central finite difference discretization [2]. Existence of a solution can be combined with the strict contraction approach in [4] to give a convergence result for a time marching method but in a ball of size $\mathrm{O}(h^3)$ in a H^1 like norm. However the compatible discretization is only first order accurate and thus we do not give details about that approach in this paper. The time marching method appears faster than Newton's method *for smooth solutions* and on fine meshes. In some cases 3 times faster. It is shown to be numerically robust for non smooth solutions of the Monge-Ampère equation with right hand side absolutely continuous with respect to the Lebesgue measure. The compatible discretization could be more efficient than the central finite difference discretization, c.f. Table 2.

1.1 Methodology for non smooth solutions

In [5], in an effort to get insight into why standard finite elements exhibit numerical convergence to viscosity solutions, we regularized the exact solution and study the behaviour of the approximation of the resulting problem as the mesh size tends to 0. The approach does not give any insight for the finite difference method. Here we use a different approach, also used in [3], which gives better results.

We regularize the data by considering functions $f_m, g_m \in C^\infty(\overline{\Omega})$ such that $0 < c_2 \leq f_m \leq c_3$, f_m converges uniformly to f on $\overline{\Omega}$ and g_m converges uniformly to $\tilde{g}$ on $\overline{\Omega}$. See [5] for an example. The second key idea of this paper is to consider a sequence of smooth uniformly convex subdomains Ω_s which converges to Ω [8].

We consider in this paper "interior" discretizations. By this, we mean that we prove convergence of the discretization in an interior domain. Values at mesh points closest to the boundary are approximated using the boundary values. Let $\delta > 0$ be a small parameter. We will need a theoretical computational domain $\widetilde{\Omega}$ chosen as a subdomain of Ω. We require that

$$\widetilde{\Omega} \subset \Omega_s, \text{ for all } s.$$

It is known, c.f. [3] or [34, Proposition 2.4], that the Aleksandrov solution of

$$\det D^2 u_m = f_m \text{ in } \Omega,\, u_m = g_m \text{ on } \partial\Omega, \tag{2}$$

converges uniformly on compact subsets of Ω to the Aleksandrov solution u of (1).

We choose $\tilde{m}$ such that $|f(x) - f_{\tilde{m}}(x)| < \delta$, $|g(x) - g_{\tilde{m}}(x)| < \delta$ and $|u(x) - u_{\tilde{m}}(x)| < \delta$ for all $x \in \overline{\Omega}$.

We show in this paper that given a mesh on Ω, an "interior" discretization (c.f. (9) below) of the problem

$$\det D^2 u_{\tilde{m}s} = f_{\tilde{m}} \text{ in } \widetilde{\Omega}, u_{\tilde{m}s} = u_{\tilde{m}} \text{ on } \partial\widetilde{\Omega}, \tag{3}$$

has a unique local solution $u_{\tilde{m}s,h}$ which is a discrete convex function, c.f. Definition 2. We discretize each component of the Hessian using a scheme which is at least first order accurate. We show that the solution u_h of the resulting discrete problem, is the limit of a subsequence in s of $u_{\tilde{m}s,h}$ where $u_{\tilde{m}s,h}$ is the finite difference approximation of the solution $u_{\tilde{m}s}$ of (3). We prove that u_h converges uniformly on compact subsets of $\widetilde{\Omega}$ to the solution $\tilde{u}$ of

$$\det D^2 \tilde{u} = f_{\tilde{m}} \text{ in } \widetilde{\Omega},\, \tilde{u} = u_{\tilde{m}} \text{ on } \partial\widetilde{\Omega}. \tag{4}$$

The solution u of (1) can then be approximated within a prescribed accuracy by first choosing $\tilde{m}$ and then h sufficiently small. We emphasize that the solution $\tilde{u}$ of (4) is not necessarily smooth.

A technical aspect of the proof is that we use *interior* second order derivative estimates of the solution u_{ms} as the latter may blow up on the boundary $\partial\Omega$ if the domain is not strictly convex. As a consequence of the interior Schauder estimates, we obtain stability on compact subsets of $\widetilde{\Omega}$ of the discretization. This is one of the main contributions of the paper and is treated in section 3.2.

The lack of a maximum principle for the discretizations analyzed in this paper is related to the difficulty of proving stability of the discretization for smooth solutions without assuming a bound on a high order norm of the solution. It is for that reason that we introduced the theoretical computational domain $\widetilde{\Omega}$ and fix the parameter $\tilde{m}$ in the regularization of the data.

