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FINITE ELEMENTS FOR SYMMETRIC TENSORS IN THREE DIMENSIONS

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Abstract. We construct finite element subspaces of the space of symmetric tensors with square-integrable divergence on a three-dimensional domain. These spaces can be used to approximate the stress field in the classical Hellinger–Reissner mixed formulation of the elasticty equations, when standard discontinuous finite element spaces are used to approximate the displacement field. These finite element spaces are defined with respect to an arbitrary simplicial triangulation of the domain, and there is one for each positive value of the polynomial degree used for the displacements. For each degree, these provide a stable finite element discretization. The construction of the spaces is closely tied to discretizations of the elasticity complex, and can be viewed as the three-dimensional analogue of the triangular element family for plane elasticity previously proposed by Arnold and Winther.

1. Introduction

The classical Lagrange finite element spaces provide natural simplicial finite element discretizations of the Sobolev space H^1. Similarly, various finite element spaces derived in the theory of mixed finite elements, such as the Raviart–Thomas and Nedelec spaces, provide the natural finite element discretizations of the spaces $H(\mathrm{div})$ and $H(\mathrm{curl})$. (These statements are made precise and treated in a uniform framework of the finite element exterior calculus in [9].) In this paper we consider the finite element discretization of the space $H(\mathrm{div}, \Omega; \mathbb{S})$ consisting of square-integrable symmetric tensors (or, given a choice of coordinates, symmetric matrix fields) with square-integrable divergence. In the classical Hellinger–Reissner mixed formulation of the elasticity equations, the stress is sought in $H(\mathrm{div}, \Omega; \mathbb{S})$ and the displacement in $L^2(\Omega; \mathbb{R}^n)$. The natural discretization of the latter space is evident—piecewise polynomial of some degree without interelement continuity constraints—but the development of an appropriate finite element subspace of $H(\mathrm{div}, \Omega; \mathbb{S})$ to use with these is a long-standing and challenging problem. For plane elasticity, the known stable mixed finite element methods have mostly involved composite elements for the stress [6, 15, 16, 21]. To avoid these, other authors have modified the standard mixed variational formulation of elasticity to a formulation that uses general, rather than symmetric, tensors for the stress, with the symmetry imposed weakly; see [2, 5, 7, 17, 18, 19, 20, 8]. Not until 2002 was a stable non-composite finite method for the classical mixed formulation of plane elasticity found [10]. This work can be seen as answering the question "what are

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the natural finite element discretizations of $H(\mathrm{div}, \Omega; \mathbb{S})$?" in the case of two dimensions. In this paper we address this question for three dimensions.

Up until recently, there were no mixed finite elements for the Hellinger–Reissner formulation in three dimensions known to be stable. In [1], a partial analogue of the lowest order element in [10] was proposed and shown to be stable. Here we will derive the full analogue of the results of [10]. We construct a family of finite element subspaces of $H(\mathrm{div}, \Omega; \mathbb{S})$ which, when used to discretize the stress in elasticity along with the obvious discontinuous piecewise polynomial discretization for the displacement, provide stable mixed finite elements for the Hellinger–Reissner principle. As in two dimensions, these spaces are related to a finite element subcomplex of the *elasticity complex*, related to it by commuting diagrams.

We recall the standard mixed formulation for the elasticity equations. Let Ω be a contractible polyhedral domain in $\mathbb{R}^3$, occupied by a linearly elastic body which is clamped on the boundary $\partial\Omega$, and let S and u denote the stress and displacement fields engendered by a force f acting on the body. The matrix field S and the vector field u can be characterized as the unique critical point of the Hellinger-Reissner functional

$$\mathcal{J}(T, v) = \int_\Omega (\frac{1}{2} AT : T + \mathrm{div}\, T \cdot v - f \cdot v)\, dx$$

over the space $H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^3)$. Here, $\mathbb{S}$ is the six dimensional space of symmetric matrices and $S : T$ denotes the Frobenius product on $\mathbb{S}$. The given compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$ is symmetric, and bounded and positive definite uniformly with respect to $x \in \Omega$. The divergence operator, div, is applied to a matrix field by taking the divergence of each row. Hence, this operator maps the space $H(\mathrm{div}, \Omega; \mathbb{S})$ into $L^2(\Omega; \mathbb{R}^3)$.

A mixed finite element method determines an approximate stress field S_h and an approximate displacement field u_h as the unique critical point (S_h, u_h) of the Hellinger–Reissner functional in a finite element space $\Sigma_h \times V_h \subset H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^3)$, where h denotes the mesh size. Equivalently, $(S_h, u_h) \in \Sigma_h \times V_h$ solves the saddle point system

$$\int_\Omega (AS_h : T + \mathrm{div}\, T \cdot u_h + \mathrm{div}\, S_h \cdot v)\, dx = \int_\Omega f v\, dx, \quad (T, v) \in \Sigma_h \times V_h. \tag{1.1}$$

To ensure that the discrete system has a unique solution and that it provides a good approximation of the true solution the finite dimensional spaces Σ_h and V_h must satisfy the stability conditions from the theory of mixed finite element methods, see [11, 12]. As is well known, see for example [10], the following two conditions are sufficient:

- $\mathrm{div}\, \Sigma_h \subset V_h$.
- There exists a linear operator $\Pi_h : H^1(\Omega; \mathbb{S}) \to \Sigma_h$, bounded in $\mathcal{L}(H^1; L^2)$ uniformly with respect to h, and such that $\mathrm{div}\, \Pi_h\, S = \Pi_h^V \mathrm{div}\, S$ for all $S \in H^1(\Omega; \mathbb{S})$, where $\Pi_h^V : L^2(\Omega; \mathbb{R}^3) \to V_h$ denotes the L^2-projection.

As mentioned above, the construction of finite element spaces which fulfill these two conditions has proved to be surprisingly hard. In this paper, we will derive a family of finite element spaces Σ_h and V_h based on tetrahedral meshes, and show that they satisfy these two stability conditions. There is one member of the family for each polynomial degree $k \geq 1$. The space V_h for the displacements is simply the space of all piecewise polynomial vector fields of degree at most k. In the lowest order case, $k = 1$, the space Σ_h contains the full space of quadratic polynomials

on each element, augmented by divergence-free polynomials of degree 3 and 4. The local dimension of Σ_h is 162, or 27 per component of stress on average. The analogous space in two dimensions, derived in [10], was of local dimension 24 (8 per component).

The complexity of the elements may very well limit their practical significance. However, we believe that the determination of the natural discretization of space $H(\operatorname{div}, \Omega; \mathbb{S})$ provides important insight, both into the obstacles to the derivation of simpler methods, and for the design of alternative procedures, such as nonconforming methods.

This paper is organized as follows. After giving some preliminaries remarks in Section 2 we present the lowest order element and establish its properties in Section 3. A family of higher order elements is then presented in Section 4. Key to the analysis of these elements is the description of the polynomial space of symmetric matrix fields with vanishing divergence and vanishing normal traces on the boundary of a simplex K. The dimension of this space is derived in Section 7 based on preliminary results derived in Sections 5 and 6. Furthermore, an explicit basis for this space, necessary for the computational procedure, is also given in the lowest order cases.

2. Notation and preliminaries

We begin with some basic notation. If $K \subset \mathbb{R}^3$ is a tetrahedron, then $\Delta_2(K)$ denotes the set of the four 2–dimensional faces of K, $\Delta_1(K)$ the set of the six 1–dimensional edges, and $\Delta_0(K)$ the set of the four vertices. Furthermore, $\Delta(K)$ is the set of all subsimplexes of K (of dimension $0, 1, 2$ or 3).

We let $\mathbb{M}$ be the space of 3×3 real matrices, and $\mathbb{S}$ and $\mathbb{K}$ the subspaces of symmetric and skew symmetric matrices, respectively. The operators $\operatorname{sym} : \mathbb{M} \to \mathbb{S}$ and $\operatorname{skw} : \mathbb{M} \to \mathbb{K}$ denote the symmetric and skew symmetric parts, respectively. Note that an element of the space $\mathbb{K}$ can be identified with its *axial vector* in $\mathbb{R}^3$ given by the map $\operatorname{vec} : \mathbb{K} \to \mathbb{R}^3$:

$$\operatorname{vec} \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

i.e., $\operatorname{vec}^{-1}(v)w = v \times w$ for any vectors v and w.

For any vector space X, we let $L^2(\Omega; X)$ be the space of square-integrable vector fields on Ω with values in X. For our purposes, X will usually either be $\mathbb{R}$, $\mathbb{R}^3$, or $\mathbb{M}$, or some subspace of one of these. In the case $X = \mathbb{R}$, we will simply write $L^2(\Omega)$. The corresponding Sobolev space of order k, i.e., the subspace of $L^2(\Omega; X)$ consisting of functions with all partial derivatives of order less than or equal to k in $L^2(\Omega; X)$, is denoted $H^k(\Omega; X)$, and its norm by $\| \cdot \|_k$. The space $H(\operatorname{div}, \Omega; \mathbb{S})$ is defined by

$$H(\operatorname{div}, \Omega; \mathbb{S}) = \{T \in L^2(\Omega; \mathbb{S}) \mid \operatorname{div} T \in L^2(\Omega; \mathbb{R}^3)\},$$

where the divergence of a matrix field is ithe vector field obtained by applying the divergence operator row–wise. For a vector field $v : \Omega \to \mathbb{R}^3$, $\operatorname{grad} v$ is the matrix field with rows the gradient of each component, and the symmetric gradient, $\epsilon(v)$,

is given by $\epsilon(v) = \operatorname{sym}\operatorname{grad} v$. Furthermore,

$$\operatorname{curl} v = -2\operatorname{vec}\operatorname{skw}\operatorname{grad} v = \begin{pmatrix} \partial_3 v_2 - \partial_2 v_3 \\ -\partial_3 v_1 + \partial_1 v_3 \\ \partial_2 v_1 - \partial_1 v_2 \end{pmatrix}.$$

If we consider a linear coordinate transformation of the form $x = B\hat{x} + b$, with corresponding vector fields v and $\hat{v}$ related by $v(x) = (B')^{-1}\hat{v}(\hat{x})$, then we have

$$\operatorname{skw}\operatorname{grad}_x v = (B')^{-1}(\operatorname{skw}\operatorname{grad}_{\hat{x}} \hat{v})B^{-1}.$$

Here B' denotes the transpose of B. In particular, if B is orthogonal, i.e., $B'B = I$, then $v = B\hat{v}$ and

$$\operatorname{skw}\operatorname{grad}_x v = B(\operatorname{skw}\operatorname{grad}_{\hat{x}} \hat{v})B'. \tag{2.1}$$

As for the divergence and the gradient operator, the operator curl acts on a matrix field by applying the ordinary curl operator to each row of the matrix. The operator curl^* is the corresponding operator obtained by taking the curl of each column. Alternatively, we have $\operatorname{curl}^* T = (\operatorname{curl} T')'$ for any matrix field T. The second order operator $\operatorname{curl}\operatorname{curl}^*$ maps symmetric matrix fields into symmetric matrix fields. Let $\Xi : \mathbb{M} \to \mathbb{M}$ be the algebraic operator $\Xi T = T' - \operatorname{tr}(T)I$, where I is the identity matrix. Then Ξ is invertible with $\Xi^{-1} T = T' - \operatorname{tr}(T)I/2$. The following identities are useful:

$$\operatorname{vec}\operatorname{skw}\operatorname{curl} T = -\frac{1}{2}\operatorname{div}\Xi T, \quad T \in C^\infty(\Omega, \mathbb{M}), \tag{2.2}$$

$$\operatorname{curl} T = \Xi\operatorname{grad}\operatorname{vec} T, \quad T \in C^\infty(\Omega, \mathbb{K}), \tag{2.3}$$

$$\operatorname{tr}\operatorname{curl} T = -2\operatorname{div}\operatorname{vec}\operatorname{skw} T, \quad T \in C^\infty(\Omega, \mathbb{M}). \tag{2.4}$$

These formulas can be verified directly, but they are also consequences of the discussions given in [8, 9], cf. Section 4 of [8] or Section 11 of [9].

