RECTANGULAR MIXED ELEMENTS FOR ELASTICITY WITH WEAKLY IMPOSED SYMMETRY CONDITION

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ABSTRACT. We present new rectangular mixed finite elements for linear elasticity. The approach is based on a modification of the Hellinger-Reissner functional in which the symmetry of the stress field is enforced weakly through the introduction of a Lagrange multiplier. The elements are analogues of the lowest order elements described in Arnold, Falk and Winther [Mixed finite element methods for linear elasticity with weakly imposed symmetry. Mathematics of Computation 76 (2007), pp. 1699–1723]. Piecewise constants are used to approximate the displacement and the rotation. The first order BDM elements are used to approximate each row of the stress field.

1. Introduction

The theory of elasticity is used to predict the response of a material to applied forces. The unknowns in the equations are the stress field, a symmetric matrix field which encodes the internal forces and the displacement, a vector field. For various reasons, mixed finite elements where one approximates both the stress and displacement are the methods of choice. One seeks the stress in the space of symmetric matrix fields with components square integrable and with divergence, taken row-wise, also square integrable. The displacement is sought in the space of square integrable vector fields. The pair forms a unique saddle point of the Hellinger-Reissner functional. It is very difficult to construct at the discrete level, finite element spaces which satisfy Brezzi's stability conditions. These conditions provide sufficient conditions for the stability of mixed finite element methods. Indeed for several decades before the work of Arnold and Winter [10, 11] the existence of such elements was an open problem. These elements have been extended to rectangular meshes in two dimension [3, 17], three dimension [13] and on tetrahedral meshes [5, 1]. Despite their relative complexity, mixed finite elements with symmetric stress fields are useful in certain situations [25]. If one desires simpler elements, one is forced to turn to nonconforming elements. Nonconformity can be introduced by weakening the symmetry condition or by weakening the requirement that the stress field is L^2 integrable. We refer to [12] for a review on nonconforming elements with symmetric stress fields and other approaches to linear elasticity.

Stable mixed finite elements with weakly imposed symmetry have been introduced in [2, 6, 26, 28, 27, 24, 7, 9, 15, 23, 22, 19], The purpose of this paper is to present elements with weakly imposed symmetry for rectangular meshes. Precisely, we will use piecewise constants to approximate the displacement and the rotation and 18 or

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12 dimensional spaces to approximate the stress field. The simplest older element on rectangular meshes in two dimensions is the one of [24] with 11 degrees of freedom for the stress, piecewise constants to approximate the displacement but a 4 dimensional space to approximate the rotation. The advantage of our element is that the rotation can be eliminated by static condensation. In three dimensions as well, our elements are simpler than Morley's elements.

The paper is organized as follows: after some preliminaries in the next section, we present our low order elements in two dimension and then in three dimension. We conclude with some remarks on higher order elements.

2. Preliminaries

Let Ω be a simply connected polygonal domain of $\mathbb{R}^n, n = 2, 3$, occupied by a linearly elastic body which is clamped on $\partial\Omega$. We denote as usual by $L^2(\Omega, \mathbb{R}^n)$ the space of square integrable vector fields with values in $\mathbb{R}^n$ and $H^k(K, X)$ the space of functions with domain $K \subset \mathbb{R}^n$, taking values in the finite dimensional space X, and with all derivatives of order at most k square integrable. We let $H(\text{div}, \Omega, X)$ be the space of square-integrable fields taking values in X and which have square integrable divergence. For our purposes, X will be either $\mathbb{M}$ the space of $n \times n$ matrices, $\mathbb{S}$ the space of $n \times n$ symmetric matrices, $\mathbb{R}^n$, or $\mathbb{R}$, and in the latter case, we simply write $H^k(X)$. The divergence operator is the usual divergence for vector fields which produces a matrix field when acting on a matrix field by taking the divergence of each row. We will also need the space $H(\text{curl}, \Omega, \mathbb{R}^n)$ of square-integrable fields with square integrable curl. We recall that in two dimension for a scalar function q, $\text{curl}(q) = (\partial_2 q, -\partial_1 q)$ and in three dimension

$$\text{curl}(q_1, q_2, q_3) = (\partial_2 q_3 - \partial_3 q_2, -\partial_1 q_3 + \partial_3 q_1, \partial_1 q_2 - \partial_2 q_1).$$

For a vector field in two dimension or a matrix field in three dimension, the curl operator produces a matrix field by taking the curl of each row. The norms in $H^k(K, X)$ and $H^k(K)$ are denoted respectively by $||\cdot||_{H^k}$ and $||\cdot||_k$. We use the usual notations of $\mathcal{P}_k(K, X)$ for the space of polynomials on K with values in X of total degree less than k and $\mathcal{P}_{k_1,k_2}(K, X)$ for the space of polynomials of degree at most k_1 in x and of degree at most k_2 in y. Similarly, $\mathcal{P}_{k_1,k_2,k_3}(K, X)$ denotes the space of polynomials of degree at most k_1 in x, of degree at most k_2 in y and of degree at most k_3 in z. We write $\mathcal{P}_k$, $\mathcal{P}_{k_1,k_2}$ and $\mathcal{P}_{k_1,k_2,k_3}$ respectively when $X = \mathbb{R}$.

The solution $(\sigma, u) \in H(\text{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^n)$ of the elasticity problem can be characterized as the unique critical point of the Hellinger-Reissner functional

$$\mathcal{J}(\sigma, v) = \int_\Omega \left(\frac{1}{2} A\tau + \text{div}\,\tau \cdot v - f \cdot v \right) dx.$$

The compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is given, bounded and symmetric positive definite uniformly with respect to $x \in \Omega$, and the body force f is also given. In the homogeneous and isotropic case,

$$A\sigma = \frac{1}{2\mu} \left(\sigma - \frac{\lambda}{2\mu + 2\lambda} \text{tr } (\sigma) I \right)$$

where I is the identity matrix and λ and μ are the positive Lame constants.