For simplicity, the dependence of $\tilde{u}$ on $\tilde{m}$ is not indicated. By unicity of the Aleksandrov solution u_m of (2), we have $\tilde{u} = u_{\tilde{m}}$ in $\widetilde{\Omega}$ and hence as $\widetilde{\Omega} \to \Omega$, $u_{\tilde{m}}|_{\partial\widetilde{\Omega}} \to g|_{\partial\Omega}$. Thus, from a practical point of view, for the implementation, we see that one can take $\widetilde{\Omega} = \Omega$, $f_m = f$ with $u_h = g$ on $\partial\Omega$. It is in that sense that the results of this paper explains the behavior of methods based on the standard finite difference method and designed to numerically converge to non-smooth solutions.

1.2 Significance of the results in relation with other work

A proven convergence proof for Aleksandrov solutions was given for the two dimensional problem for the discretization proposed in [31]. The approach through the so-called viscosity solutions was considered in [21] in the context of monotone finite difference schemes. Existence of a solution to the discrete problem for the monotone schemes is still an open problem. We refer to [6] where the Lipschitz continuity of the scheme is reduced to the Lipschitz continuity of a multilinear map. But the latter is only Lipschitz continuous on a bounded set. And it is not clear how to prove that a fixed point mapping maps the bounded set into itself when the solution of (1) is not smooth. Thus the usual argument of existence of a solution to degenerate elliptic schemes based on [30, Theorem 7] does not apply to the Monge-Ampère equation. It follows from the general approach taken in this paper that the monotone schemes introduced in [21,22], and with convergence rate for smooth solutions given in [6], are solvable in an interior domain, and converge to both viscosity solutions and Aleksandrov solutions in the sense described above. For right hand sides which approximate a combination of Dirac masses, a very good initial guess is necessary for these methods.

The distinguished feature of the methods discussed in this paper, like the ones discussed in [7], is to preserve weakly convexity in the iterations, c.f. Remark 1. In the iterations, the positivity of the discrete Laplacian is preserved. This feature allows the processes to avoid spurious solutions. Thus the results of this paper do not contradict the observations made in [19, Section 1].

Our result is important for optimal transportation problems where one has to extend the data, resulting in a discontinuous right hand side f. In that case the continuous viscosity solution approach is no longer valid.

This paper provides a blueprint which can be used to analyze the discretizations proposed in [16,29]. The main task is to understand how these schemes perform for smooth solutions. The convergence of the discretization for non smooth solutions then follows from the general approach taken in this paper. We note that the results of this paper extend immediately to the central finite difference discretization [7,2].

A standard finite difference discretization of the Dirichlet problem for the Monge-Ampère equation was introduced in [17]. Finite element discretizations have also been proposed, e.g. [23,9,20,10,28,15,11].

1.3 Organization of the paper

We organize the paper as follows. In the next section we introduce our compatible discretization of the Monge-Ampère equation. In section 3 we recall key results on the Aleksandrov theory of the Monge-Ampère equation and give a general framework of convergence of standard discretizations to the Aleksandrov solution. The last section is devoted to a discussion of numerical results.

2 Standard finite difference discretizations of the Monge-Ampère equation

We recall that Ω is a bounded convex domain of $\mathbb{R}^2$. For $0 < h < 1$, we define

$$\mathbb{Z}_h = \{x = (x_1, x_2)^T \in \mathbb{R}^2 : x_i/h \in \mathbb{Z}\}, \Omega_h = \overline{\Omega} \cap \mathbb{Z}_h.$$

Let $\mathcal{M}(\Omega_h)$ denote the space of grid functions, i.e. mappings from Ω_h to $\mathbb{R}$. We denote by $e_i, i = 1, 2$ the i-th unit vector of $\mathbb{R}^2$ and consider first order difference operators defined on $\mathbb{Z}_h$ by

$$\partial^i_+ v_h(x) \coloneqq \frac{v_h(x + he_i) - v_h(x)}{h}, \qquad \partial^i_- v_h(x) \coloneqq \frac{v_h(x) - v_h(x - he_i)}{h}.$$

We have for $x \in \mathbb{Z}_h$

$$\partial^j_- \partial^i_+ v_h(x) = \frac{v_h(x + he_i) - v_h(x) - v_h(x + he_i - he_j) + v_h(x - he_j)}{h^2}. \quad (5)$$