For $K \subset \mathbb{R}^3$ we let $\mathcal{P}_k(K; X)$ be the space of polynomials of degree k, defined on K and with values in X. We write $\mathcal{P}_k$ or $\mathcal{P}_k(K)$ for $\mathcal{P}_k(K; \mathbb{R})$. The de Rham complex has a polynomial analogue of the form

$$\mathbb{R} \hookrightarrow \mathcal{P}_{k+3} \xrightarrow{\operatorname{grad}} \mathcal{P}_{k+2}(K; \mathbb{R}^3) \xrightarrow{\operatorname{curl}} \mathcal{P}_{k+1}(K; \mathbb{R}^3) \xrightarrow{\operatorname{div}} \mathcal{P}_k \to 0. \tag{2.5}$$

In fact, this complex is an exact sequence [9].

In recent years differential complexes have come to play a significant role in the design of mixed finite element methods [3, 10, 8, 9]. For the equations of elasticity, the relevant differential complex is the *elasticity complex*. In three space dimensions, the elasticity complex takes the form

$$\mathbb{T} \hookrightarrow C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\epsilon} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\operatorname{curl}\operatorname{curl}^*} C^\infty(\Omega; \mathbb{S}) \xrightarrow{\operatorname{div}} C^\infty(\Omega; \mathbb{R}^3) \to 0,$$

where $\mathbb{T}$ is the six-dimensional space of infinitesimal rigid motions, i.e., the space of linear polynomial functions of the form $x \mapsto a + b \times x$ for some $a, b \in \mathbb{R}^3$. It is straightforward to verify that the elasticity complex is a complex, i.e., the composition of two successive operators is zero. In fact, if the domain Ω is contractible, then the elasticity complex is an exact sequence; see [8, 9].

An analogous complex with less smoothness is

$$\mathbb{T} \hookrightarrow H^1(\Omega; \mathbb{R}^3) \xrightarrow{\epsilon} H(\operatorname{curl}\operatorname{curl}^*, \Omega; \mathbb{S}) \xrightarrow{\operatorname{curl}\operatorname{curl}^*} H(\operatorname{div}, \Omega; \mathbb{S}) \xrightarrow{\operatorname{div}} L^2(\Omega; \mathbb{R}^3) \to 0,$$

where $H(\operatorname{curl}\operatorname{curl}^*, \Omega; \mathbb{S}) = \{\, S \in L^2(\Omega; \mathbb{S}) \mid \operatorname{curl}\operatorname{curl}^* S \in L^2(\Omega; \mathbb{S}) \,\}$.

There is also a polynomial analogue of the elasticity complex. Let $K \subset \mathbb{R}^3$ be a tetrahedron and $k \geq 0$. The polynomial elasticity complex is given by

$$\mathbb{T} \hookrightarrow \mathcal{P}_{k+4}(K;\mathbb{R}^3) \xrightarrow{\epsilon} \mathcal{P}_{k+3}(K;\mathbb{S}) \xrightarrow{\operatorname{curl}\operatorname{curl}^*} \mathcal{P}_{k+1}(K;\mathbb{S}) \xrightarrow{\operatorname{div}} \mathcal{P}_k(K;\mathbb{R}^3) \to 0. \tag{2.6}$$

This complex is an exact sequence. To prove the exactness, we first show that if S is a matrix field in $\mathcal{P}_{k+3}(\Omega;\mathbb{S})$, with $\operatorname{curl}\operatorname{curl}^* S = 0$, then $S = \epsilon(u)$ for $u = (u_1, u_2, u_3)^T \in \mathcal{P}_{k+4}(\Omega;\mathbb{R}^3)$. Clearly $S = \epsilon(u)$ for $u \in C^\infty(K;\mathbb{R}^3)$. It is enough to show that all second derivatives of u_k, $k = 1, 2, 3$, are in $\mathcal{P}_{k+2}(\Omega;\mathbb{R})$. This follows from the identity $\partial_{ij} u_k = \partial_i \epsilon(u)_{jk} + \partial_j \epsilon(u)_{ik} - \partial_k \epsilon(u)_{ij}$. Next, we show that if $S \in \mathcal{P}_{k+1}(\Omega;\mathbb{S})$, and $\operatorname{div} S = 0$, then $S = \operatorname{curl}\operatorname{curl}^* T$ for some $T \in \mathcal{P}_{k+3}(\Omega;\mathbb{S})$. First we observe that since $\operatorname{div} S = 0$ it follows from the fact that (2.5) is exact that $S = \operatorname{curl} U$ for some $U \in \mathcal{P}_{k+2}(K;\mathbb{M})$. Furthermore, since S is symmetric it follows from (2.2) that $\operatorname{div}\Xi U = 0$, and as a consequence, using (2.5) once more, we obtain that $\Xi U = \operatorname{curl} T$ for some $T \in \mathcal{P}_{k+3}(K;\mathbb{M})$, or $S = \operatorname{curl} U = \operatorname{curl}\Xi^{-1}\operatorname{curl} T$. However, by (2.3) we have $\operatorname{curl}\Xi^{-1}\operatorname{curl}\operatorname{skw} T = \operatorname{curl}\operatorname{grad}\operatorname{vec}\operatorname{skw} T = 0$. Hence, we can take $T \in \mathcal{P}_{k+3}(K;\mathbb{S})$. Finally, we observe that if T is symmetric, then (2.4) implies that $\operatorname{tr}\operatorname{curl} T = 0$, and therefore $S = \operatorname{curl}\Xi^{-1}\operatorname{curl} T = \operatorname{curl}\operatorname{curl}^* T$. To establish the surjectivity of the last map, one can use the fact that $\dim \mathcal{P}_k = (k+1)(k+2)(k+3)/6$ to verify that the alternating sum of the dimensions of the spaces in the sequence is zero. The same arguments show that (2.6) is exact for $k = -1$, -2, or -3, if $\mathcal{P}_j$ is interpreted as the zero space for $j < 0$.

Let $\{\mathcal{T}_h\}$ denote a family of triangulations of Ω by tetrahedra with diameter bounded by h. We assume that the intersection of any two tetrahedra in $\mathcal{T}_h$ is either empty or a common subsimplex of each. The family $\{\mathcal{T}_h\}$ is also assumed to be shape regular in the sense that the ratio of the radii of the circumscribed and inscribed spheres of all the tetrahedra can be bounded by a fixed constant. Furthermore, we will use the notation $\Delta_j(\mathcal{T}_h)$, for $j = 0, 1, 2$, to denote the set of vertices, edges, and faces, respectively, associated with the mesh $\mathcal{T}_h$. In Section 4 we will define a family of finite element spaces $\Sigma_h \subset H(\operatorname{div},\Omega;\mathbb{S})$ and $V_h \subset L^2(\Omega;\mathbb{R})$ for the elasticity problem consisting of piecewise polynomial spaces with repect to $\mathcal{T}_h$ of arbitrarily high polynomial order. However, we will first consider the lowest order case of this family in Section 3 below. All our spaces will have the property that $\operatorname{div}\Sigma_h \subset V_h$. Furthermore, we will identify a corresponding projection operator $\Pi_h : H^1(\Omega;\mathbb{S}) \to \Sigma_h$ satisfying the commutativity relation

$$\operatorname{div}\Pi_h T = \Pi_h^V \operatorname{div} T, \qquad T \in H^1(\Omega;\mathbb{S}), \tag{2.7}$$

and the bound

$$\|\Pi_h T\|_0 \leq C\|T\|_1, \qquad T \in H^1(\Omega;\mathbb{S}), \tag{2.8}$$

with constant C independent of h. Here $\Pi_h^V : L^2(\Omega;\mathbb{R}^3) \to V_h$ is the L^2 projection. It is a consequence of the general error bounds derived in [14], cf. also [10], that the properties above imply that (Σ_h, V_h) is a stable pair of elements for the discretization (1.1), and that the error bounds

$$\|S - S_h\|_0 \leq \|(I - \Pi_h)S\|_0 \tag{2.9}$$

$$\|u - u_h\|_0 \leq \|(I - \Pi_h^V)u\|_0 + c\|(I - \Pi_h)S\|_0 \tag{2.10}$$

holds, with a constant c independent of h. Here (S, u) is the unique critical point of the Hellinger-Reissner functional over $H(\operatorname{div},\Omega;\mathbb{S}) \times L^2(\Omega;\mathbb{R}^2)$ and $(S_h, u_h) \in \Sigma_h \times V_h$ the corresponding finite element solution. In addition, $\operatorname{div} S_h = \Pi_h^V \operatorname{div} S$.

3. The lowest order element

We first describe the restriction of the lowest order spaces Σ_h and V_h to a single tetrahedron $K \in \mathcal{T}_h$. Define

$$\Sigma_K = \{\, T \in \mathcal{P}_4(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_1(K;\mathbb{R}^3) \,\}, \quad V_K = \mathcal{P}_1(K;\mathbb{R}^3).$$

The space V_K has dimension 12 and a complete set of degrees of freedom is given by the zero and first order moments with respect to K. The space Σ_K has dimension at least 162 since the dimension of $\mathcal{P}_4(K;\mathbb{S})$ is 210 and the condition $\operatorname{div} T \in \mathcal{P}_1(K;\mathbb{R}^3)$ represents 48 linear constraints. We will show that $\dim \Sigma_K = 162$ by exhibiting 162 degrees of freedom which determine the elements uniquely. Define

$$\mathcal{M}(K) = \{\, T \in \mathcal{P}_4(K;\mathbb{S}) \mid \operatorname{div} T = 0,\ Tn = 0 \quad \text{on } \partial K \,\}.$$

It will follow from Theorem 7.2 below that the dimension of $\mathcal{M}(K)$ is 6. Furthermore, in Section 7 we will also give an explicit basis for this space. Using this basis, we can state the 162 degrees of freedom for the space Σ_K.