To treat both two and three dimensional problems in a unified framework, one possibility is to use finite element differential forms [8]. However, for $n = 2, 3$ a simple device will suffice. We define $\mathbb{P}$ to be $\mathbb{R}$ when $n = 2$ and $\mathbb{P} = \mathbb{R}^3$ for $n = 3$. Then we define $\operatorname{as}\tau = \tau_{12} - \tau_{21}$ for a 2×2 matrix and $\operatorname{as}\tau = (\tau_{32} - \tau_{23}, \tau_{13} - \tau_{31}, \tau_{21} - \tau_{12})'$ in three dimension. For a symmetric matrix field, $\operatorname{as}\tau = 0$. Next, we define $\mathbb{H}$ to be $\mathbb{R}^2$ when $n = 2$ and $\mathbb{H} = \mathbb{M}$ for $n = 3$. For the formulation with weakly imposed symmetry condition, a critical point of the extended functional

$$\mathcal{J}(\sigma, v) + \int_\Omega \eta \cdot \operatorname{as}\tau$$

is sought over $H(\operatorname{div}, \Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{P})$. The unique solution (σ, u, γ) satisfies

$$\begin{aligned} (A\sigma, \tau) + (\operatorname{div}\tau, u) + (\operatorname{as}\tau, \gamma) = 0, &\quad \tau \in H(\operatorname{div}, \Omega, \mathbb{M}), \\ (\operatorname{div}\sigma, v) = (f, v), &\quad v \in L^2(\Omega, \mathbb{R}^n), \\ (\operatorname{as}\sigma, q) = 0, &\quad q \in L^2(\Omega, \mathbb{P}). \end{aligned} \tag{2.1}$$

For the associated discrete system with finite element spaces $\Sigma_h \times V_h \times Q_h \subset H(\operatorname{div}, \Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{R}^n) \times L^2(\Omega, \mathbb{P})$, the symmetry condition will be enforced only weakly. The Brezzi's conditions for stability are

- There exists a positive constant c_1 independent of h such that $||\tau||_{H(\operatorname{div})} \leq c_1(A\tau, \tau)$, if $\tau \in \Sigma_h$, $(\operatorname{div}\tau, v) = 0$ for all $v \in V_h$ and $(\operatorname{as}\tau, q) = 0, \forall q \in Q_h$,
- There exists a positive constant c_2 independent of h such that $\forall\ (v, q) \in V_h \times Q_h, (v, q) \neq (0, 0), \exists\, \tau \in \Sigma_h, \tau \neq 0$ with $(\operatorname{div}\tau, v) + (\operatorname{as}\tau, q) \geq c_2 ||\tau||_{H(\operatorname{div})}(||v||_{L^2} + ||q||_{L^2})$.

To fulfill these conditions, we construct Σ_h, V_h and Q_h such that

1- $\operatorname{div}\Sigma_h \subset V_h$
2- Given $(v, q) \in V_h \times Q_h, (v, q) \neq (0, 0), \exists\, \tau \in \Sigma_h, \tau \neq 0$ such that

$$||\tau||_{H(\operatorname{div})} \leq C(||v||_{L^2} + ||q||_{L^2}), \tag{2.2}$$

and $\operatorname{div}\tau = v$, $P_{Q_h}\operatorname{as}\tau = q$, where P_{Q_h} is the L^2 projection operator.

The first Brezzi condition follows from the condition $\operatorname{div}\Sigma_h \subset V_h$. It is easy to see that the second follows from condition (2) above. To construct elements which satisfy (1) and (2), we follow the constructive approach of Arnold, Falk and Winther, [7, 9], using discrete versions of the de Rham sequence. In addition to the spaces Σ_h, V_h and Q_h, we also construct finite element spaces $R_h \subset H(\operatorname{div}, \Omega, \mathbb{H})$ and $\Theta_h \subset H(\operatorname{curl}, \Omega, \mathbb{H})$ in such a way that the following diagrams commute:

$$\begin{array}{ccccc} H(\operatorname{div}, \Omega, \mathbb{H}) & \xrightarrow{\operatorname{div}} & L^2(\Omega, \mathbb{P}) & \longrightarrow & 0 \\ \downarrow \Pi_{R_h} & & \downarrow \Pi_{Q_h} & & \\ R_h & \xrightarrow{\operatorname{div}} & Q_h & \longrightarrow & 0, \end{array}$$

$$
\begin{array}{ccccccc}
H(\mathrm{curl},\Omega,\mathbb{H}) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},\Omega,\mathbb{M}) & \xrightarrow{\mathrm{div}} & L^2(\Omega,\mathbb{R}^n) & \longrightarrow & 0 \\
\Big\downarrow \Pi_{\Theta_h} & & \Big\downarrow \Pi_{\Sigma_h} & & \Big\downarrow \Pi_{V_h} & & \\
\Theta_h & \xrightarrow{\mathrm{curl}} & \Sigma_h & \xrightarrow{\mathrm{div}} & V_h & \longrightarrow & 0.
\end{array}
$$

We note that the commutativity of the far left side of the diagram above will not be used. For a finite dimensional space X_h, Π_{X_h} is a bounded projection operator. We recall that

$$\Pi_{X_h} v = v, \ \forall \ v \in X_h. \tag{2.3}$$

Next, we define an operator $S : C^\infty(\Omega, \mathbb{H}) \to C^\infty(\Omega, \mathbb{H})$ which connects the two diagrams above. In two dimension, S is simply the identity operator, while in three dimension, for $q = (q_{ij})_{i,j=1,\dots,3}$, we define

$$S(q) = \begin{pmatrix} q_{22}+q_{33} & -q_{21} & -q_{31} \\ -q_{12} & q_{11}+q_{33} & -q_{32} \\ -q_{13} & -q_{23} & q_{11}+q_{22} \end{pmatrix}. \tag{2.4}$$

In that case, S is also invertible with $S(q) = \mathrm{tr}(q)I - q^T, S^{-1}(q) = 1/2\,\mathrm{tr}(q)I - q^T$, [15], where q^T denotes the transpose of q, I is the 3×3 identity matrix and $\mathrm{tr}(q)$ denotes the trace of q. The following fundamental relation holds in both dimension:

$$\mathrm{as}\,\mathrm{curl}\, q = -\,\mathrm{div}\, S(q). \tag{2.5}$$

We summarize the elements of the constructive approach of [7, 9] in the following theorem, the proof of which is reproduced below for convenience.

Theorem 2.1. *Under the commutativity assumptions*

$$\Pi_{Q_h} \mathrm{div}\, q = \mathrm{div}\, \Pi_{R_h} q, \ \forall q \in C^\infty(\Omega, \mathbb{H}), \tag{2.6}$$

$$\mathrm{div}\, \Pi_{\Sigma_h} \sigma = \Pi_{V_h} \mathrm{div}\, \sigma, \ \forall\, \sigma \in C^\infty(\Omega, \mathbb{M}), \tag{2.7}$$

and

$$\Pi_{R_h} S \Pi_{\Theta_h} S^{-1} = \Pi_{R_h}, \tag{2.8}$$

$$||\Pi_{\Sigma_h} u||_{L^2} \leq c||u||_{H^1}, \ \ \forall \tau \in H^1(\Omega, \mathbb{M}), \tag{2.9}$$

$$||\,\mathrm{curl}\, \Pi_{\Theta_h} \rho||_{L^2} \leq c||\rho||_{H^1}, \ \ \forall \rho \in H^1(\Omega, \mathbb{H}). \tag{2.10}$$

the second Brezzi condition holds.