We will also need the central second order accurate first order operator defined for $i = 1, 2$ by

$$\partial^i_h v_h(x) \coloneqq \frac{v_h(x + he_i) - v_h(x - he_i)}{2h}.$$

We use the notation $A = (a_{ij})_{i,j=1,2}$ to denote the matrix A with entries a_{ij}. Several discrete analogues of the Hessian $D^2 v$ of a C^2 function v can be defined for $x \in \mathbb{Z}_h$ and a grid function v_h. One possibility is to define the discrete Hessian as the non symmetric matrix field $\mathcal{H}_d(v_h)$ with components

$$(\mathcal{H}_d(v_h)(x))_{ij} = \partial^j_- \partial^i_+ v_h(x), i, j = 1, 2.$$

Definition 1 A 2×2 matrix A is said to be positive definite if and only if $z^T A z > 0$ for $z \in \mathbb{R}^2, z \neq 0$. The matrix A is said to be positive if and only if $z^T A z \geq 0$ for $z \in \mathbb{R}^2$.

Decomposing a matrix A into its symmetric and skew symmetric part, i.e. $A = (A + A^T)/2 + (A - A^T)/2$, one concludes that A is (positive) definite if and only if its symmetric part is (positive) definite. We will use the notation $\operatorname{sym} A$ to denote the symmetric part of A.

Another discretization of the Hessian matrix which has been used in previous work [7,27,14,13] is to consider for a grid function v_h, the matrix field $\overline{\mathcal{H}}_d(v_h)$ with components

$$(\overline{\mathcal{H}}_d(v_h)(x))_{ii} = \partial^i_+ \partial^i_- v_h(x), i, j = 1, 2$$
$$(\overline{\mathcal{H}}_d(v_h)(x))_{ij} = \partial^i_h \partial^j_h v_h(x), i, j = 1, 2, i \neq j.$$

We denote by Ω_h^0 the subset of Ω_h consisting of grid points x for which $x \pm he_i \pm he_j \in \overline{\Omega}$ for $i,j=1,2$ and put $\partial \Omega_h = \Omega_h \setminus \Omega_h^0$.

For the study of the convergence of numerical methods for non smooth solutions, we will consider the set of interior mesh points

$$\Omega_h^{00} = \{\, x \in \widetilde{\Omega} \cap \mathbb{Z}_h, x \pm 2he_i \pm 2he_j \in \overline{\Omega}, \text{ for } i,j=1,2 \,\},$$

and define $\partial \Omega_h^0 = (\overline{\widetilde{\Omega}} \cap \mathbb{Z}_h) \setminus \Omega_h^{00}$.

The restriction map is defined as a mapping

$$r_h : C(\Omega) \to \mathcal{M}(\Omega_h), r_h(v)(x) = v(x), x \in \Omega_h,$$

and is extended canonically to vector fields and matrix fields. The restriction to a subset of $\overline{\Omega}$ is defined analogously.

For a vector valued grid function v_h with components $v_{h,i}, i=1,2$, the divergence of v_h is defined as the grid function $\operatorname{div}_h v_h = \sum_{i=1}^2 \partial_-^i v_{h,i}$. The operator div_h is extended to matrix fields by taking the divergence of each row. We define two discrete versions of the gradient: $D_h v_h$ and $\overline{D}_h v_h$ as:

$$D_h v_h := (\partial_+^i v_h)_{i=1,2}, \qquad \overline{D}_h v_h := (\partial_-^i v_h)_{i=1,2}.$$

If $v_h = (v_{h,i})_{i=1,2}$ is a vector field, we define $\overline{D}_h v_h$ as the matrix field obtained by applying $\overline{D}_h$ to each row, i.e. $\overline{D}_h v_h = (\partial_-^j v_{h,i})_{i,j=1,2}$. Thus for a scalar field v_h $\overline{D}_h D_h v_h = \mathcal{H}_d(v_h)$. The discrete Laplacian Δ_h is defined as $\Delta_h v_h := \sum_{i=1}^2 \partial_+^i \partial_-^i v_h$. With the above definitions, we have $\operatorname{div}_h D_h v_h = \Delta_h v_h$.

Definition 2 A mesh function v_h is said to be discrete convex if $\mathcal{H}_d(v_h)(x)$ is a positive matrix for all $x \in \Omega_h^0$. The function v_h is said to be discrete strictly convex if $\mathcal{H}_d(v_h)(x)$ is a positive definite matrix for all $x \in \Omega_h^0$.