Lemma 3.1. *A matrix field $T \in \Sigma_K$ is uniquely determined by the following degrees of freedom*

1. *the values of T at the vertices of K, $4 \times 6 = 24$ degrees of freedom,*
2. *for each edge $e \in \Delta_1(K)$ with unit tangent vector s and linearly independent normal vectors n_- and n_+, the constant, linear and quadratic moments over e of $n_-'Tn_-$, $n_+'Tn_+$, $n_-'Tn_+$, $6 \times 3 \times 3 = 54$ degrees of freedom,*
3. *for each face $f \in \Delta_2(K)$, with normal n, and each edge $e \subset \partial f$ with tangent s, the constant, linear and quadratic moments over e of $s'Tn$, $4 \times 3 \times 3 = 36$ degrees of freedom,*
4. *for each face $f \in \Delta_2(K)$, with normal n, the constant and linear moments over f of Tn, $4 \times 3 \times 3 = 36$ degrees of freedom,*
5. *the average of T over K, 6 degrees of freedom,*
6. *the value of the moments $\int_K T : U \ dx$, $U \in \mathcal{M}(K)$, 6 degrees of freedom.*

Proof. We assume that all degrees of freedom vanish and show that $T = 0$. Since $T = 0$ at the vertices, the second and third set of degrees of freedom imply that $Tn = 0$ on each edge for both faces meeting the edge. By the fourth set of degrees of freedom we obtain that $Tn = 0$ on each face of K. For $v = \operatorname{div} T \in \mathcal{P}_1(K;\mathbb{R}^3)$ we have

$$\int_K v^2 \, dx = -\int_K T : \epsilon(v) \, dx + \int_{\partial K} Tn \cdot v \, dx_f = -\int_K T : \epsilon(v) \, dx = 0$$

by the fifth set of degrees of freedom. Here and below, dx_f denotes the surface measure on ∂K. We conclude that $\operatorname{div} T = 0$, and, by the last set of degrees of freedom, that $T = 0$. □

We now describe the finite element spaces on the triangulation $\mathcal{T}_h$. We denote by V_h the space of vector fields which belong to V_K for each $K \in \mathcal{T}_h$ and by Σ_h the space of matrix fields which belong piecewise to Σ_K, and with the continuity conditions induced by the degrees of freedom. In particular, for $T \in \Sigma_h$, the normal components Tn are continuous across all faces $f \in \Delta_2(\mathcal{T}_h)$ and, hence, $\Sigma_h \subset H(\operatorname{div}, \Omega; \mathbb{S})$. In addition, if $T \in \Sigma_h$, $e \in \Delta_1(\mathcal{T}_h)$, and n_+, n_- are two vectors normal to e, then $n_+'Tn_-$ is continuous on e and of course, T is continuous at the vertices.

It remains to define an interpolation operator $\Pi_h : H^1(\Omega;\mathbb{S}) \to \Sigma_h$ which satisfies (2.7) and (2.8). The technique used here is standard and can be found for example in [12, § VI-4]. Because of the vertex and edge degrees of freedom, the canonical interpolation operator for Σ_h, Π_h^Σ, defined directly from the degrees of freedom, is not bounded on $H^1(\Omega;\mathbb{S})$. In order to overcome this difficulty we introduce the operator $\Pi_h^0 : H^1(\Omega;\mathbb{S}) \to \Sigma_h$ defined from the degrees of freedom above, but where the vertex and edge degrees of freedom are set equal to zero, i.e., we have

$$\Pi_h^0 T(x) = 0, \quad x \in \Delta_0(\mathcal{T}_h), \tag{3.1}$$

$$\int_e n'_- \Pi_h^0 T n_+ v \, ds = 0, \quad e \in \Delta_1(\mathcal{T}_h),\ v \in \mathcal{P}_2(e), \text{ where } n_-, n_+ \in e^\perp, \tag{3.2}$$

$$\int_e s' \Pi_h^0 T n v \, ds = 0, \quad v \in f \in \Delta_2(\mathcal{T}_h),\ e \in \Delta_1(f),\ n \perp f,\ s \parallel e,\ v \in \mathcal{P}_2(e), \tag{3.3}$$

$$\int_f (T - \Pi_h^0 T) n \cdot v \, dx_f = 0, \quad f \in \Delta_2(\mathcal{T}_h),\ v \in \mathcal{P}_1(f;\mathbb{R}^3), \tag{3.4}$$

$$\int_K (T - \Pi_h^0 T) \, dx = 0, \quad K \in T_h, \tag{3.5}$$

$$\int_K (T - \Pi_h^0 T) : U \, dx = 0, \quad K \in T_h,\ U \in \mathcal{M}(K). \tag{3.6}$$

The commutativity property (2.7) for Π_h^0 follows from (3.4) and (3.5) since

$$\int_K (\operatorname{div} \Pi_h^0 T - \operatorname{div} T) \cdot v \, dx = -\int_K (\Pi_h^0 T - T) : \epsilon(v) \, dx + \int_{\partial K} (\Pi_h^0 T - T) n \cdot v \, dx_f = 0.$$

The uniform boundedness (2.8) can be seen from a standard scaling argument using the matrix Piola transform. Let $\hat{K}$ be a fixed reference tetrahedron and $F = F_K : \hat{K} \to K$ be an affine isomorphism of the form $F\hat{x} = B\hat{x} + b$. Given a matrix field $\hat{T} : \hat{K} \to \mathbb{S}$, define $T : K \to \mathbb{S}$ by the matrix Piola transform $T(x) = B\hat{T}(\hat{x})B^T$, with $x = F\hat{x}$. Using $\operatorname{div} T(x) = B \operatorname{div} \hat{T}(\hat{x})$, it is easy to verify that $T \in \Sigma_K$ if and only if $\hat{T} \in \Sigma_{\hat{K}}$. Furthermore, as in [4, 10] a scaling argument can be used to verify the uniform boundedness condition (2.8) for the operator Π_h^0. We can therefore conclude that the operator Π_h^0 satisfies the two conditions (2.7) and (2.8). However, the operator Π_h^0 lacks good approximation properties. Therefore, in order to obtain error estimates from the general bounds (2.9) and (2.10) a more accurate interpolation operator is needed.

Consider the modified interpolation operator $\Pi_h : H^1(\Omega;\mathbb{S}) \to \Sigma_h$ of the form

$$\Pi_h = \Pi_h^0 (I - R_h) + R_h, \tag{3.7}$$

where $R_h : L^2(\Omega;\mathbb{S}) \to \Sigma_h$ is the Clément operator onto the continuous piecewise quadratic subspace of Σ_h [13]. This operator satifies the bounds

$$\|R_h T - T\|_j \le c h^{m-j} \|T\|_m, \quad 0 \le j \le 1, \quad j \le m \le 3.$$

As a consequence of this bound, and the boundedness (2.8) of Π_h^0, we obtain the estimate

$$\|\Pi_h T - T\|_0 \le c h^m \|T\|_m, \quad 1 \le m \le 3 \tag{3.8}$$

for the interpolation error. Furthermore, since Π_h^0 satisfies (2.7) and $\Pi_h^V \operatorname{div} R_h = \operatorname{div} R_h$, we conclude that Π_h satisfies (2.7).

We also recall that the projection operator $\Pi_h^V : L^2(\Omega;\mathbb{R}^3) \to V_h$ satisfies the error estimate

$$\|\Pi_h^V v - v\|_0 \leq ch^m \|v\|_m, \quad 0 \leq m \leq 2. \tag{3.9}$$

The estimates (3.8) and (3.9), combined with the basic error bounds (2.9) and (2.10), and the fact that $\operatorname{div} S_h = \Pi_h^V \operatorname{div} S$, imply the following error estimates for the finite element method generated by $\Sigma_h \times V_h$.

Theorem 3.2. *Let (S, u) denote the unique critical point of the Hellinger-Reissner functional over $H(\operatorname{div}, \Omega; \mathbb{S}) \times L^2(\Omega;\mathbb{R}^2)$ and let (S_h, u_h) be the unique critical point over $\Sigma_h \times V_h$. Then*

$$\begin{aligned} \|S - S_h\|_0 &\leq ch^m \|S\|_m, \qquad 1 \leq m \leq 3, \\ \|\operatorname{div} S - \operatorname{div} S_h\|_0 &\leq ch^m \|\operatorname{div} S\|_m, \qquad 0 \leq m \leq 2, \\ \|u - u_h\|_0 &\leq ch^m \|u\|_{m+1}, \qquad 1 \leq m \leq 2. \end{aligned}$$

Remark. It is not possible to lower the polynomial degree of the stress space Σ_h introduced above. That is, if $\Sigma_K = \{\, T \in \mathcal{P}_k(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_s(K;\mathbb{R}^3) \,\}$, then we must require that $k \geq 4$ and $s \leq k-3$. The argument generalizes a similar argument given in [10]. Any element $T \in \Sigma_K$ must be uniquely determined by an arbitrary specification of degrees of freedom which are associated with the vertices, edges, faces, and interior of K. Since we require the assempled finite element space to be contained in $H(\operatorname{div}, \Omega; \mathbb{S})$, the degrees of freedom associated with a face, its edges, and its vertices must determine Tn on that face. Now, consider two faces of K meeting an an edge e with normals n_+ and n_-. The quantity $n_-'Tn_+$ is determined on e by the degrees of freedom associated to the first face and its edges and vertices, and, since this quantity is identical to $n_+'Tn_-$, it is also determined by the degrees of freedom associated to the second face and its edges and vertices. But, by definition, the degrees of freedom are independent, so this is only possible if $n_-'Tn_+$ is determined on e by the degrees of freedom associated to e or associated to one of its two vertices. But the edge e is shared by other simplices which will have different values for the normals n_+ and n_-, from which it follows that the degrees of freedom associated to e and those associated to its two vertices must determine the quantities $m'Tn$ for every pair of vectors m, $n \in e^\perp$ (assuming there is no restriction put on the triangulations). Next, consider a vertex x belonging to an edge e and a face with normal n. Then we have just seen that $n'T(x)n$ is determined by the degrees of freedom associated to e and its vertices. But it is similarly determined by the corresponding quantities for the other edge in the face containing x. By independence of the degrees of freedom, we conclude that $n'T(x)n$ is determined by the degrees of freedom associated to x. This is true for each face containing x, so for arbitrary values of n. It follows that T itself is determined at a vertices by the degrees of freedom associated to the vertex (a symmetric matrix is determined by the associated quadratic form).

Moreover, for the commutativity relation (2.7) to hold, we need to have as degrees of freedom on a face the moment of Tn times the divergence of the stress elements. Since we already have enough degrees of freedom at the vertices and on edges to determine two components of Tn there, we can give at most the moments of degree $k-3$ of these components. Thus our polynomial space must incorporate the restriction $s \leq k-3$. It follows that $k < 3$ is impossible. For $k = 3$, the above argument leads to 24 degrees of freedom at the vertices and 36 degrees of

freedom on the edges. Next, for each face, and each edge, we take as degrees of freedom the constant and linear moments of $s'Tn$, 24 degrees of freedom, as well as the constant moments of Tn on faces, which are another 12 degrees of freedom for $H(\mathrm{div})$ continuity. In total 96 degrees of freedom, but the space of cubics with constant divergence has dimension 93, from the exactness of (2.6). It follows that $k \geq 4$.