Proof. By elliptic regularity, given $v \in V_h, \exists\, \eta \in H^1(\Omega, \mathbb{M})$ such that

$$\mathrm{div}\, \eta = v \quad \text{and} \quad ||\eta||_{H^1} \leq ||v||_{L^2}. \tag{2.11}$$

Given $q \in Q_h \subset L^2(\Omega, \mathbb{P})$, there exists $\phi \in H^1(\Omega, \mathbb{H})$ such that

$$\mathrm{div}\, \phi = q - \Pi_{Q_h} \mathrm{as}\, \Pi_{\Sigma_h} \eta \text{ and } ||\phi||_{H^1} \leq C||q - \Pi_{Q_h} \mathrm{as}\, \Pi_{\Sigma_h} \eta||_{L^2}. \tag{2.12}$$

We set $\tau = \Pi_{\Sigma_h} \eta + \mathrm{curl}\, \Pi_{\Theta_h} S^{-1} \phi$ and by (2.7) and (2.3) we have

$$\mathrm{div}\, \tau = \mathrm{div}\, \Pi_{\Sigma_h} \eta = \Pi_{V_h} \mathrm{div}\, \eta = \Pi_{V_h} v = v.$$

By (2.5) and (2.6) it follows that

$$\Pi_{Q_h} \mathrm{as}\, \mathrm{curl}\, q = \Pi_{Q_h} \mathrm{div}\, Sq = \mathrm{div}\, \Pi_{R_h} Sq,$$

We therefore have using (2.8), (2.6) and (2.3),

$$\begin{aligned}\Pi_{Q_h} \operatorname{as} \tau &= \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta + \Pi_{Q_h} \operatorname{as} \operatorname{curl} \Pi_{\Theta_h} S^{-1} \phi \\ &= \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta + \operatorname{div} \Pi_{R_h} S \Pi_{\Theta_h} S^{-1} \phi \\ &= \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta + \operatorname{div} \Pi_{R_h} \phi \\ &= \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta + \Pi_{Q_h} \operatorname{div} \phi \\ &= \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta + \Pi_{Q_h} q - \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta \\ &= q.\end{aligned}$$

It remains to prove the inequality (2.2). We have by (2.11) and (2.9)

$$||\Pi_{\Sigma_h} \eta||_{L^2} \leq C||\eta||_{H^1} \leq C||v||_{L^2},$$

and by (2.11), (2.3), (2.11), (2.9) and (2.12)

$$\begin{aligned}|| \operatorname{curl} \Pi_{\Theta_h} S^{-1} \phi||_{L^2} &\leq ||S^{-1}\phi||_{H^1} \leq C||\phi||_{H^1} \leq ||q - \Pi_{Q_h} \operatorname{as} \Pi_{\Sigma_h} \eta||_{L^2} \\ &\leq C(||q||_{L^2} + || \operatorname{as} \Pi_{\Sigma_h} \eta||_{L^2}) \leq C(||q||_{L^2} + ||\eta||_{H^1}) \\ &\leq C(||q||_{L^2} + ||v||_{L^2}).\end{aligned}$$

It follows that $||\tau||_{L^2} = ||\Pi_{\Sigma_h} \eta + \operatorname{curl} \Pi_{\Theta_h} \phi||_{L^2} \leq C(||q||_{L^2} + ||v||_{L^2})$. Since $\operatorname{div} \tau = v$, this proves the result. □

Let $\mathcal{T}_h$ denote a conforming partition of Ω into rectangles of diameter bounded by h, which is quasi-uniform in the sense that the aspect ratio of the rectangles is bounded by a fixed constant. Let $\hat{R} = [0,1]^n$ be the reference rectangle and let $F : \hat{R} \to R$ be an affine mapping onto R, $F(\hat{x}) = B\hat{x} + b$, with $b \in \mathbb{R}^n$ and B a $n \times n$ diagonal matrix. Our goal in the next section is to construct spaces Σ_h, V_h and Θ_h such that the conditions of Theorem (2.1) hold. If (σ, u, p) denotes the solution of problem (2.1) and $(\sigma_h, u_h, p_h) \in \Sigma_h \times V_h \times \Theta_h$ is the solution of the associated discrete system, the optimality condition

$$\begin{aligned}||\sigma - \sigma_h||_{H(\operatorname{div})} + ||u - u_h||_{L^2} + ||\gamma - \gamma_h||_{L^2} &\leq C \inf\nolimits_{\tau_h \in \Sigma_h, v_h \in V_h, \rho_h \in Q_h} \\ &(||\sigma - \tau_h||_{H(\operatorname{div})} + ||u - v_h||_{L^2} + ||\gamma - \rho_h||_{L^2}),\end{aligned} \tag{2.13}$$

holds.

As with [7, 5, 15], the following refined error estimates hold

$$\begin{aligned}||\sigma - \sigma_h||_{H(\operatorname{div})} + ||u_h - \Pi_{V_h} u||_{L^2} + ||\gamma - \gamma_h||_{L^2} &\leq C(||\sigma - \Pi_{\Sigma_h}\sigma||_{H(\operatorname{div})} + ||\gamma - \Pi_{Q_h}\gamma||_{L^2}), \\ ||u - u_h||_{L^2} &\leq C(||\sigma - \Pi_{\Sigma_h}\sigma||_{H(\operatorname{div})} + ||\gamma - \Pi_{Q_h}\gamma||_{L^2} + ||u - \Pi_{V_h}u||_{L^2}), \\ || \operatorname{div}(\sigma - \sigma_h)||_{L^2} &= || \operatorname{div} \sigma - \Pi_{V_h} \operatorname{div} \sigma||_{L^2}.\end{aligned}$$

3. Two dimensional elements

We recall the lowest order BDM element,

$$BDM_1(K) = \{\, q \,|\, q = p_1(x,y) + r \operatorname{curl}(x^2 y) + s \operatorname{curl}(x y^2), p_1 \in \mathcal{P}_1 \times \mathcal{P}_1 \,\}, \tag{3.1}$$

and an element $q \in BDM_1(K)$ is uniquely determined by the conditions $\int_e q \cdot n \, p_1 \, ds$, for each edge e of K, $\forall\, p_1 \in \mathcal{P}_1(e)$.