We recall that the cofactor matrix $\operatorname{cof} A$ of the matrix A is defined by $(\operatorname{cof} A)_{ij} = (-1)^{i+j} \det(A)_i^j$ where $\det(A)_i^j$ is the determinant of the matrix obtained from A by deleting the ith row and the jth column.

We are interested in the following discretization of (1) which is a standard compatible discretization in the sense that the divergence of the transpose of the cofactor matrix of the discrete Hessian vanishes

$$\frac{1}{2} \operatorname{div}_h [(\operatorname{cof} \mathcal{H}_d u_h) D_h u_h] = r_h(f) \text{ in } \Omega_h^0, u_h = r_h(g) \text{ on } \partial \Omega_h. \tag{6}$$

The discrete analogue of the maximum norm is given by

$$|v_h|_{0,\infty,h} = \max\{\, |v_h(x)|, x \in \Omega_h^0 \,\}. \tag{7}$$

We have under smoothness assumptions of the solution u of (1)

Proposition 1 *Problem* (6) *has a unique local solution* u_h *with* $\lambda_1(\mathcal{H}_d(u_h)) \geq c > 0$ *for a constant* c *independent of* h *and*

$$|u_h - r_h u|_{0,\infty,h} \leq Ch,$$

with a constant C *which can be taken as a multiple of* $||u||_{C^4(\overline{\Omega})}$. *Thus* u_h *converges uniformly on* Ω *to the unique smooth convex solution of* (1).

The above result can be proven using the continuity of the eigenvalues of the Hessian as a function of its entries and Schauder estimates as for the central finite difference discretization [2]. That can then be combined with the strict contraction approach in [4] to give a convergence result for the time marching method

$$\begin{aligned} -\nu \Delta_h u_h^{k+1} &= -\nu \Delta_h u_h^k + \frac{1}{2} \operatorname{div}_h [(\operatorname{cof} \mathcal{H}_d u_h^k) D_h u_h^k] - r_h(f) \operatorname{in} \Omega_h^0 \\ u_h^{k+1} &= r_h(g) \operatorname{on} \partial\Omega_h, \end{aligned} \tag{8}$$

for $\nu > 0$ sufficiently large and an initial guess u_h^0 in a ball of size $\mathrm{O}(h^3)$ in a H^1 like norm.

Remark 1 If one takes ν large in (8), one gets that the left hand side is negative, i.e. discrete subharmonicity is preserved in the iterations.

For the situation where (1) does not have a smooth solution, we consider the related problem

$$\frac{1}{2} \operatorname{div}_h [(\operatorname{cof} \mathcal{H}_d u_h) D_h u_h] = r_h(f_{\tilde{m}}) \operatorname{in} \Omega_h^{00}, u_h = r_h(u_{\tilde{m}}) \operatorname{on} \partial\Omega_h^0, \tag{9}$$

with corresponding time marching method

$$\begin{aligned} -\nu \Delta_h u_h^{k+1} &= -\nu \Delta_h u_h^k + \frac{1}{2} \operatorname{div}_h [(\operatorname{cof} \mathcal{H}_d u_h^k) D_h u_h^k] - r_h(f_{\tilde{m}}) \operatorname{in} \Omega_h^{00} \\ u_h^{k+1} &= r_h(u_{\tilde{m}}) \operatorname{on} \partial\Omega_h^0. \end{aligned}$$

We recall that the parameter $\tilde{m}$ was defined in section 1.1. Intuitively Problem (9) discretizes the Monge-Ampère equation in the interior of the domain where the non smooth solution can be approximated by smooth functions which solve related Monge-Ampère equations. It is clear that since (9) is very close to (6), and with the choice of the small parameter δ introduced in section 1.1, numerical experiments with the latter would indicate convergence for non smooth solutions.

The following lemma is essential to our methodology

Lemma 1 *A sequence of (discrete) convex functions which is locally uniformly bounded has a subsequence which converges uniformly on compact subsets to a (discrete) convex function.*

Proof We consider separately the cases of a sequence u_m of convex functions, a sequence $(u_{mh})_m$ of discrete convex functions and a sequence u_{h_l} of discrete convex functions.

A sequence u_m of convex functions is locally equicontinuous by [24, Lemma 3.2.1], c.f. [3] for details. If the sequence is also locally uniformly bounded, the result follows from the Arzela-Ascoli theorem [33, p. 179].