However, as in [10], a minor simplification is possible. On each tetrahedron $K \in \mathcal{T}_h$ we take the restricted displacement space V_K to be the rigid motions $\mathbb{T} \subset \mathcal{P}_1(K;\mathbb{R}^3)$ and the corresponding stress space to be

$$\tilde{\Sigma}_K = \{\, T \in \mathcal{P}_4(K;\mathbb{S}) \mid \operatorname{div} T \in \mathbb{T} \,\}.$$

Clearly $\dim \tilde{\Sigma}_K \geq 210-(60-6) = 156$. In fact, $\dim \tilde{\Sigma}_K = 156$ and a complete set of degrees of freedom is obtained by removing the six average values of T represented by (4) in Lemma 3.1. The proof of the fact that these degrees of freedom are unisolvent for $\tilde{\Sigma}_K$ follows by a simple modification of the proof of Lemma 3.1 above. Just observe that if $v = \operatorname{div} T \in \mathbb{T}$ then $\epsilon(v) = 0$. However, the simplified element is less accurate, since the stress space lacks some quadratics, and the displacement space some linears. Instead of the error estimates given in Theorem 3.2 we obtain at most $O(h^2)$ convergence for $||S - S_h||_0$, and at most first order convergence for $||\operatorname{div}(S - S_h)||_0$ and $||u - u_h||_0$.

4. A Family of Higher Order Elements

In this section we describe a family of stable element pairs, one for each degree $k \geq 1$. The lowest order case $k = 1$ is the one treated above. We first describe the elements on a single tetrahedron. Define

$$\Sigma_K = \{\, T \in \mathcal{P}_{k+3}(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_k(K;\mathbb{R}^3) \,\}, \quad V_K = \mathcal{P}_k(K;\mathbb{R}^3).$$

Then

$$\dim V_K = 3\binom{k+3}{3} = \frac{(k+3)(k+2)(k+1)}{2},$$

$$\begin{aligned}\dim \Sigma_K \geq d_k :&= \dim \mathcal{P}_{k+3}(K;\mathbb{S}) - [\dim \mathcal{P}_{k+2}(K;\mathbb{R}^3) - \dim \mathcal{P}_k(K;\mathbb{R}^3)] \\ &= 6\binom{k+6}{3} - 3\binom{k+5}{3} + 3\binom{k+3}{3} = k^3 + 12k^2 + 56k + 93.\end{aligned}$$

Notice that the space $\epsilon[\mathcal{P}_k(K;\mathbb{R}^3)]$ has dimension $(k+3)(k+2)(k+1)/2 - 6$. Analoguous to the lowest order case, we define the space

$$\mathcal{M}_k(K) = \{\, T \in \mathcal{P}_k(K;\mathbb{S}) \mid \operatorname{div} T = 0, \quad Tn = 0 \quad \text{on } \partial K \,\}.$$

We will prove in Section 7, Theorem 7.2, that $\dim \mathcal{M}_k(K)$ is $(k+2)(k-2)(k-3)/2$ for $k \geq 4$.

The degrees of freedom for V_K are the moments of degree less than or equal to k with respect to K. A unisolvent set of degrees of freedom for Σ_K are given by

(1) the values of T at the vertices of K, $4 \times 6 = 24$ degrees of freedom,
(2) for each edge $e \in \Delta_1(K)$ with unit tangent vector s and linearly independent normal vectors n_- and n_+, the moments of degree at most $k+1$ over e of n'_-Tn_-, n'_+Tn_+, n'_-Tn_+, $6 \times (k+2) \times 3 = 18k+36$ degrees of freedom,

(3) for each face $f \in \Delta_2(K)$, with normal n, and each edge $e \subset \partial f$ with tangent s, the moments of degree at most $k+1$ over e of $s'Tn$, $4\times3\times(k+2) = 12k+24$ degrees of freedom,
(4) for each $f \in \Delta_2(K)$ with normal vector n, the moments of degree at most k over f for Tn, $3 \times 4 \times (k+2)(k+1)/2 = 6k^2 + 18k + 12$ degrees of freedom,
(5) $\int_K T : U\, dx$, $U \in \epsilon(V_K)$, $(k+3)(k+2)(k+1)/2 - 6$ degrees of freedom,
(6) $\int_K T : U\, dx$, $U \in \mathcal{M}_{k+3}(K)$, $(k+5)(k+1)k/2$ degrees of freedom.

The proof that this set of functionals is unisolvent for the space Σ_K is almost identical to the lowest order case, and it is easily checked that their numbers sum up to d_k. Hence, we have shown that $\dim \Sigma_k = d_k$. Furthermore, in Section 7 we will give an explicit basis for the space $\mathcal{M}_k(K)$ when $k = 4$ and $k = 5$.

The finite element space $V_h \subset L^2(\Omega; \mathbb{R}^3)$ consists of all vector fields which belong to $\mathcal{P}_k(K; \mathbb{R}^3)$ for each $K \in \mathcal{T}_h$, while the corresponding stress space Σ_h is the space of matrix fields which belong piecewise to Σ_K, and with the continuity conditions induced by the degrees of freedom. In particular, this implies that the normal components Tn, for $T \in \Sigma_h$, are continuous over all faces in $\Delta_2(\mathcal{T}_h)$. Hence, as in the lowest order case we have that $\Sigma_h \subset H(\mathrm{div}, \Omega; \mathbb{S})$.

The L^2 projection Π_h^V onto V_h satisfies the estimate

$$\|\Pi_h^V v - v\|_0 \le ch^m \|v\|_m, \quad 0 \le m \le k+1. \tag{4.1}$$

We also introduce the Clément interpolant $R_h : L^2(\Omega; \mathbb{S}) \to \Sigma_h$ defined as the $\mathbb{S}$–valued version of the standard scalar Clément interpolant into continuous piecewise polynomials of order $k+1$. Hence, the operator R_h satisfies

$$\|R_h T - T\|_j \le ch^{m-j} \|T\|_m, \quad 0 \le j \le 1,\ j \le m \le k+2.$$

Furthermore, we define the modified canonical interpolation operator Π_h by (3.7), where the operator Π_h^0 is defined in complete analogy with the lowest order case, by setting the degrees freedom associated with $\Delta_0(\mathcal{T}_h)$ and $\Delta_1(\mathcal{T}_h)$ equal to zero. Then the operator Π_h satisfies (2.7) and (2.8), and the error bound

$$\|\Pi_h T - T\|_0 \le ch^m \|T\|_m, \quad 1 \le m \le k+2. \tag{4.2}$$

As above, the interpolation estimates (4.1) and (4.2), and the error bounds (2.9) and (2.10), leads to the following error estimates.

Theorem 4.1. *Let (S, u) denote the unique critical point of the Hellinger-Reissner funtional over $H(\mathrm{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^2)$ and let (S_h, u_h) be the unique critical point over $\Sigma_h \times V_h$. Then*

$$\begin{aligned} \|S - S_h\|_0 &\le ch^m \|S\|_m, \qquad 1 \le m \le k+2, \\ \|\operatorname{div} S - \operatorname{div} S_h\|_0 &\le ch^m \|\operatorname{div} S\|_m, \qquad 0 \le m \le k+1, \\ \|u - u_h\|_0 &\le ch^m \|u\|_{m+1}, \qquad 1 \le m \le k+1. \end{aligned}$$

5. Some properties of vector fields and matrix fields

It remains to prove the claimed dimension formula for $\mathcal{M}_k(K)$, given in Theorem 7.2 below. To do so we introduce a related space, $\mathcal{N}_k(K)$, in the next section and determine its dimension. For the analysis we need some basic notations and properties of differential operators on vector fields and matrix fields, which are the subject of the present section.

5.1. **Identities for vector fields.** Let n denote a fixed unit vector in $\mathbb{R}^3$, $P_n = nn'$ the orthogonal projection onto $\mathbb{R}n$, $f = n^\perp$ the plane orthogonal to n, and $Q_n = I - P_n$ is the orthogonal projection onto f. Furthermore, set

$$C_n = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} = -\operatorname{vec}^{-1} n,$$

so that $C_n v = v \times n$. The following identities are easily checked:

$$C_n' = -C_n, \quad C_n^2 = -Q_n, \quad C_n P_n = 0, \quad C_n Q_n = Q_n C_n = C_n.$$

For any vector field $v = (v_1, v_2, v_3)'$ in $\mathbf{R}^3$, we obviously have $v = P_n v + Q_n v$ and

$$\operatorname{curl} v = \operatorname{curl} P_n v + \operatorname{curl} Q_n v = P_n \operatorname{curl} P_n v + P_n \operatorname{curl} Q_n v + Q_n \operatorname{curl} P_n v + Q_n \operatorname{curl} Q_n v.$$

It is elementary to verify that

$$P_n \operatorname{curl} P_n v = 0, \quad Q_n \operatorname{curl} Q_n v = C_n \frac{\partial v}{\partial n},$$

if $n = e_3 = (0, 0, 1)'$. In view of the transformation formula (2.1), these identities hold for an arbitrary unit vector n. Furthermore, we define

$$\operatorname{rot}_f v = P_n \operatorname{curl} v = P_n \operatorname{curl} Q_n v = -(\operatorname{div} C_n v) n, \quad \operatorname{curl}_f v = Q_n \operatorname{curl} P_n v.$$

With this notation we obtain the decomposition

$$\operatorname{curl} v = \operatorname{rot}_f v + \operatorname{curl}_f v + C_n \frac{\partial v}{\partial n}. \tag{5.1}$$

We also define the tangential gradient $\operatorname{grad}_f \phi = Q_n \operatorname{grad} \phi$, for a scalar field ϕ. For a vector field v, $\operatorname{grad}_f v = (\operatorname{grad} v) Q_n$ is the matrix field with rows equal to the tangential gradients of the components of v, and we let

$$\epsilon_f(v) = \frac{1}{2}\{\operatorname{grad}_f(Q_n v) + [\operatorname{grad}_f(Q_n v)]'\} = Q_n\, \epsilon(v) Q_n$$

be the tangential part of the symmetric gradient. Note that the definitions of $\operatorname{grad}_f v$, $\epsilon_f(v)$, $\operatorname{curl}_f v$, and $\operatorname{rot}_f v$ do not depend on the choice of the unit normal n to f. The identity

$$\operatorname{curl}_f v = -C_n \ \operatorname{grad}(n'v) = -C_n \ \operatorname{grad}_f(n'v) \tag{5.2}$$

can be easily verified in the special case $n = e_3$, and holds in general.

5.2. **Identities for matrix fields.** We extend the operators curl, curl_f and rot_f to act on and yield 3×3 matrix fields by applying the vector operations row-wise. More precisely, $\operatorname{rot}_f S = (\operatorname{curl} S) P_n = (\operatorname{curl} S Q_n) P_n$ and $\operatorname{curl}_f S = (\operatorname{curl} S P_n) Q_n$. We notice that, for any constant matrix A, $\operatorname{curl} AS = A \operatorname{curl} S$. We also recall that $\operatorname{curl}^* S = (\operatorname{curl} S')'$ is the corresponding operator obtained by applying the curl operation to each column. The corresponding column operators rot_f^* and curl_f^* are defined similarly, i.e., $\operatorname{rot}_f^* S = P_n \operatorname{curl}^* S = P_n \operatorname{curl}^* Q_n S$ and $\operatorname{curl}_f^* S = Q_n \operatorname{curl}^* P_n S$. For a given row vector v, $\operatorname{grad}_f^* v = (\operatorname{grad}_f v')' = Q_n (\operatorname{grad} v')'$, which is the matrix whose columns are the tangential gradients of the components of v.