We choose $V_h = \mathcal{P}_0(\mathcal{T}_h)$, $Q_h = \mathcal{P}_0(\mathcal{T}_h)$, with degrees of freedom the value at an interior point in each element K and

$$\Sigma_K = \{\, \tau, \tau(x,y) \in \mathbb{M}, (\tau_{i1}, \tau_{i2}) \in BDM_1(K), i = 1, 2 \,\}.$$

A matrix field $\tau \in \Sigma_K$ is uniquely determined by the first two moments of τn on each edge, ($2 \times 2 \times 4 = 16$ degrees of freedom). The stress field space Σ_h is therefore the space of matrix fields which belong piecewise to Σ_K and have normal components which are continuous across mesh edges.

We will also need the serendipity finite element space S_h, defined on a single element K by

$$S_K = \mathcal{P}_2(K) + \operatorname{span}\{x^2 y, xy^2\},$$

and with degrees of freedom for $q \in S_K$

(1) the values of q at the vertices (4 degrees of freedom),
(2) the average of q on each edge (4 degrees of freedom).

It is not difficult to check that the sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} S_K \xrightarrow{\text{curl}} BDM_1(K) \xrightarrow{\text{div}} \mathcal{P}_0(K) \longrightarrow 0.$$

is exact. One checks that each space is mapped in the one that follows. Then one notes that the alternating sum of the dimensions is zero and that the polynomial de Rham sequence is exact.

We therefore define the space Θ_h as follows: on each element K, $\Theta_K = S_K \times S_K$ and the space Θ_h is the space of vector fields which belong piecewise to Θ_K and are continuous across mesh edges.

Finally we take for R_h the lowest order Raviart-Thomas element, i.e. $R_h = RT_0(\mathcal{T}_h)$. We recall that $RT_0(K) = \mathcal{P}_{1,0}(K) \times \mathcal{P}_{0,1}(K)$ with degrees of freedom the average of the normal component of $q \in RT_0(K)$ on each edge.

The projection operator Π_{Σ_h} is taken as the canonical interpolation operator and defined by

$$\int_e \Pi_{\Sigma_h}(\sigma) n \cdot q \, ds = \int_e \sigma n \cdot q \, ds, \qquad \text{for all edges } e \text{ and for all } q \in \mathcal{P}_1(e) \times \mathcal{P}_1(e).$$

Similarly we define Π_{R_h} by

$$\int_e \Pi_{R_h}(q) \cdot n \, ds = \int_e q \cdot n \, ds, \qquad \text{for all edges} \quad e.$$

It remains to define the interpolation operator Π_{Θ_h}. For this we first define $\Pi^0_K : H^1(K, \mathbb{R}^2) \to \Theta_K$ by

$$\Pi^0_K \psi(v) = 0 \quad \text{for each vertex } v \text{ of } K,$$

$$\int_e \Pi^0_K \psi(s) \, ds = \int_e \psi(s) \, ds \quad \text{for each edge } e \subset \partial K,$$

and $\Pi^0_h : H^1(\Omega, \mathbb{R}^2) \to \Theta_h$ by $(\Pi^0_h \tau)|_K = \Pi^0_K \tau$. Next, let L_h be a Clement interpolation operator [14, 18] which maps $L^2(\Omega, \mathbb{R})$ into

$$\{\, \theta_h \in C^0(\bar{\Omega}) \,|\, \theta_{h|K} \in \mathcal{P}_{1,1}, \forall K \in \mathcal{T}_h \,\},$$

and denote as well by L_h the corresponding operator which maps $L^2(\Omega,\mathbb{R}^2)$ into the subspace Θ_h of continuous vector fields whose components are piecewise in $\mathcal{P}_{1,1}$. We have

$$\|L_h\tau-\tau\|_j \le ch^{m-j}\|\tau\|_m,\quad 0\le j\le 1,\quad j\le m\le 2, \tag{3.2}$$

with c independent of h. We define our interpolation operator Π_{Θ_h} by

$$\Pi_{\Theta_h} = \Pi_h^0(I-L_h)+L_h. \tag{3.3}$$

Theorem 3.1. *For the triple* $(\Sigma_h, V_h, \Theta_h)$ *the conditions of Theorem* (2.1) *hold and we have the optimality condition* (2.13). *Moreover if* σ *and* u *are sufficiently smooth,*

$$||\sigma-\sigma_h||_{H(\mathrm{div})}+||u-u_h||_{L^2}+||\gamma-\gamma_h||_{L^2}\le C\,h||u||_3. \tag{3.4}$$

Proof. Let $q\in C^\infty(\Omega,\mathbb{R}^2)$. We have using the definition of Π_{R_h} and Green's theorem,

$$\int_\Omega \operatorname{div}\Pi_{R_h}q\,dx = \sum_K\int_K \operatorname{div}\Pi_{R_h}q\,dx = \sum_K\int_{\partial K}\Pi_{R_h}q\cdot n\,ds$$
$$= \sum_K\int_{\partial K} q\cdot n\,ds = \int_\Omega \operatorname{div} q,$$

which proves (2.6).

Next, let $\sigma\in C^\infty(\Omega,\mathbb{M})$. Again using the definition of Π_{Σ_h} and Green's theorem,

$$\int_\Omega \operatorname{div}\sigma-\operatorname{div}\Pi_{\Sigma_h}\sigma\,dx = \sum_K\int_K \operatorname{div}(\sigma-\Pi_{\Sigma_h}\sigma)\,dx = \sum_K\int_{\partial K}(\sigma-\Pi_{\Sigma_h}\sigma)n\,ds = 0,$$

which proves (2.7).

For $q\in C^\infty(\Omega,\mathbb{R}^2)$, put $u=\Pi_h^0 q$. We have using the definition of Π_h^0

$$\int_e (u-q)\cdot n\,ds = \int_e(\Pi_h^0 q-q)\cdot n\,ds = 0.$$

It follows that $\Pi_{R_h}(u-q)=0$ i.e. $\Pi_{R_h}\Pi_h^0 q=\Pi_{R_h}q$. Finally, $\Pi_{R_h}\Pi_{\Theta_h}=\Pi_{R_h}\Pi_h^0(I-L_h)+\Pi_{R_h}L_h=\Pi_{R_h}(I-L_h)+\Pi_{R_h}L_h=\Pi_{R_h}$, that is (2.8) holds.