If we consider a sequence $(u_{mh})_m$ of discrete convex functions, for fixed h the number of grid points is finite and the result follows from the Bolzano-Weierstrass theorem.

If the sequence u_{h_l} is a sequence of discrete convex mesh functions in the sense that $\overline{\mathcal{H}}_d(u_{h_l})(x)$ is a positive matrix for all $x \in \Omega_h^0$, the result is given by [1, Corollary 4.8] and the Arzela-Ascoli theorem (which requires only local uniform boundedness). Since $\mathcal{H}_d(u_{h_l})(x)$ and $\overline{\mathcal{H}}_d(u_{h_l})(x)$ have the same diagonal elements, the discrete analogue of local equicontinuity [1, (2.2) and p. 22] also holds when one requires that $\mathcal{H}_d(u_{h_l})(x)$ is a positive matrix for all $x \in \Omega_h^0$, that is the result also holds in that case.

We make the usual abuse of notation of denoting by C a generic constant which does not depend on h.

3 General framework for convergence of standard discretizations to the Aleksandrov solution

3.1 The Aleksandrov solution

Let $K(\Omega)$ denote the cone of convex functions on Ω and let us denote by $B(\Omega)$ the space of Borel measures on Ω. We define the mapping

$$M : C^2(\Omega) \cap K(\Omega) \to B(\Omega), M[v](B) = \int_B \det D^2 v(x)\, dx,$$

where B is a Borel set.

The topology on $K(\Omega)$ is the topology of compact convergence, i.e. for $v_m, v \in K(\Omega)$, v_m converges to v if and only if v_m converges to v uniformly on compact subsets of Ω. The topology on $B(\Omega)$ is induced by the weak convergence of measures.

Definition 3 A sequence μ_m of Borel measures converges weakly to a Borel measure μ if and only if

$$\int_\Omega p(x)\, d\mu_m \to \int_\Omega p(x)\, d\mu,$$

for every continuous function p with compact support in Ω.

If the measures μ_m have density a_m, and μ has density a, we have

Definition 4 Let $a_m, a \geq 0$. The sequence a_m converges weakly to a as measures if and only if

$$\int_{\Omega} a_m p \, dx \to \int_{\Omega} a p \, dx,$$

for all continuous functions p with compact support in Ω.

The mapping M extends uniquely to a continuous operator on $K(\Omega)$, [32, Proposition 3.1]. This notion of Monge-Ampère measure can be shown to be equivalent to the one used in [24,26]. The proof is given by [32, Proposition 3.4]. We have

Lemma 2 (Lemma 1.2.3 [24]) *Let v_m be a sequence of convex functions in Ω such that $v_m \to v$ uniformly on compact subsets of Ω. Then the associated Monge-Ampère measures $M[v_m]$ tend to $M[v]$ weakly.*

Definition 5 A convex function $u \in C(\overline{\Omega})$ is said to be an Aleksandrov solution of (1) if $u = g$ on $\partial\Omega$ and $M[u]$ has density f.

We have

Theorem 1 (Theorem 1.1 [26]) *Let Ω be a bounded convex domain of $\mathbb{R}^2$ and assume that g can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in Ω. Then if $f \in L^1(\Omega)$, (1) has a unique convex Aleksandrov solution in $C(\overline{\Omega})$ which assumes the boundary condition in the classical sense.*

3.2 Convergence of the discretization

Let Ω_s denote a sequence of smooth uniformly convex domains increasing to Ω, i.e. $\Omega_s \subset \Omega_{s+1} \subset \Omega$ and $d(\partial\Omega_s, \partial\Omega) \to 0$ as $s \to \infty$. Here $d(\partial\Omega_s, \partial\Omega)$ denotes the distance between $\partial\Omega_s$ and $\partial\Omega$. For the special case $\Omega = (0,1)^2$, a construction was done in [35]. A general construction follows from the approach in [8].

We recall that f_m and g_m are $C^{\infty}(\overline{\Omega})$ functions such that $0 < c_2 \leq f_m \leq c_3, f_m \to f$ and $g_m \to \tilde{g}$ uniformly on $\overline{\Omega}$. The sequences f_m and g_m can be constructed by a standard mollification.

Recall from section 1.1 that we choose $\tilde{m}$ such that $|u(x) - u_{\tilde{m}}(x)| < \delta$ for all $x \in \Omega$, where δ is a small parameter. And we are interested in convergence of the discretization to the solution $u_{\tilde{m}}$ of (4).