We now extend the decomposition (5.1) to curl and curl^*. It is easy to see that $C_n S$ results in C_n applied to the columns of S, while $-SC_n$ is C_n applied row-wise. It follows that

$$\operatorname{curl} S = \operatorname{curl}_f S + \operatorname{rot}_f S - \frac{\partial S}{\partial n} C_n,$$

and

$$\operatorname{curl}^* S = \operatorname{curl}_f^* S + \operatorname{rot}_f^* S + C_n \frac{\partial S}{\partial n}, \tag{5.3}$$

where $\partial S/\partial n$ is obtained by taking the directional derivative of each component. Furthermore, the identities

$$\operatorname{curl}_f S = (\operatorname{grad}_f Sn)C_n, \quad \operatorname{curl}_f^* S = -C_n \ \operatorname{grad}_f^* n'S \tag{5.4}$$

are just matrix analogues of (5.2). Note also that from the definitions of the operators rot_f and rot_f^* we have

$$\begin{aligned} P_n(\operatorname{curl}\operatorname{curl}^* S)P_n &= P_n(\operatorname{rot}_f \operatorname{curl}^* S) = \operatorname{rot}_f(P_n \operatorname{curl}^* S) \\ &= \operatorname{rot}_f \operatorname{rot}_f^* S = \operatorname{rot}_f \operatorname{rot}_f^*(Q_n S Q_n). \end{aligned} \tag{5.5}$$

We will also need exact sequences relating spaces of functions defined on a two dimensional space. Let $f = n^\perp$. In analogy with (2.6) the following two dimensional complexes are exact:

$$\mathbb{T}_f \hookrightarrow \mathcal{P}_{k+3}(f; Q_n\mathbb{R}^3) \xrightarrow{\epsilon_f} \mathcal{P}_{k+2}(f; Q_n\mathbb{S}Q_n) \xrightarrow{\operatorname{rot}_f \operatorname{rot}_f^*} \mathcal{P}_k(f; \mathbb{R}P_n) \to 0, \tag{5.6}$$

(5.7)

$$\mathcal{P}_1(f;\mathbb{R}) \hookrightarrow \mathcal{P}_{k+3}(f;\mathbb{R}) \xrightarrow{\operatorname{grad}_f \operatorname{grad}_f^*} \mathcal{P}_{k+1}(f; Q_n\mathbb{S}Q_n) \xrightarrow{\operatorname{rot}_f} \mathcal{P}_k(f; Q_n\mathbb{S}P_n) \to 0.$$

Here, $\mathbb{T}_f$ is the 3-dimensional space of vector fields on f of the form $v(x) = Q_n w(x)$ for some $w \in \mathbb{T}$.

To close this section, we introduce a useful operator Λ_f. Let f be a plane with unit normal n. For any symmetric matrix field S defined on a neighborhood on f, we define $\Lambda_f(S) : f \to Q_n\mathbb{S}Q_n$ by

$$\Lambda_f(S) = 2\,\epsilon_f(Sn) - Q_n \partial_n S Q_n,$$

where $\partial_n S := \partial S/\partial n$. Hence, $\Lambda_f(S)$ is a tangential symmetric matrix field defined on f. Note that $\Lambda_f(S)$ depends on the choice of normal: if we reverse the sign of n, we reverse the sign of $\Lambda_f(S)$ as well. For future reference, we note that if $T = \epsilon(v)$, where v is a vector field then we have

$$2\,\epsilon_f(Tn) = \operatorname{grad}_f \operatorname{grad}_f^*(n'v) + Q_n \partial_n\, \epsilon(v) Q_n.$$

Hence, we obtain that

$$\Lambda_f\big(\epsilon(v)\big) = \operatorname{grad}_f \operatorname{grad}_f^*(n'v). \tag{5.8}$$

The tangential–normal components of the matrix field $\operatorname{curl}\operatorname{curl}^* S$ on f can be expressed in terms of $\Lambda_f(S)$. Indeed, by the definition of the operator rot_f and (5.3) we have

$$C_n(\operatorname{curl}\operatorname{curl}^* S)P_n = C_n \operatorname{rot}_f \operatorname{curl}^* S = \operatorname{rot}_f C_n(\operatorname{curl}_f^* S + \operatorname{rot}_f^* S + C_n \partial_n S).$$

However, $C_n \operatorname{rot}_f^* S = 0$ and, by (5.3), $C_n \operatorname{curl}_f^* S = Q_n \operatorname{grad}_f^* n'S$. Hence,

$$C_n(\operatorname{curl}\operatorname{curl}^* S)P_n = \operatorname{rot}_f Q_n(\operatorname{grad}_f^* n'S - \partial_n S)Q_n = \operatorname{rot}_f \Lambda_f(S). \tag{5.9}$$

6. Polynomial matrix fields on a single tetrahedron

In this section we will compute the dimension of the polynomial space

$$\mathcal{N}_k = \mathcal{N}_k(K) = \{\, S \in \mathcal{P}_k(K;\mathbb{S}) \,|\, Q_n S Q_n = \Lambda_f(S) = 0,\ f \in \Delta_2(K)\}. \tag{6.1}$$

where $K \subset \mathbb{R}^3$ is a fixed tetrahedron and $k \geq 3$. The final result is obtained in Theorem 6.6, and applied in the next section to obtain the dimension of the space $\mathcal{M}_k(K)$. We start by introducing some additional notation.

If $f \in \Delta_2(K)$, we denote by h_f the perpendicular distance from the opposite vertex to f and by $n = n_f$ the outward normal vector to f. If e is an edge, we let $s = s_e$ denote one of the unit vectors parallel to e. When the edge e belongs to the face f, we write $m = m_{e,f}$ for the unit vector in f, normal to e, pointing from e into f. See Figure 1. When the notations f_+ and f_- are used to denote two faces, the corresponding normals will be denoted n_+ and n_-, and the perpendicular distances h_+ and h_-, respectively. The notations m_+ and m_- will also be used to denote m_{e,f_+} and m_{e,f_-} where e is the edge common to f_+ and f_-.

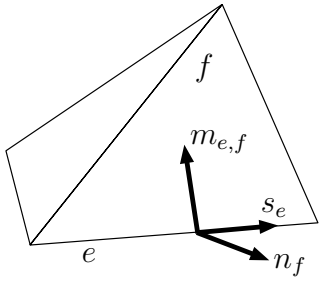

FIGURE 1. The $(n_f, s_e, m_{e,f})$ coordinate system for a face f and edge e of the tetrahedron K.

The barycentric coordinates on K will be labelled by the faces. That is, they are $\lambda_f \in \mathcal{P}_1(K;\mathbb{R})$ determined by $\lambda_f \equiv 0$ on f and $\sum_f \lambda_f \equiv 1$ on K. We recall that $\operatorname{grad} \lambda_f = -n_f/h_f$. Let $g \in \Delta(K)$ be a face of dimension m with vertices $x_{i_0}, x_{i_1}, \ldots, x_{i_m}$. For $m = 0$, g is a vertex, for $m = 1, g$ is an edge, and so on. We define the bubble functions $b_g = \lambda_{f_{i_0}} \lambda_{f_{i_1}} \cdots \lambda_{f_{i_m}}$, where f_{i_k} is the face opposite vertex x_{i_k}. For $d > 0$, $\mathcal{P}_d^g(K) = \operatorname{span}\{\, \lambda_{f_{i_0}}^{j_0} \lambda_{f_{i_1}}^{j_1} \cdots \lambda_{f_{i_m}}^{j_m} \,|\, j_0 + \cdots + j_m = d \,\}$, so that $\dim \mathcal{P}_d^g(K) = \dim \mathcal{P}_d(g)$. Note that if x is a vertex opposite face f, then $b_x = \lambda_f$ and $\mathcal{P}_d^x(K) = \mathbb{R}\lambda_f^d$, while $b_K = \prod_f \lambda_f$ and $\mathcal{P}_d^K = \mathcal{P}_d$. For a given face $f_0 \in \Delta_2(K)$, $b_{f_0} = b_K/\lambda_{f_0} = \prod_{f \neq f_0} \lambda_f$, and for a given edge $e \in \Delta_1(K)$, $b_e = b_K/(\lambda_{f_-}\lambda_{f_+})$, where f_- and f_+ are the faces containing e.

The monomials of degree k in the barycentric coordinates form a basis for $\mathcal{P}_k(K)$, and by grouping together terms according to which coordinates enter the monomial, we can uniquely represent any $p \in \mathcal{P}_k$ as

$$p = \sum_{g \in \Delta(K)} b_g p_g, \quad p_g \in \mathcal{P}_{k-1-\dim g}^g(K). \tag{6.2}$$

The standard Lagrangian degrees of freedom for $p \in \mathcal{P}_k(K)$ are the values of p at the vertices, the moments of p of degree at most $k-2$ on each of the edges of K, the moments of degree at most $k-3$ on each of the faces, and the moments of degree

at most $k-4$ on K. From the vertex values of p we may determine the polynomials p_g in (6.2) for $g \in \Delta_0(K)$. From these and the edge moments we may determine as well the p_g for $g \in \Delta_1(K)$, etc.

Of course analogous considerations apply to $\mathcal{P}_k(K;X)$ for X a vector space. In particular, we have the representation $p = \sum_{g\in\Delta(K)} b_g p_g$ for $p \in \mathcal{P}_k(K;X)$ where now $p_g \in \mathcal{P}^g_{k-1-\dim g}(K;X)$, the space of X-valued polynomials on K whose components with respect to a basis of X belong to $\mathcal{P}^g_{k-1-\dim g}(K)$.

If $e = f_- \cap f_+ \in \Delta_1(K)$, with $f_-, f_+ \in \Delta_2(K)$, we let $G_e \in \mathbb{S}$ be the matrix

$$G_e = n_- n'_+ + n_+ n'_-.$$

We note that $G_e s = 0$ and $m'_- G_e m_- = m'_+ G_e m_+ = 0$.

Lemma 6.1. *For $k \geq 0$, the dimension of the space*

$$\mathcal{N}^0_k = \mathcal{N}^0_k(K) := \{\, S \in \mathcal{P}_k(K;\mathbb{S}) \,|\, Q_n S Q_n = 0, \quad f \in \Delta_2(K) \,\}$$

is $(k+1)k(k-1)$.

Proof. For a subsimplex $g \in \Delta(K)$, let us first define

$$N^g = \{\, S \in \mathbb{S} \,|\, Q_n S Q_n = 0 \text{ for all faces containing } g \,\}.$$

Clearly, if $g = K$ then $\dim N^g = 6$, if $g \in \Delta_2(K)$ then $\dim N^g = 3$, and if $g \in \Delta_0(K)$ then $\dim N^g = 0$. Finally, if $g \in \Delta_1(K)$, i.e., $g = e$ is an edge, then $\dim N^g = 1$. In fact, the space N^e is then spanned by the matrix G_e introduced above.