By the trace theorem, one shows that $(\Pi_{\Sigma_h})|_{\hat K}$ is bounded on $H^1(\hat K,\mathbb{M})$. Moreover if we define for a matrix field $\hat M$, $P_F(\hat M)(x)=1/\det(B)\hat M(\hat x)B^T, x=F(\hat x)$, then it is not difficult to verify that $P_F((\Pi_{\Sigma_h})|_{\hat K}\hat\sigma)=(\Pi_{\Sigma_h})|_K P_F\hat\sigma$, hence (2.9) follows from a standard scaling argument.

Let $\hat\rho\in H^1(\hat K,\mathbb{R}^2)$. We define its Piola transform by $P_F\hat\rho=(P_F\hat\rho_1,P_F\hat\rho_2)$ where for a scalar function $\hat u$, $P_F\hat u=\hat u\circ F^{-1}$.

Since $\hat{\mathrm{curl}}\Pi_{\hat K}^0\hat\rho\in\Sigma_{\hat K}$,

$$||\hat{\mathrm{curl}}\Pi_{\hat K}^0\hat\rho||_{L^2(\hat T)}\le C\sum_{\hat e\subset\partial\hat K}\sum_{i=0}^1\left|\int_{\hat e}\hat{\mathrm{curl}}\Pi_{\hat K}^0\hat\rho\cdot\hat n\hat s^i\,d\hat s\right|,$$

where $\hat{e}$ is an edge of $\partial\hat{K}$. Next, $\operatorname{curl} q \cdot n = \partial q/\partial s$ and using the definition of $\Pi^0_{\hat{K}}$,

$$\int_{\hat{e}} \hat{\operatorname{curl}} \Pi^0_{\hat{K}} \hat{\rho} \cdot \hat{n} \, d\hat{s} = \int_{\hat{e}} \frac{\partial}{\partial \hat{s}} \Pi^0_{\hat{K}} \hat{\rho} \, d\hat{s} = 0$$

$$\int_{\hat{e}} \hat{\operatorname{curl}} \Pi^0_{\hat{K}} \hat{\rho} \cdot \hat{n} \; \hat{s} \, d\hat{s} = \int_{\hat{e}} \frac{\partial}{\partial s} (\Pi^0_{\hat{K}} \hat{\rho}) \hat{s} \, d\hat{s} = - \int_{\hat{e}} \Pi^0_{\hat{K}} \hat{\rho} \, d\hat{s} = - \int_{\hat{e}} \hat{\rho} \, d\hat{s}.$$

By the trace theorem, it follows that

$$||\hat{\operatorname{curl}} \Pi^0_{\hat{K}} \hat{\rho}||_{L^2(\hat{T})} \leq C ||\hat{\rho}||_{1,\hat{T}},$$

and scaling to an arbitrary rectangle K, we get

$$|| \operatorname{curl} \Pi^0_K \rho||_{L^2(K)} \leq C(h^{-1}|\rho|_{0,K} + C|\rho|_{1,K}).$$

We therefore have

$$\begin{aligned} || \operatorname{curl} \Pi_{\Theta_h} \rho||_{L^2} &\leq || \operatorname{curl} \Pi^0_h (I - L_h)\rho||_{L^2} + || \operatorname{curl} L_h \rho||_{L^2} \\ &\leq c(h^{-1}||(I - L_h)\rho||_{L^2} + ||(I - L_h)\rho||_{H^1}) + c||L_h \rho||_{H^1} \\ &\leq c||\rho||_{H^1}, \end{aligned}$$

that is (2.10) holds. Since $\operatorname{div} \Sigma_h \subset V_h$, the Brezzi conditions hold and the error estimates follow from the optimality error estimate from the theory of mixed methods, properties of the canonical interpolation operator for BDM elements, [16] p. 132, and error estimates of the L^2 projection operator. □

3.1. **Simplified element of low order.** Analogous to the simplified element of [7], we can develop elements simpler than the lowest order BDM type elements. The key point is that for (2.8) to hold, we only need Θ_h to have normal components continuous across edges. We start the construction by taking as Θ_h the rectangular version of a space introduced by Fortin, [20] and [21] p. 153. The spaces R_h, V_h and Q_h are the same. To define the space Θ_h, let i, j be the unit vectors in the x and y directions respectively. We put

$$\begin{aligned} p_1 &= -x(1-x)(1-y)\, i \\ p_2 &= -y(1-y)(1-x)\, j \\ p_3 &= x(1-x)y\, i \\ p_4 &= xy(1-y)\, j, \end{aligned}$$

and define on each element K,

$$\Theta_K = \mathcal{P}_{1,1}(K) \times \mathcal{P}_{1,1}(K) \oplus \operatorname{span}\{ p_1, p_2, p_3, p_4 \}$$

with degrees of freedom

(1) the values of q at the vertices ($4 \times 2 = 8$ degrees of freedom),
(2) the average of $q \cdot n$ on each edge (4 degrees of freedom).

The stress space $\overline{\Sigma}_K$ is defined as

$$\begin{pmatrix} \mathcal{P}_{1,0}(K) & \mathcal{P}_{0,1}(K) \\ \mathcal{P}_{1,0}(K) & \mathcal{P}_{0,1}(K) \end{pmatrix} \oplus \operatorname{span}\{ \operatorname{curl} p_1, \operatorname{curl} p_2, \operatorname{curl} p_3, \operatorname{curl} p_4 \},$$

where $\begin{pmatrix} \mathcal{P}_{1,0}(K) & \mathcal{P}_{0,1}(K) \\ \mathcal{P}_{1,0}(K) & \mathcal{P}_{0,1}(K) \end{pmatrix}$ is the space of matrix fields with components in the indicated spaces. Explicitly, we have $\operatorname{curl} p_1 = \begin{pmatrix} x(1-x) & (1-2x)(1-y) \\ 0 & 0 \end{pmatrix}$, $\operatorname{curl} p_2 = \begin{pmatrix} 0 & 0 \\ (-1+2y)(1-x) & -y(1-y) \end{pmatrix}$, $\operatorname{curl} p_3 = \begin{pmatrix} x(1-x) & -(1-2x)y \\ 0 & 0 \end{pmatrix}$ and $\operatorname{curl} p_4 = \begin{pmatrix} 0 & 0 \\ x(1-2y) & -y(1-y) \end{pmatrix}$.

For $\tau \in \begin{pmatrix} \mathcal{P}_{1,0}(K) & \mathcal{P}_{0,1}(K) \\ \mathcal{P}_{1,0}(K) & \mathcal{P}_{0,1}(K) \end{pmatrix}$, $\tau n \in \mathcal{P}_0(e) \times \mathcal{P}_0(e)$ on each edge e but $(\operatorname{curl} p_i) n \cdot t \in \mathcal{P}_1(e), i = 1, \ldots, 4$. The following degrees of freedom are unisolvent:

(1) $\int_e \tau n \cdot n \, ds$ for each edge e
(2) $\int_e \tau n \cdot t \, p \, ds$ for each edge e and $p \in \mathcal{P}_1(e)$.