By [12], the problem (3) has a unique convex solution $u_{\tilde{m}s} \in C^{\infty}(\overline{\Omega}_s)$. As $s \to \infty$, the sequence $u_{\tilde{m}s}$ converges uniformly on compact subsets of $\widetilde{\Omega}$ to the unique convex solution $u_{\tilde{m}} \in C(\overline{\widetilde{\Omega}})$ of the problem (4) [3].

We have by the interior Schauder estimates, [18, Theorem 4] and [3] for details,

$$||u_{\tilde{m}s}||_{C^2(K)} \leq C_{\tilde{m}}, \tag{10}$$

where the constant $C_{\tilde{m}}$ depends on $\tilde{m}, c_2, \widetilde{\Omega}, d(K, \partial\Omega), f_{\tilde{m}}$ and $\max_{x\in\Omega} |u_{\tilde{m}s}(x)|$. Moreover, by a bootstrapping argument we have

$$||u_{\tilde{m}s}||_{C^4(K)} \leq C_{\tilde{m}}, \tag{11}$$

as well. Let us use the notation $M_h[v_h]$ for the discrete Monge-Ampère operator applied to the grid function v_h, i.e.

$$M_h[v_h] = \frac{1}{2} \operatorname{div}_h[(\operatorname{cof} \mathcal{H}_d v_h) D_h v_h]. \tag{12}$$

We can now prove the main result of this paper

Theorem 2 *The problem* (9) *has a unique local discrete convex solution* u_h *which converges uniformly on compact subsets of* $\widetilde{\Omega}$ *to the unique convex solution* $\tilde{u}$ *of* (4) *as* $h \to 0$.

Proof Recall that

$$\Omega_h^{00} \subset \widetilde{\Omega} \cap \mathbb{Z}_h \subset \Omega_s \text{ and } \partial\Omega_h^0 \subset \overline{\widetilde{\Omega}} \cap \mathbb{Z}_h \subset \Omega_s.$$

Part 1: Existence of a discrete convex solution u_h

By Proposition 1, applied to the problem (3), there exists a unique local solution $u_{\tilde{m}s,h}$ to the problem

$$M_h[u_{\tilde{m}s,h}] = r_h(f_{\tilde{m}}) \,\text{in}\, \Omega_h^{00}, u_{\tilde{m}s,h} = r_h(u_{\tilde{m}}) \,\text{on}\, \partial\Omega_h^0. \tag{13}$$

For fixed h, the number of grid points is finite. Thus by Lemma 1, there exist a subsequence s_q such that $u_{\tilde{m}s_q,h}$ converges pointwise (and hence uniformly on compact subsets of Ω_h^{00}) to a mesh function u_h.

By construction $\partial\Omega_h^0 \subset \Omega_s$ and hence for $x \in \partial\Omega_h^0$, $u_h(x) = r_h(u_{\tilde{m}})(x)$. By taking pointwise limits in (13), we get that u_h solves (9).

Since $f_m \geq c_2 > 0$, we have $\lambda_1(\mathcal{H}_d u_{\tilde{m}s,h}(x)) \geq c_4 > 0$ for all $x \in \Omega_h^{00}$ for a constant c_4 independent of h and for s sufficiently large, as a consequence of the approach taken in [2], that. But $\lambda_1(\mathcal{H}_d u_{\tilde{m}s,h}(x))$ is the solution of a polynomial equation with coefficients which are combinations of entries of $(\mathcal{H}_d u_{\tilde{m}s,h}(x))_{i,j=1,2}$. By continuity of the roots of a polynomial as a function of its coefficients [25], taking a limit as $s_q \to \infty$, we obtain that $\lambda_1(\mathcal{H}_d u_h(x)) \geq 0$ for all $x \in \Omega_h^{00}$. That is, u_h is also discrete convex. Since $r_h(f) \geq c_0 > 0$, u_h is discrete strictly convex.

For the local uniqueness of the discrete solution u_h, we note that the techniques for proving local uniqueness of the discrete solution when (1) has a strictly convex smooth solution, [2,4], also applies in the case of a discrete strictly convex solution. We conclude that $u_{\tilde{m}s,h}$ converges uniformly on compact subsets of Ω_h^{00} to u_h as $s \to \infty$.

Part 2: Uniform convergence on compact subsets of Ω of a subsequence u_{h_l} to a convex function $v \in C(\widetilde{\Omega})$.