If $S \in \mathcal{N}^0_k$ then from (6.2) we obtain the representation

$$S = \sum_{g\in\Delta(K)} b_g S_g, \quad S_g \in \mathcal{P}^g_{k-1-\dim g}(K; N^g). \tag{6.3}$$

As a consequence

$$\begin{aligned}\dim \mathcal{N}^0_k &= \sum_{g\in\Delta(K)} \dim \mathcal{P}^g_{k-1-\dim g}(K;N^g) \\ &= 6(k-1) + 6(k-1)(k-2) + (k-1)(k-2)(k-3) = (k+1)k(k-1). \qquad \square\end{aligned}$$

Before we are able to compute $\dim \mathcal{N}_k$, we need to establish several lemmas.

Lemma 6.2. *If $S \in \mathcal{N}_k$, S is zero on each edge.*

Proof. As above let $e = f_- \cap f_+ \in \Delta_1(K)$, with $f_-, f_+ \in \Delta_2(K)$. Then

$$m'_- n_+ = m'_+ n_- < 0.$$

In fact, the transformation

$$\begin{pmatrix} m_- \\ n_- \end{pmatrix} \mapsto \begin{pmatrix} n_+ \\ m_+ \end{pmatrix} \tag{6.4}$$

is a rotation in the plane orthogonal to e.

Let $S \in \mathcal{N}_k$. Since $\mathcal{N}_k \subset \mathcal{N}^0_k$ we know that $S|_e = \rho G_e$, where $\rho \in \mathcal{P}_k(e)$. Therefore, if we can show that

$$m'_+ S n_+ + m'_- S n_- = 0 \tag{6.5}$$

on e, then $\rho(m'_+ n_- + m'_- n_+) = 2\rho m'_+ n_- = 0$, and, as a consequence, S is zero on e. It therefore suffices to show (6.5).

On e, we must have $s'\Lambda_f(S)m = 0$, i.e.,

$$\partial_s(m'Sn) + \partial_m(s'Sn) - \partial_n(s'Sm) = 0,$$

where f is either f_- or f_+. By adding this property for the two faces we obtain

$$\begin{aligned}-\,&\partial_s(m'_+Sn_+ + m'_-Sn_-)\\ &= [\partial_{m_-}(s'Sn_-) - \partial_{n_-}(s'Sm_-)] - [\partial_{n_+}(s'Sm_+) - \partial_{m_+}(s'Sn_+)].\end{aligned}$$

However, the right hand side here is zero as a consequence of the fact that the transformation (6.4) is a rotation. In fact, this property implies that

$$\partial_{m_-}(v'n_-) - \partial_{n_-}(v'm_-) = \partial_{n_+}(v'm_+) - \partial_{m_+}(v'n_+)$$

for any smooth vector field v on K. Hence, we can conclude that $m'_+Sn_+ + m'_-Sn_-$ is a constant along e, and since it is zero at the vertices, (6.5) holds. □

Lemma 6.3. *Let*

$$\begin{aligned}\mathcal{N}_{k,\partial K} := \{\, U \in \mathcal{N}_k^0 \,|\, U = &\sum_{f\in\Delta_2(K)} b_f U_f,\ U_f \in \mathcal{P}_{k-3}^f(K;\mathbb{S}) \text{ and}\\ &\Lambda_f(U)|_e = 0 \text{ for each face } f \text{ and each edge } e \text{ of } f\,\}.\end{aligned}$$

Then $\dim \mathcal{N}_{k,\partial K} = 6(k^2 - 6k + 10)$.

Proof. If $U = \sum_f b_f U_f$, then $U \in \mathcal{N}_k^0$ if and only if each coefficient $U_f \in \mathcal{P}_{k-3}^f(K;\mathbb{S})$ satisfies

$$Q_n U_f Q_n = 0 \quad \text{on } f.$$

Hence, this property is assumed to hold. We have $\Lambda_f(U) = 0$ on an edge $e \subset f$ if and only if the three terms $s'\Lambda_f(U)s$, $s'\Lambda_f(U)m$ and $m'\Lambda_f(U)m$ vanish there. For any fixed unit vector t and $e \in \Delta_1(K)$, we have

$$\partial_t U = \sum_f (\partial_t b_f) U_f = -b_e\Big(\frac{t'n_+}{h_+}U_{f_-} + \frac{t'n_-}{h_-}U_{f_+}\Big) \quad \text{on } e, \tag{6.6}$$

where f_- and f_+ are the two faces meeting the edge e and we have used that $\operatorname{grad} b_{f_-} = -n_+ b_e/h_+$ on e.

Recall that $s'\Lambda_f(U)s = 2\partial_s(s'Un) - \partial_n(s'Us)$. Since $U = 0$ on e,

$$s'\Lambda_f(U)s = -\partial_n(s'Us) \quad \text{on } e.$$

However, since $Q_n U_f Q_n = 0$ on the face f, we have $s'U_{f_-}s = s'U_{f_+}s = 0$ on e. By (6.6), with $t = n$, we conclude that $s'\Lambda_f(U)s = 0$ on e.

Next, similar considerations for $s'\Lambda_f(U)m = \partial_s(m'Un) + \partial_m(s'Un) - \partial_n(s'Um)$ give

$$s'\Lambda_{f_+}(U)m_+ = \partial_{m_+}(s'Un_+) - \partial_{n_+}(s'Um_+) \quad \text{on } e.$$

Furthermore, from (6.6) we have

$$\partial_{m_+}(s'Un_+) = -m'_+ n_- b_e \frac{s'U_{f_+}n_+}{h_-}$$

on e, and, using $s'U_{f_-}m_- = s'U_{f_+}m_+ = 0$, we obtain

$$\partial_{n_+}(s'Um_+) = -b_e\frac{s'U_{f_-}m_+}{h_+} = -m'_+ n_- b_e \frac{s'U_{f_-}n_-}{h_+}.$$

It follows that

$$s'\Lambda_{f_+}(U)m_+ = m'_+n_-b_e(\frac{s'U_{f_-}n_-}{h_+} - \frac{s'U_{f_+}n_+}{h_-}) \quad \text{on } e.$$

We have therefore shown that $s'\Lambda_f(U)m = 0$ on all edges if and only if,

$$\frac{s'U_{f_-}n_-}{h_+} = \frac{s'U_{f_+}n_+}{h_-} \tag{6.7}$$

on all edges of K.

Finally, we consider $m'\Lambda_f(U)m = 2\partial_m(m'Un) - \partial_n(m'Um)$. Using the fact that both $m'_-U_{f_-}m_-$ and $m'_+U_{f_+}m_+$ vanish on e, we obtain from (6.6) that, on e,

$$\begin{aligned}\partial_{n_+}m'_+Um_+ &= -b_e\frac{m'_+U_{f_-}m_+}{h_+}\\ &= -\frac{b_e}{h_+}[(m'_+n_-)^2n'_-U_{f_-}n_- + 2m'_+m_-m'_+n_-m'_-U_{f_-}n_-],\end{aligned}$$

so

$$m'_+\Lambda_{f_+}(U)m_+ = m'_+n_-b_e[m'_+n_-\frac{n'_-U_{f_-}n_-}{h_+} - 2(\frac{m'_+U_{f_+}n_+}{h_-} - m'_+m_-\frac{m'_-U_{f_-}n_-}{h_+})].$$

Hence, $m'_+\Lambda_{f_+}(U)m_+$ vanishes on e if and only if,

$$n'_-U_{f_-}n_- = \frac{2h_+}{m'_+n_-}(\frac{m'_+U_{f_+}n_+}{h_-} - m'_+m_-\frac{m'_-U_{f_-}n_-}{h_+}) \quad \text{on } e.$$

Note that this condition is not symmetric in f_- and f_+. We thus obtain two conditions for each edge e.

Since $U_f \in \mathcal{P}^f_{k-3}(K;\mathbb{S})$, with $Q_nU_fQ_n = 0$ on f, it follows that U_f is uniquely determined by the vector field $v_f := U_fn \in \mathcal{P}_{k-3}(f;\mathbb{R}^3)$. The analysis above shows that $U = \sum_f b_fU_f \in \mathcal{N}_{k,\partial K}$ if and only if these vector fields satisfy

(A) $\dfrac{s'v_{f_-}}{h_+} = \dfrac{s'v_{f_+}}{h_-}$ on e, and

(B) $n'_-v_{f_-} = \dfrac{2h_+}{m'_+n_-}(\dfrac{m'_+v_{f_+}}{h_-} - m'_+m_-\dfrac{m'_-v_{f_-}}{h_+})$ on e,

whenever an edge e is shared by faces f_- and f_+. Therefore, there is an isomorphism between $\mathcal{N}_{k,\partial K}$ and

$$\{\,(v_f) \in \prod_{f\in\Delta_2(K)} \mathcal{P}_{k-3}(f;\mathbb{R}^3) \,|\, \text{the } v_f \text{ satisfy (A) and (B)}\,\}. \tag{6.8}$$

To compute the dimension of the space (6.8) we consider the relations (A) and (B) at a fixed vertex x of K. If (v_f) is an element of the space (6.8) define $z \in \mathbb{R}^3$ by

$$s'z = h_fs'v_f(x) \tag{6.9}$$

for s chosen as tangents to each edge e meeting x, and where f is a face meeting e. Note that the vector z is well–defined as a consequence of condition (A), and that for each face f containing x we have $h_ft'v_f(x) = t'z$ for all vectors t which are tangential to the face f. Using the expansion $n_- = [m_+ - (m'_+m_)) - m_-]/m'_+n_-$ we can then rewrite condition (B) at the vertex x as

$$h_fn'_fv_f(x) = 2n'_fz. \tag{6.10}$$

From this discussion we can conclude that the dimension of the space (6.8), and hence $\dim \mathcal{N}_{k,\partial K}$, is at least $6(k^2-6k+10)$. To see this observe that

$$\dim \prod_f \mathcal{P}_{k-3}(f,\mathbb{R}^3) = 6(k-2)(k-1).$$

Furthermore, the conditions (A) and (B) represent a total of $6\cdot 3\cdot(k-4) = 18(k-4)$ constraints in the interior of the edges and $4\cdot 6 = 24$ constraints at the vertices. Since $6(k-2)(k-1) - 18(k-4) - 24 = 6(k^2-6k+10)$, this is a lower bound for $\dim \mathcal{N}_{k,\partial K}$.