To see this, let $\tau = \eta + a_1 \operatorname{curl} p_1 + a_2 \operatorname{curl} p_2 + a_3 \operatorname{curl} p_3 + a_4 \operatorname{curl} p_4 \in \overline{\Sigma}_K$ such that all the above degrees of freedom vanish. Since the normal component of $(\tau_{i1}, \tau_{i2}), i = 1, 2$ vanish on each edge, we have

$$\tau_{i1} = x(1-x)c_{i1}, \tau_{i2} = y(1-y)c_{i2}, i = 1, 2, c_{i,j} \in \mathbb{R}, i, j = 1, 2.$$

Since

$$\begin{aligned}
\tau_{11} &= \eta_{11} + a_1 x(1-x) + a_3 x(1-x), \eta_{11} \in \mathcal{P}_{10}(K) \\
\tau_{12} &= \eta_{12} + a_1(1-2x)(1-y) - a_3(1-2x)y, \eta_{12} \in \mathcal{P}_{01}(K) \\
\tau_{21} &= \eta_{21} + a_2(-1+2y)(1-x) - a_4 x(1-2y), \eta_{21} \in \mathcal{P}_{10}(K) \\
\tau_{22} &= \eta_{21} - a_4 y(1-y) - a_4 y(1-y), \eta_{22} \in \mathcal{P}_{01}(K),
\end{aligned}$$

we conclude that $a_1 = a_2 = a_3 = a_4 = 0$ and $\eta = 0$, that is: $\tau = 0$ and the claim follows.

From the approximation properties of the lowest order Raviart-Thomas element, the estimate (3.4) still holds.

4. Three dimensional elements

The de Rham complex in three dimensions is

$$\mathbb{R} \xrightarrow{\subset} C^\infty(\Omega, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega, \mathbb{R}) \longrightarrow 0.$$

We choose the following form of BDM elememt, [16], p.124

$$BDM_1(K) = \mathcal{P}_1(K, \mathbb{R}^3) + \operatorname{curl}\operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ xy^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x^2 y \end{pmatrix}, \begin{pmatrix} y^2 z \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} yz^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xz^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 z \\ 0 \end{pmatrix} \right\}.$$

Clearly $\operatorname{div} BDM_1(K) = \mathcal{P}_0(K)$. We define $V_K = \mathcal{P}_0(K)^3$ and

$$\Sigma_K = \{ \tau, \tau(x, y, z) \in \mathbb{M}, (\tau_{i1}, \tau_{i2}, \tau_{i3}) \in BDM_1(K), i = 1, 2, 3 \}.$$

The degrees of freedom on V_K are the values of each component at an interior point while a matrix field τ in Σ_K is uniquely determined by the moments of order 0 and 1 of τn on each face ($3 \times 3 \times 6$ degrees of freedom).

We now define two spaces S_K and U_K such that the sequence below is exact.

$$\mathbb{R} \xrightarrow{\subset} S_K \xrightarrow{\text{grad}} U_K \xrightarrow{\text{curl}} BDM_1(K) \xrightarrow{\text{div}} \mathcal{P}_0(K,\mathbb{R}) \longrightarrow 0\cdot$$

The space S_K is not directly used in the construction but helped discover U_K. We take the space S_K as the three dimensional serendipity space of order 2 defined as

$$S_K = \mathcal{P}_2(K,\mathbb{R}) + \text{span}\{\, x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, xyz, x^2yz, xy^2z, xyz^2 \,\},$$

with degrees of freedom

(1) the values of $q \in S_K$ at the vertices (8 degrees of freedom),
(2) the average of $q \in S_K$ on each edge (12 degrees of freedom).

The unisolvency of these degrees of freedom is proven for example in [4]. We define the space U_K as

$$U_K = \mathcal{P}_{1,1,1}(K,\mathbb{R}^3) + \text{span}\{\, y^2z, yz^2, y^2, z^2 \,\} \times \text{span}\{\, x^2z, xz^2, x^2, z^2 \,\} \times \text{span}\{\, x^2y, xy^2, x^2, y^2 \,\},$$

with degrees of freedom for $u \in U_K$,

(1) the first two moments of $u \cdot t$ on each edge, where t is a tangential vector to the edge ($12 \times 2 = 24$ degrees of freedom),
(2) the average of $u \wedge n$ on each face with unit outward normal n ($6 \times 2 = 12$ degrees of freedom).

It is not very difficult to verify that the sequence above is exact. One checks that each space is mapped in the one that follows. Then one notes that the alternating sum of the dimensions is zero and that the polynomial de Rham sequence is exact. We then only need to verify either that the kernel of the curl operator is the image of the grad operator or that the kernel of the div operator is the image of the curl operator. We verify the last one. Let $u \in BDM_1(K)$ such that $\operatorname{div} u = 0$. We write $u = w + \operatorname{curl} z, w \in \mathcal{P}_1(K,\mathbb{R}^3)$ and z in the span of the extra monomials in the definition of $BDM_1(K)$. Note that $z \in U_K$ and $\operatorname{div} u = \operatorname{div} w = 0$. By the exactness of the polynomial de Rham sequence, $w = \operatorname{curl} a, a \in \mathcal{P}_2(K,\mathbb{R}^3)$. Since for $\alpha,\beta,\gamma \in \mathbb{R}, \operatorname{curl}(\alpha x^2, \beta y^2, \gamma z^2) = 0$, we may assume that $a \in U_K$ which completes the proof of the claim.

We can now describe the space Θ_h as

$$\Theta_h = \{\, q, q(x,y,z) \in \mathbb{M}, (q_{i1}, q_{i2}, q_{i3}) \in U_h, i = 1,2,3 \,\},$$

with the degrees of freedom for $q \in \Theta_h$

(1) $\int_e q\, t\, s^i, i = 0,1$ for each edge e, where t is a tangential vector to the edge ($12 \times 2 \times 3 = 72$ degrees of freedom),
(2) $\int_f q \wedge n \, dx_f$ for each face f with unit outward normal n ($6 \times 2 \times 3 = 36$ degrees of freedom). For a matrix field q with row vectors $q_i, i = 1,2,3$, $q \wedge n$ is defined as the matrix field with rows $q_i \wedge n, i = 1,2,3$.