This is a direct consequence of the error estimates of Proposition 1, the interior Schauder estimate $||u_{\tilde{m}s}||_{C^4(\widetilde{\Omega})} \leq C_{\tilde{m}}$ and Lemma 1. The continuity of v on $\widetilde{\Omega}$ follows from its convexity, Proposition 1 and (11) which imply that $u_{\tilde{m}s,h}$ and u_h, hence v are locally finite.

Part 3: The continuous convex function v is equal to the Aleksandrov solution $\tilde{u}$ of (4).

Let K be a compact subset of $\widetilde{\Omega}$ and let $\epsilon > 0$. Since u_{h_l} converges uniformly on K to v, $\exists l_0$ such that $\forall l \geq l_0$ $|u_{h_l}(x) - v(x)| < \epsilon/6$ for all $x \in K \cap \Omega_h^{00}$.

By definition u_{h_l} is the uniform limit on $K \cap \Omega_h^{00}$ of $u_{\tilde{m}s,h_l}$ as $s \to \infty$. Thus $\exists s_l$ such that $\forall s \geq s_l$ $|u_{\tilde{m}s,h_l}(x) - u_{h_l}(x)| < \epsilon/6$ for all $x \in K \cap \Omega_h^{00}$.

By Proposition 1 and (11) we have on K $|u_{\tilde{m}s,h_l}(x) - u_{\tilde{m}s}(x)| \leq C h_l$ for all $x \in K \cap \Omega_h^{00}$. We recall that the constant C is independent of s but depends on $\tilde{m}$ and $\widetilde{\Omega}$.

By the uniform convergence of $u_{\tilde{m}s}$ to $u_{\tilde{m}}$, we may assume that $|u_{\tilde{m}}(x) - u_{\tilde{m}s}(x)| < \epsilon/6$ for all $x \in K$.

We conclude that for $\forall l \geq l_0$, $\exists s_l$ such that $\forall s \geq s_l$ $|u_{\tilde{m}}(x) - v(x)| < \epsilon/2 + C h_l$ for all $x \in K \cap \Omega_h^{00}$.

For $x \in K$, if necessary by choosing a sequence x_{h_l} such that $x_{h_l} \to x$ as $l \to \infty$, we get for all $\epsilon > 0$ $|u_{\tilde{m}}(x) - v(x)| < \epsilon$. We conclude that $\tilde{u} = u_{\tilde{m}} = v$ on K. We have by construction $\tilde{u} = v$ on $\partial\widetilde{\Omega}$. This proves that $\tilde{u} = v$.

Part 4: Finishing up.

By the unicity of the solution $\tilde{u}$ of (4) we conclude that u_h converges uniformly on compact subsets of $\widetilde{\Omega}$ to $\tilde{u}$.

4 Numerical results

The computational domain is the unit square $[0,1]^2$. The initial guess for the iterations was taken as the finite difference approximation of the solution of $\Delta u = 2\sqrt{f}$ with boundary condition $u = g$. Numerical errors are in the maximum norm.

The scheme (6), when solved with the time marching method (8), performs well for the standard tests for convex solutions of the Monge-Ampère equation. The results are given on Tables 1 (Test 1), 2 (Test 2) and Figures 1 (Test 3) and 2 (Test 4). We note the high accuracy for the non smooth solution of Table 2. For $h = 1/2^8$, it only took 805 seconds on a 2.5 GHz MacBook Pro. That's less time it took Newton's method to find the solution for a more regular problem, c.f. Table 3.

For smooth solutions, (8) appears to be faster than Newton's method on fine meshes, Table 3. We also give on Table 3 computational time for the iterative method

$$\Delta_h u_h^{k+1} = \sqrt{(\Delta_h u_h^k)^2 + 4(f - M_h[u_h^k])}, \tag{14}$$

where the operator M_h is defined by (12). The above iterative method is the analogue of a variant of an iterative method introduced in [7] where therein $M_h[v_h] = \det \overline{\mathcal{H}}_d(v_h)$. The iterative method (14), unlike the one proposed in

h						
$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$
$5.52\ 10^{-3}$	$5.19\ 10^{-3}$	$3.15\ 10^{-3}$	$1.73\ 10^{-3}$	$9.01\ 10^{-4}$	$4.60\ 10^{-4}$	$2.32\ 10^{-4}$