We complete the proof by showing that elements of the space (6.8) are determined by $6(k^2-6k+10)$ degrees of freedom, in fact by degrees of freedom corresponding to the space

$$\prod_{x\in\Delta_0(K)} \mathbb{R}^3 \times \prod_{e\in\Delta_1(K)} \mathcal{P}_{k-5}(e;\mathbb{R}^3) \times \prod_{f\in\Delta_2(K)} \mathcal{P}_{k-6}(f;\mathbb{R}^3).$$

To see this, for each vertex x pick a vector $z = z(x) \in \mathbb{R}^3$ and choose $v_f(x)$ such that the relations (6.9) and (6.10) hold for all faces meeting x. This determines the vectors $v_f(x)$ for all vertices $x \in \partial f$. We then define $h_f s' v_f$ on each edge by the standard interior degrees of freedom, and $m' v_f$ with respect to both faces meeting e are determined similarly. These degrees of freedom on the edges correspond to the space $\prod_e \mathcal{P}_{k-5}(e;\mathbb{R}^3)$. The normal components $n_f' v_f$ are determined on each edge by condition (B). Finally, we must apply the interior degrees of freedom to v_f on f. We conclude that elements of the space $\mathcal{N}_{k,\partial K}$ are determined by $12 + 18\dim \mathcal{P}_{k-5}(e) + 12\dim \mathcal{P}_{k-6}(f) = 6(k^2-6k+10)$ degrees of freedom. □

For $U \in \mathcal{N}_{k,\partial K}$ and $f \in \Delta_2(K)$, $\Lambda_f(U)$ is a polynomial vanishing on ∂f, and so the quotient $\Lambda_f(U)/b_f$ is a polynomial. We define $T_f : \mathcal{N}_{k,\partial K} \to \mathcal{P}_{k-4}(f, Q_n \mathbb{S} Q_n)$ by

$$T_f(U) = -h_f \Lambda_f(U)/b_f.$$

Lemma 6.4. *If $f_-, f_+ \in \Delta_2(K)$, $e = f_- \cap f_+$, and s is a unit vector parallel to e, then*

$$s' T_{f_+}(U) s = s' T_{f_-}(U) s \quad \text{on } e, \quad U \in \mathcal{N}_{k,\partial K}.$$

Proof. First we show that

$$\partial_{m_+} s' \Lambda_{f_+}(U) s = \partial_{m_-} s' \Lambda_{f_-}(U) s \quad \text{on } e. \tag{6.11}$$

Recall that $\partial_{m_+} b_{f_+} = -m_+' n_- b_e/h_-$, $\partial_{m_-} b_{f_-} = -m_-' n_+ b_e/h_+$ on e and $m_+' n_- = m_-' n_+$. We have on an edge e, $\partial_m s' \Lambda_f(U) s = 2\partial_s \partial_m s' U n - \partial_m \partial_n s' U s$. Using (6.6) we obtain

$$\partial_{m_+} s' U n_+ = -b_e \frac{m_+' n_-}{h_-} s' U_{f_+} n_+,$$

which is symmetric in f_- and f_+ as a consequence of (6.7) and $m_+' n_- = m_-' n_+$.

The identity (6.11) will follow if we show that $\partial_{m_+}\partial_{n_+} s' U s = \partial_{m_-}\partial_{n_-} s' U s$. Consider first the term

$$V = b_{f_-} U_{f_-} + b_{f_+} U_{f_+}.$$

Since $Q_n U_f Q_n = 0$ for $f = f_-, f_+$ and $\operatorname{grad} b_{f_-} = -\frac{b_e}{h_+} n_+$ on f_+ we derive that at the edge e

$$\begin{aligned}\partial_{m_+}\partial_{n_+} s' V s &= -b_e[\frac{1}{h_+} s'\partial_{m_+} U_{f_-} s + \frac{m'_+ n_-}{h_-} s'\partial_{n_+} U_{f_+} s] \\ &= -(m'_+ n_-) b_e[\frac{1}{h_+} s'\partial_{n_-} U_{f_-} s + \frac{1}{h_-} s'\partial_{n_+} U_{f_+} s],\end{aligned}$$

and this expression is symmetric in f_- and f_+. Finally, consider terms of the form $W = b_f U_f$, where f is neither f_- nor f_+. In this case $b_f = \lambda_{f_-}\lambda_{f_+}\lambda$, where λ is the barycentic coordinate associated the fourth face ($\neq f, f_-, f_+$) of K, and on e we have

$$\partial_{m_+}\partial_{n_+} s' W s = \frac{m'_+ n_-}{h_- h_+} \lambda s' U_f s.$$

This is again symmetric in f_- and f_+. We have therefore established (6.11).

Now, by definition, $h_+ \Lambda_{f_+}(U) = -b_{f_+} T_{f_+}(U)$. Therefore

$$(m'_+ n_-)^{-1} h_+ h_- \partial_{m_+} s' \Lambda_{f_+}(U) s = b_e s' T_{f_+}(U) s \quad \text{on } e.$$

By (6.11), the left-hand side is unchanged if we interchange the subscripts + and −, so the same must be true of the right-hand side. □

Lemma 6.5. *Let $(T_f) \in \prod_{f\in\Delta_2(K)} \mathcal{P}_k(f; Q_n \mathbb{S} Q_n)$ be such that $s' T_{f_-} s = s' T_{f_+} s$ on e, whenever $e = f_- \cap f_+$, $f_-, f_+ \in \Delta_2(K)$. Then there exist an $S \in \mathcal{P}_k(K;\mathbb{S})$ such that $Q_n S Q_n = T_f$ for all $f \in \Delta_2(K)$.*

Proof. We will define $S \in \mathcal{P}_k(K;\mathbb{S})$ by first specifying its vertex values, then specifying its moments of degree at most $k-2$ on the edges, then its moments of degree at most $k-3$ on faces, and then the moments of degree at most $k-4$ over the interior of K.

Let x be a vertex. We define the matrix $S(x) \in \mathbb{S}$ by specifying the values $s_i' S(x) s_j$ where the s_i are the tangents to the edges e_i meeting at x (and so the s_i form a basis for $\mathbb{R}^3$). Namely we take $s_i' S(x) s_j = s_i' T_f s_j$ with f the face containing e_i and e_j. If $i = j$ there are two possible choices of the face f, but they give the same result by assumption.

For the interior degrees of freedom on an edge e we use the basis s, m_-, m_+ of $\mathbb{R}^3$, and let $T_e \in \mathcal{P}_k(e;\mathbb{S})$ be given by

$$\begin{aligned} &s' T_e s = s' T_{f_-} s = s' T_{f_+} s, \quad s' T_e m_- = s' T_{f_-} m_-, \quad s' T_e m_+ = s' T_{f_+} m_+, \\ &m'_- T_e m_- = m'_- T_{f_-} m_-, \quad m'_+ T_e m_+ = m'_+ T_{f_+} m_+, \quad m'_- T_e m_+ = 0. \end{aligned}$$

Then we define $S|_e$ by

$$\int_e (S - T_e) V \, ds = 0, \quad V \in \mathcal{P}_{k-2}(e;\mathbb{S}).$$

Similarly, for the interior degrees of freedom on each face we let $Q_n S Q_n$ inherit the moments from T_f, while the data for $S P_n$ is taken to be zero. The interior degrees of freedom on K are all taken to be zero. □

As a consequence of the two previous lemmas, there is a map

$$\mathcal{N}_{k,\partial K} \to \mathcal{P}_{k-4}(K;\mathbb{S}), \quad U \mapsto S(U), \tag{6.12}$$

such that $Q_n S(U) Q_n = T_f(U)$ for all faces $f \in \Delta_2(K)$.

We are finally ready to compute the dimension of the space $\mathcal{N}_k$ defined in (6.1).

Theorem 6.6. *For $k \geq 3$, the dimension of the space $\mathcal{N}_k$ is $k(k^2-6k+11)$.*

Proof. Let $S \in \mathcal{N}_k$. By Lemma 6.2, S must be zero on each edge and so can be written $S = \sum_f b_f S_f + b_K S_K$, where $S_f \in \mathcal{P}^f_{k-3}(K;\mathbb{S})$ and $S_K \in \mathcal{P}_{k-4}(K;\mathbb{S})$. Now $\epsilon_f(b_K S_K n_f)$ vanishes on f since b_K does, while

$$\partial_n(b_K Q_n S_K Q_n) = (\partial_n b_K) Q_n S_K Q_n = -\frac{b_f}{h_f} Q_n S_K Q_n \quad \text{on } f.$$

Thus

$$\Lambda_f(b_K S_K) = \frac{b_f}{h_f} Q_n S_K Q_n. \tag{6.13}$$

In particular, $\Lambda_f(b_K S_K)$ vanishes on ∂f. It follows that if $S \in \mathcal{N}_k$ and we define $U = \sum_f b_f S_f$ then $U \in \mathcal{N}_{k,\partial K}$. Therefore, the map $(U, S_K) \mapsto U + b_K S_K$ defines an isomorphism from

$$\{\, (U, S_K) \in \mathcal{N}_{k,\partial K} \times \mathcal{P}_{k-4}(K;\mathbb{S}) \,|\, Q_n S_K Q_n = T_f(U) \quad \text{on each face } f \,\}$$

onto $\mathcal{N}_k$.

Finally, note that a matrix field of the form $b_K V$, $V \in \mathcal{P}_{k-4}(K;\mathbb{S})$, belongs to $\mathcal{N}_k$ if and only if $V \in \mathcal{N}^0_{k-4}$. Therefore, using the map (6.12), the mapping

$$\mathcal{N}_{k,\partial K} \times \mathcal{N}^0_{k-4} \to \mathcal{N}_k, \quad (U, V) \mapsto (U, S(U) + V),$$

is an isomorphism. It follows that $\dim \mathcal{N}_k = \dim \mathcal{N}_{k,\partial K} + \dim \mathcal{N}^0_{k-4}$ and using Lemma 6.3 and Lemma 6.1 we get $\dim \mathcal{N}_k = 6(k^2-6k+10)+(k-3)(k-4)(k-5) = k(k^2-6k+11)$. □

7. The space of divergence-free matrix fields with vanishing normal traces

Recall that the space

$$\mathcal{M}_k = \mathcal{M}_k(K) = \{\, S \in \mathcal{P}_k(K;\mathbb{S}) \,|\, \operatorname{div} S = 0 \quad \text{on } K, \quad P_n S = 0, \quad f \in \Delta_2(K) \,\}$$

appears in the degrees of freedom for the finite element space $\Sigma_h \subset H(\operatorname{div}, \Omega; \mathbb{S})$ introduced in Sections 3 and 4. Therefore, a derivation of the dimension of this space is fundamental for our theory, while a construction of a (dual) basis for the space $\mathcal{M}_k$ is necessary for the implementation of the method. The dimension formula will be a simple consequence of the following lemma, in which $\mathcal{P}^0_{k+3}(K;\mathbb{R}^3) := \{v \in \mathcal{P}_{k+3}(K;\mathbb{R}^3) \,|\, v \equiv 0 \quad \text{on } \partial K \,\} = b_K \mathcal{P}_{k-1}(K;\mathbb{R}^3)$.