Next we define the space Q_h. We take $Q_K = \mathcal{P}_0(K)^3$ with degrees of freedom the values of each component at an interior point.

Finally we describe the space R_h as

$$\{ q, q(x,y,z) \in \mathbb{M}, (q_{i1}, q_{i2}, q_{i3})_{|K} \in RT_0(K), i = 1,2,3 \},$$

where

$$RT_0(K) = \mathcal{P}_{1,0,0}(K) \times \mathcal{P}_{0,1,0}(K) \times \mathcal{P}_{0,0,1}(K),$$

is the lowest order Raviart-Thomas element in three dimensions with degrees of freedom the average of the normal component on each face, ($1 \times 1 \times 6$=6 degrees of freedom).

4.0.1. *Unisolvency.* The unisolvency of the degrees of freedom for V_K, Σ_K and S_K are well known. Similarly unisolvency for the degrees of freedom of R_h is immediate. We only study the case of U_K. Let $v = (v_1, v_2, v_3) \in U_K$ and assume that all degrees of freedom vanish. We show that $v_1 = 0$. On each edge e, $v \cdot t \in \mathcal{P}_1(e)$ and hence we get $v \cdot t = 0$ on each edge. This implies that on the face $z = 0$ for example,

$$\begin{aligned} v_1 &= y(1-y)w_1, w_1 \in \mathcal{P}_{1,0} \\ v_2 &= x(1-x)w_2, w_2 \in \mathcal{P}_{0,1}. \end{aligned}$$

However, if w_1 has a linear term in x, xy^2 would be the highest degree monomial in v_1. We conclude that w_1 is constant. The face degrees of freedom imply that the average of w_1 vanish on the face $z = 0$, that is: $w_1 = 0$. Similarly $w_2 = 0$. We conclude that v has expression

$$\begin{aligned} v_1 &= y(1-y)z(1-z)r_1, \\ v_2 &= x(1-x)z(1-z)r_2, \\ v_3 &= x(1-x)y(1-y)r_3, \end{aligned}$$

for constants r_1, r_2 and r_3 which must vanish given the form of the highest degree monomial in the expression of $v_i, i = 1,2,3$.

4.0.2. *Definition of interpolation operators.* For $q \in C^\infty(\Omega, \mathbb{M})$, we define Π_{R_h} by

$$\int_f (\Pi_{R_h} q) n \, dx = \int_f qn \, dx, \quad \text{for all faces } f.$$

The interpolation operator Π_{Σ_h} is defined by

$$\int_f \Pi_{\Sigma_h}(\sigma) n \cdot q \, ds = \int_f \sigma n \cdot q \, ds, \quad \text{for all faces } f \text{ and for all } q \in \mathcal{P}_1(f) \times \mathcal{P}_1(f) \times \mathcal{P}_1(f).$$

It remains to define the interpolation operator Π_{Θ_h}. For this we first define $\Pi_K^0 : H^1(K, \mathbb{M}) \to \Theta_K$ by

$$\begin{aligned} &\int_e (\Pi_K^0 q) \, t \, s^i \, ds = 0, \; i = 0,1 \quad \text{for each edge } e \subset \partial K, \\ &\int_f (\Pi_K^0 q) \wedge n \, dx_f = \int_f q \wedge n \, dx_f, \quad \text{for each face } f \subset \partial K \end{aligned}$$

and $\Pi^0_h : H^1(\Omega, \mathbb{M}) \to \Theta_h$ by $(\Pi^0_h \tau)|_K = \Pi^0_K \tau$. Next, let L_h be a Clement interpolation operator [14, 18] which maps $L^2(\Omega, \mathbb{R})$ into

$$\{ \theta_h \in C^0(\bar{\Omega}) \,|\, \theta_{h|K} \in \mathcal{P}_{1,1,1}, \forall K \in \mathcal{T}_h \},$$

and denote as well by L_h the corresponding operator which maps $L^2(\Omega, \mathbb{M})$ into the subspace of Θ_h of continuous matrix fields whose components are piecewise in $\mathcal{P}_{1,1,1}$. We have

$$\|L_h \tau - \tau\|_j \leq c h^{m-j} \|\tau\|_m, \quad 0 \leq j \leq 1, \quad j \leq m \leq 2, \tag{4.1}$$

with c independent of h. We define our interpolation operator Π_{Θ_h} by

$$\Pi_{\Theta_h} = \Pi^0_h (I - L_h) + L_h. \tag{4.2}$$

4.0.3. *Commutativity and surjectivity assumptions.* The commutativity assumption (2.6) and (2.7) are proven as in the 2D case. We verify the surjectivity assumption $\Pi_{R_h} S \Pi_{\Theta_h} = \Pi_{R_h} S$. We first show that $\Pi_{R_h} S \Pi_{\Theta_h} = \Pi_{R_h} S$. For this let $q \in C^\infty(\Omega, \mathbb{M})$, put $\omega = q - \Pi^0_h q$. We need to show that $\Pi_{R_h} S \omega = 0$, that is

$$\int_f (S\omega)(x) n \, dx_f = 0, \quad \text{for each face } f.$$

Since $\Pi^0_h w = 0$,

$$\int_f \omega \wedge n = 0, \quad \text{for each face } f.$$

Next for $q = (q_{ij})_{i,j=1,2,3}$,

$$q \wedge n = \begin{pmatrix} q_{13}n_1 - q_{11}n_3 & q_{11}n_2 - q_{12}n_1 & q_{12}n_3 - q_{13}n_2 \\ q_{23}n_1 - q_{21}n_3 & q_{21}n_2 - q_{22}n_1 & q_{22}n_3 - q_{23}n_2 \\ q_{33}n_1 - q_{31}n_3 & q_{31}n_2 - q_{32}n_1 & q_{32}n_3 - q_{33}n_2 \end{pmatrix},$$

and

$$(Sq)n = \begin{pmatrix} q_{22}n_1 + q_{33}n_1 - q_{21}n_2 - q_{31}n_3 \\ -q_{12}n_1 + q_{11}n_2 + q_{33}n_2 - q_{32}n_3 \\ -q_{13}n_1 - q_{23}n_2 + q_{11}n_3 + q_{22}n_3 \end{pmatrix} = \begin{pmatrix} -(q\wedge n)_{22} + (q \wedge n)_{31} \\ (q\wedge n)_{12} - (q\wedge n)_{33} \\ -(q \wedge n)_{11} + (q\wedge n)_{23} \end{pmatrix}.$$

This shows that $\int_f \omega \wedge n = 0$ implies $\int_f (S\omega) n = 0$ and the result follows using the definition of Π_h.