Table 1 Smooth solution $u(x,y) = e^{(x^2+y^2)/2}, g(x,y) = e^{(x^2+y^2)/2}$ and $f(x,y) = (1 + x^2 + y^2)e^{x^2+y^2}$ with the iterative method (8) and $\nu = 50$

h					
$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$
$3.94\ 10^{-3}$	$2.85\ 10^{-3}$	$1.67\ 10^{-3}$	$8.97\ 10^{-4}$	$4.65\ 10^{-4}$	$2.37\ 10^{-4}$

Table 2 Non smooth solution (not in $H^2(\Omega)$) $u(x,y) = -\sqrt{2-x^2-y^2}, g(x,y) = -\sqrt{2-x^2-y^2}$ and $f(x,y) = 2/(2-x^2-y^2)^2$ with the iterative method (8) and $\nu = 150$

	h						
	$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$
Newton	0.02	0.03	0.09	0.42	2.66	4.51	994
$\nu = 4$	0.16	0.14	0.34	1.38	0.67	3.10	1.52
Method (14)	0.03	0.05	0.15	0.63	1.06	1.56	7.12

Table 3 Computation times for Newton's method, the time marching method (8) and the iterative method (14) for $u(x,y) = e^{(x^2+y^2)/2}$

[7], may not be well defined for some initial guesses. Nethertheless it finds the solution to (6) given by (8) and Newton's method for Test 1. The variant

$$\Delta_h u_h^{k+1} = \sqrt{(\Delta_h u_h^k)^2 + 2(f - M_h[u_h^k])},$$

works for Test 3 but not for Test 4. It also converges to a solution different from the one found by (8) for Tests 1 and 2. The iterative method

$$\Delta_h u_h^{k+1}(x) = \bigg(2f(x) + (\partial_-^1 \partial_+^1 u_h^k(x))^2 + (\partial_-^2 \partial_+^2 u_h^k(x))^2 + (\partial_-^2 \partial_+^1 u_h^k(x))\partial_-^2 \partial_+^1 u_h^k(x - he_1) + (\partial_-^1 \partial_+^2 u_h^k(x))(\partial_-^1 \partial_+^2 u_h^k(x - he_2))\bigg)^{\frac{1}{2}},$$

modeled after the one introduced in [7] and suggested by how the compatible discretization treats the terms $(\partial^2 u/\partial x_1^2)(\partial^2 u/\partial x_2^2)$ and $\partial^2 u/(\partial x_1 \partial x_2)$, performs well for all our tests but returns the solution to a discretization different from the one studied in this paper. These numerical experiments suggest that the time marching approach is easier to generalize than the one proposed in [7]. Also, no convergence result is known for the latter.

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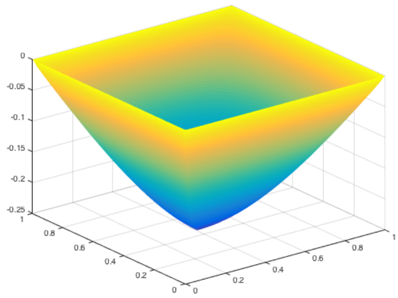

Fig. 1 No known exact solution, $f(x, y) = 1, g(x, y) = 0, h = 1/2^7, \nu = 500$ with the iterative method (8)

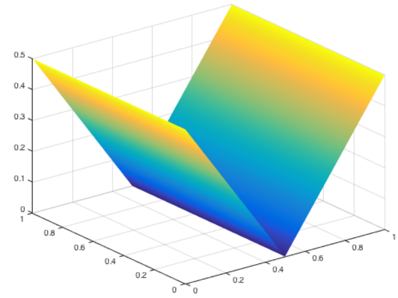

Fig. 2 $u(x, y) = |x - 1/2|$ with $g(x, y) = |x - 1/2|$ and $f(x, y) = 0$, $h = 1/2^2, \nu = 5$ with the iterative method (8)

	h						
	$1/2^2$	$1/2^3$	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$	$1/2^8$
Newton	0.08	0.03	0.14	0.63	4.62	69.54	*
$\nu = 150$	2.94	5.28	3.74	8.15	33.38	166.4	805.3
Method (14)	0.09	0.05	0.15	0.55	0.83	4.70	25.10

Table 4 Computation times for Newton's method, the time marching method (8) and the iterative method (14) for $u(x, y) = -\sqrt{2 - x^2 - y^2}$. * means Newton's method did not pick the solution of (6)

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