Lemma 7.1. (1) *The operator* $\operatorname{curl}\operatorname{curl}^*$ *maps* $\mathcal{N}_{k+2}(K)$ *onto* $\mathcal{M}_k(K)$.
(2) $\{\, T \in \mathcal{N}_{k+2} \,|\, \operatorname{curl}\operatorname{curl}^* T = 0 \,\} = \epsilon[\mathcal{P}^0_{k+3}(K;\mathbb{R}^3)]$.
(3) *The following sequence is exact:*

$$0 \to \mathcal{P}^0_{k+3}(K;\mathbb{R}^3) \xrightarrow{\epsilon} \mathcal{N}_{k+2}(K) \xrightarrow{\operatorname{curl}\operatorname{curl}^*} \mathcal{M}_k(K) \to 0. \tag{7.1}$$

Proof. It follows directly from (5.5) and (5.9) that $\operatorname{curl}\operatorname{curl}^* \mathcal{N}_{k+2} \subset \mathcal{M}_k$. Hence, to prove the first statement we need only show that $\mathcal{M}_k \subset \operatorname{curl}\operatorname{curl}^* \mathcal{N}_{k+2}$. Let $S \in \mathcal{M}_k$. Since $\operatorname{div} S = 0$, it follows from the exactness of the complex (2.6) that there is a $T \in \mathcal{P}_{k+2}(K;\mathbb{S})$ such that $S = \operatorname{curl}\operatorname{curl}^* T$. The proof will be completed by constructing a vector field $u \in \mathcal{P}_{k+3}(K;\mathbb{R}^3)$ such that

$$Q_n\big(T - \epsilon(u)\big)Q_n = 0, \quad \Lambda_f\big(T - \epsilon(u)\big) = 0, \quad \text{on each face } f. \tag{7.2}$$

Note that since $S \in \mathcal{M}_k$ it follows from (5.5) that

$$\operatorname{rot}_f \operatorname{rot}_f^* Q_n T Q_n = P_n(\operatorname{curl}\operatorname{curl}^* T)P_n = P_n S P_n = 0 \quad \text{on each face } f.$$

Hence, from the exact sequence (5.6) we conclude that for each face $f \in \Delta_2(K)$ there is a vector field $v_f \in \mathcal{P}_{k+3}(f;\mathbb{R}^3)$, with $P_n v_f = 0$, such that $Q_n T Q_n = \epsilon_f(v_f)$. The vector fields v_f are uniquely determined up to a 2D rigid motion and hence we may normalize them so that $\int_e s' v_f \, ds = 0$ on each edge $e \subset f$. Since $\partial_s(s' v_{f_-}) = s' T s = \partial_s(s' v_{f_+})$ on each edge, we obtain that $P_s v_{f_-} = P_s v_{f_+}$ on each edge $e = f_+ \cap f_-$. As a consequence, there is $v \in \mathcal{P}_{k+3}(K;\mathbb{R}^3)$ such that $Q_n v Q_n = v_f$ on each face f. Then $Q_n\, \epsilon(v) Q_n = \epsilon_f(v_f) = Q_n T Q_n$, i.e., $Q_n U Q_n = 0$ on each face f, where $U = T - \epsilon(v)$. This implies, in particular, that Us and $\operatorname{grad}(s'Us)$ vanish on each edge $e \in \Delta_1(K)$. Therefore,

$$s' \Lambda_f(U) s = 2\partial_s(n' U s) - \partial_n(s' U s) = 0 \quad \text{on } \partial f \tag{7.3}$$

for each face f.

Next, observe that, by (5.9), $\operatorname{rot}_f \Lambda_f(U) = C_n S P_n = 0$. Hence, (5.7) implies that there is a scalar field $q_f \in \mathcal{P}_{k+4}(f;\mathbb{R})$, uniquely determined up to a linear function on f, such that $\Lambda_f(U) = \operatorname{grad}_f \operatorname{grad}_f^* q_f$. On each edge $e \in \partial f$, we have by (7.3) that $0 = s' \Lambda_f(U) s = \partial_s^2 q_f$. It follows that we can assume that $q_f \equiv 0$ on ∂f. Hence, there exists another vector field w in $\mathcal{P}_{k+3}(K;\mathbb{R}^3)$ such that $Q_n w = 0$ and $n' w = q_f$ on each face. Recall by (5.8) that $\Lambda_f\big(\epsilon(w)\big) = \operatorname{grad}_f \operatorname{grad}_f^* q_f = \Lambda_f(U)$. Hence, if we let $u = v + w$, then the relation (7.2) holds. This proves the first statement.

We now prove the second statement. If $T = \epsilon(v)$ for some $v \in \mathcal{P}^0_{k+3}(K;\mathbb{R}^3)$, then $\operatorname{curl}\operatorname{curl}^* T = 0$, and, by (5.8),

$$Q_n T Q_n = \epsilon_f(Q_n v), \quad \Lambda_f(T) = \operatorname{grad}_f \operatorname{grad}_f^*(n' v) \quad \text{on } f, \tag{7.4}$$

for each face f. Since v vanishes on f, the right hand sides of these equations vanish, and so T belongs to $\mathcal{N}_{k+2}(K)$. Conversely, if $T \in \mathcal{N}_{k+2}(K)$, then, by the exactness of the sequence (2.6), $T = \epsilon(v)$ for some $v \in \mathcal{P}_{k+3}(K;\mathbb{R}^3)$, which is determined uniquely if we require that

$$\int_e s' v \, ds = 0, \quad e \in \Delta_1(K). \tag{7.5}$$

(The functionals $v \mapsto \int_e s' v \, ds$, $e \in \Delta_1(K)$, form a set of degrees of freedom for $\mathbb{T}$, the null space of ϵ.) From the first equation in (7.4) and (7.5), we find that $Q_n v$ vanishes on each face f. Therefore the entire vector v vanishes on each edge e. Using the second equation in (7.4), we see that $n' v$ vanishes on each face as well, so $v \in \mathcal{P}^0_{k+3}(K;\mathbb{R}^3)$. This completes the proof of the second statement.

The third statement is an immediate consequence of the first two and the fact that $\mathbb{T} \cap \mathcal{P}^0_{k+3}(K;\mathbb{R}^3) = 0$. □

Theorem 7.2. *For $k \geq 4$ the space $\mathcal{M}_k(K)$ has dimension $(k+2)(k-2)(k-3)/2$.*

Proof. Using first the short exact sequence in the lemma and then the dimension formula in Theorem 6.6, we get

$$\begin{aligned}\dim \mathcal{M}_k &= \dim \mathcal{N}_{k+2} - \dim \epsilon[\mathcal{P}^0_{k+3}(K;\mathbb{R}^3)] \\ &= (k+2)(k^2-2k+3) - \dim \mathcal{P}_{k-1}(K;\mathbb{R}^3) = (k+2)(k-2)(k-3)/2. \end{aligned}$$ □

To conclude, we construct a basis for the space $\mathcal{M}_k(K)$ for $k = 4$ and $k = 5$. (Alternatively a basis could be constructed for any k using computational algebra software.) For this we use the following lemma, similar to Lemma 7.1.

Lemma 7.3. (1) *The operator* $\operatorname{curl}\operatorname{curl}^*$ *maps* $b_K\mathcal{N}^0_{k-2}(K)$ *onto* $\mathcal{M}_k(K)$.
(2) $\{\, T \in b_K\mathcal{N}^0_{k-2} \mid \operatorname{curl}\operatorname{curl}^* T = 0 \,\} = \epsilon[b_K^2\mathcal{P}_{k-5}(K;\mathbb{R}^3)]$.
(3) *The following sequence is exact:*

$$0 \to b_K^2\mathcal{P}_{k-5}(K;\mathbb{R}^3) \xrightarrow{\epsilon} b_K\mathcal{N}^0_{k-2}(K) \xrightarrow{\operatorname{curl}\operatorname{curl}^*} \mathcal{M}_k(K) \to 0. \tag{7.6}$$

Proof. Note that $b_K\mathcal{N}^0_{k-2} \subset \mathcal{N}_{k+2}$ by (6.13), and so $\operatorname{curl}\operatorname{curl}^* b_K\mathcal{N}^0_{k-2} \subset \mathcal{M}_k(K)$.

First we prove 2. Let $w \in \mathcal{P}_{k-5}(K;\mathbb{R}^3)$. By the Leibniz rule

$$\epsilon(b_K^2 w) = b_K^2\,\epsilon(w) + b_K[(\operatorname{grad} b_K)w' + w(\operatorname{grad} b_K)'].$$

Clearly $b_K\,\epsilon(w) \in \mathcal{N}^0_{k-2}$, and, recalling that $\operatorname{grad} b_K = -b_f n_f/h_f$, we see that $(\operatorname{grad} b_K)w' + w(\operatorname{grad} b_K)' \in \mathcal{N}^0_{k-2}$. Thus $\epsilon(b_K^2 w) \in b_K\mathcal{N}^0_{k-2}$, giving the inclusion $\supset$. Conversely, if $T \in b_k\mathcal{N}^0_{k-2}$ with $\operatorname{curl}\operatorname{curl}^* T = 0$, then, by Lemma 7.1, $T = \epsilon(b_K v)$ for some $v \in \mathcal{P}_{k-1}(K;\mathbb{R}^3)$ and we need to show that $v = 0$ on ∂K. Using the Leibniz rule and the fact that T vanishes on ∂K, we get that $nv' + vn'$ vanishes on each face. We conclude that v vanishes on the face, using the elementary identity $v = (I + Q_n)(nv' + vn')n/2$.

It follows that

$$\dim[\operatorname{curl}\operatorname{curl}^*(b_K\mathcal{N}^0_{k-2})] = \dim\mathcal{N}^0_{k-2} - \dim\mathcal{P}_{k-5}(K;\mathbb{R}^3) = \dim\mathcal{M}_k,$$

where we have used Lemma 6.1 and Theorem 7.2. The exactness of (7.6), and so also the first statement of the lemma, follows. □

Thus for $k = 4$, $\operatorname{curl}\operatorname{curl}^*$ is injective on $b_K\mathcal{N}^0_2$, and so a basis for $\mathcal{M}_4 = \operatorname{curl}\operatorname{curl}^*(b_K\mathcal{N}^0_2)$ is computable directly from a basis for $\mathcal{N}^0_2$, which may be obtained directly from (6.3).

Now let $k = 5$. The map $\operatorname{curl}\operatorname{curl}^*$ is not injective on $b_K\mathcal{N}^0_3$, but has a kernel of dimension 3. In this case, the representation (6.3) presents an arbitrary element $S \in \mathcal{N}^0_3$ as

$$S = \sum_{e\in\Delta_1(K)} b_e S_e + \sum_{f\in\Delta_2(K)} b_f S_f, \quad S_e \in \mathcal{P}^e_1(K;N^e),\ S_f \in N^f.$$

Fix a particular face $f_0 \in \Delta_2(K)$ and define $\mathcal{N}^{00}_3$ as the subspace of $S \in \mathcal{N}^0_3$ for which $S_{f_0} = 0$ in this representation, clearly a subspace of codimension 3. We claim that $\operatorname{curl}\operatorname{curl}^*$ is injective on the space $b_K\mathcal{N}^{00}_3$, and hence a basis for $\mathcal{M}_5$ can be computed from a corresponding basis of $\mathcal{N}^{00}_3$. The injectivity follows since if $w \in \mathbb{R}^3$, with $\epsilon(b_K^2 w) \in b_K\mathcal{N}^{00}_3$, then we get $w = 0$ arguing as in the proof of Lemma 7.3.

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