We notice that for $q \in \Theta_h$, for the surjectivity assumption to hold, the following degrees of freedom were not used: $\int_f q_{12}n_3 - q_{13}n_2 \, dx_f = \int_f (q \wedge n)_{13}$, $\int_f q_{23}n_1 - q_{21}n_3 \, dx_f = \int_f (q\wedge n)_{12}$, $\int_f q_{31}n_2 - q_{32}n_1 \, dx_f = \int_f (q\wedge n)_{32}$. However since the faces of a rectangle are parallel to the axes, one of these degrees of freedom is identically zero for each face, hence two degrees of freedom per face are unnecessary.

4.0.4. *Boundedness of the interpolation operators.* By the trace theorem, one shows that $(\Pi_{\Sigma_h})|_{\hat{K}}$ is bounded on $H^1(\hat{K}, \mathbb{M})$. Moreover if we define for a matrix field $\hat{M}$, $P_F(\hat{M})(x) = 1/\det(B) \hat{M}(\hat{x}) B^T, x = F(\hat{x})$, then it is not difficult to verify that $P_F((\Pi_{\Sigma_h})|_{\hat{K}} \hat{\sigma}) = (\Pi_{\Sigma_h})|_K P_F \hat{\sigma}$, hence (2.9) follows from a standard scaling argument.

Let $\hat{\rho} \in H^1(\hat{K}, \mathbb{R}^3)$. We define its Piola transform by $P_F \hat{\rho} = (P_F \hat{\rho}_1, P_F \hat{\rho}_2, P_F \hat{\rho}_3)$ where for a scalar function $\hat{u}$, $P_F \hat{u} = \hat{u} \circ F^{-1}$.

Since $\hat{\text{curl}}\Pi^0_{\hat{K}}\hat{\rho} \in \Sigma_{\hat{K}}$,

$$||\hat{\text{curl}}\Pi^0_{\hat{K}}\hat{\rho}||_{L^2(\hat{T})} \leq C \sum_{\hat{f}\subset\partial\hat{K}} \sum_{i=0}^{1} \left| \int_{\hat{f}} \hat{\text{curl}}\Pi^0_{\hat{K}}\hat{\rho} \cdot \hat{n}\hat{s}^i \, d\hat{s} \right|,$$

where $\hat{f}$ is a face of $\partial\hat{K}$. Next, using the definition of $\Pi^0_{\hat{K}}$, for $q \in \mathcal{P}_{1,1}(f) \times \mathcal{P}_{1,1}(f) \times \mathcal{P}_{1,1}(f)$,

$$\int_{\hat{f}} (\hat{\text{curl}}(\Pi^0_{\hat{K}}\hat{\rho})\hat{n}) \cdot q \, dx_f = \int_{\hat{f}} (\Pi^0_{\hat{K}}\hat{\rho}) \wedge \hat{n}\nabla q \, dx_f = \int_{\hat{f}} \hat{\rho} \wedge \hat{n}\nabla q \, dx_f.$$

By the trace theorem, it follows that

$$||\hat{\text{curl}}\Pi^0_{\hat{K}}\hat{\rho}||_{L^2(\hat{T})} \leq C||\hat{\rho}||_{1,\hat{T}},$$

and scaling to an arbitrary rectangle K, we get

$$||\,\text{curl}\,\Pi^0_K \rho||_{L^2(K)} \leq C(h^{-1}|\rho|_{0,K} + C|\rho|_{1,K}).$$

We therefore have

$$\begin{aligned} ||\,\text{curl}\,\Pi_{\Theta_h}\rho||_{L^2} &\leq ||\,\text{curl}\,\Pi^0_h(I - L_h)\rho||_{L^2} + ||\,\text{curl}\,L_h\rho||_{L^2} \\ &\leq c(h^{-1}||(I - L_h)\rho||_{L^2} + ||(I - L_h)\rho||_{H^1}) + c||L_h\rho||_{H^1} \\ &\leq c||\rho||_{H^1}, \end{aligned}$$

that is (2.10) holds. Since $\text{div}\,\Sigma_h \subset V_h$, the Brezzi conditions hold. From the optimality error estimate from the theory of mixed methods (2.13), properties of the canonical interpolation operator for BDM elements, [16] p. 132, and error estimates of the L^2 projection operator, we have the following error estimate.

Theorem 4.1. *For the triple* $(\Sigma_h, V_h, \Theta_h)$ *the conditions of Theorem* (2.1) *hold and we have the optimality condition* (2.13). *Moreover if* σ *and* u *are sufficiently smooth,*

$$||\sigma - \sigma_h||_{H(\text{div})} + ||u - u_h||_{L^2} + ||\gamma - \gamma_h||_{L^2} \leq C\,h||u||_3. \tag{4.3}$$

5. Higher order elements

Except the simplified element in two dimension, the elements we have described do not have optimal rate of convergence for the stress. It does not seem possible to simplify the three dimensional element using the framework described here. In two dimension, for higher order approximation, $H(\text{div})$ elements can be constructed based on the sequence,

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} \mathcal{P}_{k+1,k+1} \xrightarrow{\text{curl}} \mathcal{P}_{k+1,k} \times \mathcal{P}_{k,k+1} \xrightarrow{\text{div}} \mathcal{P}_{k,k} \longrightarrow 0.$$

Take V_h to be the space of piecewise continuous vector fields which belong locally to $P_{k,k}(K) \times P_{k,k}(K)$, Q_h the space of piecewise continuous functions which belong locally to $Q_K = P_{k-1,k-1}(K)$ and $\Sigma_K = \{\, \tau \in \mathbb{M}, (\tau_{i1}, \tau_{i2}) \in \mathcal{P}_{k+1,k} \times \mathcal{P}_{k,k+1}, i = 1, 2 \,\}$ with degrees of freedom

(1) $\int_e \tau n \cdot p_k \, ds$, for each edge e of K, $\forall\, p_k \in \mathcal{P}_k(e)$,

(2) $\int_K \tau : \phi \, dx$, $\forall\, \phi \in \begin{pmatrix} \mathcal{P}_{k,k-1}(K) & \mathcal{P}_{k-1,k}(K) \\ \mathcal{P}_{k,k-1}(K) & \mathcal{P}_{k-1,k}(K) \end{pmatrix}$,

for $k \geq 1$. The space R_h is taken to be the Raviart-Thomas space of order $k-1$ and finally the space Θ_h is the space of continuous vector fields with components in $\mathcal{P}_{k+1,k+1}(K)$ on each element K. Again, there one does not have optimal convergence rate for the stress. We leave the details of the three dimensional analogue to the interested reader.

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