# ROBUSTNESS OF A SPLINE ELEMENT METHOD WITH CONSTRAINTS

GERARD AWANOU

Abstract. The spline element method with constraints is a discretization method where the unknowns are expanded as polynomials on each element and Lagrange multipliers are used to enforce the interelement conditions, the boundary conditions and the constraints in numerical solution of partial differential equations. Spaces of piecewise polynomials with global smoothness conditions are known as multivariate splines and have been extensively studied using the Bernstein-Bezier representation of polynomials. It is used here to write the constraints mentioned above as linear equations. In this paper, we illustrate the robustness of this approach on two singular perturbation problems, a fourth order problem and a Stokes-Darcy flow. It is shown that the method converges uniformly in the perturbation parameter.

## 1. Introduction

In this paper, we use piecewise polynomials of arbitrary degree to approximate the solutions of two singular perturbation problems. It is shown that the method, called here spline element method, and described in [1], [5], [2], [6], [14] and [17] is robust with respect to the perturbation parameter. We consider plane problems but the method extends easily to 3D problems.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain with Lipschitz continuous boundary $\partial\Omega$ and let us denote by $\epsilon \in (0,1]$ a real small parameter.

The first problem we solve is a fourth order elliptic singular perturbation problem

$$
\begin{aligned}
&\epsilon^2 \Delta^2 u - \Delta u = f \text{ in } \Omega \\
&u = 0,\ \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega,
\end{aligned} \tag{1}
$$

where $\Delta$ denotes the Laplace operator and $\frac{\partial u}{\partial n}$ denotes the normal derivative on $\partial\Omega$. Robust numerical methods for the above problem which exhibit convergence rates uniform with respect to the perturbation parameter have attracted recently a lot of attention, [19], [8], [21], [22], [23] and [10]. As $\epsilon \to 0$, the problem formally degenerates to Poisson equation and hence can be considered as a model for a plate equation which degenerates to a membrane equation. For $\epsilon$ non zero, conforming elements require $C^1$ elements which are complicated, even in two dimensions. As explained in [19], nonconforming elements for the biharmonic equation may not be suitable in the limit where only $C^0$ elements are needed. Robust numerical methods must accommodate both regimes.

The second problem we consider is a Stokes-Darcy flow

$$
\begin{aligned}
\mathbf{u} - \epsilon^2 \mathbf{\Delta u} - \nabla p &= \mathbf{f} \text{ in } \Omega \\
\operatorname{div} \mathbf{u} &= g \text{ in } \Omega \\
\mathbf{u} &= 0 \text{ on } \partial\Omega.
\end{aligned} \tag{2}
$$

The unknowns here are the velocity vector $\mathbf{u}$ and the pressure $p$. Are given the vector field $\mathbf{f}$ and the scalar field $g$. In the singular limit, the problem tends to a Darcy flow, while for $\epsilon$ not too small, and $g = 0$, it is a Stokes equation with an additional term and can be seen as modeling flow through an almost porous medium. It was shown in [19] mainly by numerical experiments that most of the finite elements proposed for the Stokes equations or the mixed formulation of the Poisson equation do not exhibit convergence properties uniform in $\epsilon$. A finite element with a smooth transition between these two regimes was then introduced in the 2D case and later extended to 3D in [18]. Another nonconforming approach is [9], while a least squares formulation based on the nonconforming element introduced in [20], was presented in [11].

As we shall see, the spline element method is robust for all the above singular problems in the two dimensional setting. The extension to three dimensions is straightforward. The advantages of this approach are that it can be applied to a wide range of PDEs in science and engineering in both two and three dimensions; constraints and smoothness are enforced exactly and there is no need to implement basis functions with the required properties; it is particularly suitable for fourth order PDEs; no inf-sup condition are needed to approximate Lagrange multipliers which arise due to the constraints, e.g. the pressure term in Darcy-Stokes flow; one gets in a single implementation approximations of variable order.

Other advantages of the method include the flexibility of using polynomials of different degrees on different triangles and the simplicity of a posteriori error estimates since the method is conforming for the problems considered. The issue of adaptivity is not addressed in this paper though.

The paper is organized as follows: In the first section, we present an abstract setting for the application of the method and briefly review the Bernstein-Bezier representation of polynomials and associated matrix notations. The following section is devoted to the robustness of the fourth order elliptic equation, while the last section is devoted to the Darcy-Stokes equation. Polynomials of low degrees 3 and 4 are used in the numerical illustrations with MATLAB.

## 2. Preliminaries

2.1. **Spline element method.** We consider the variational problem: Find $u \in W$ such that

$$
a(u, v) = (l, v) \quad \text{for all } v \in V, \tag{3}
$$

where $W$ is the set where the solution is sought and $V$ is a Hilbert space, a space of test functions. The form $l$ is bounded and linear and $a$ is a continuous mapping in some sense on $W \times V$ which is linear in the argument $v$. Additional assumptions are needed for the well-posedness of the variational problem. For example, for the weak formulation of the Poisson equation with homogeneous boundary conditions: Find $u$ in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega),$$

the form $a$ need to be bounded and coercive on $W = V = H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$. For the weak formulation of the Navier-Stokes equations in $\mathbf{R^n}$: Find $\mathbf{u} \in H^1(\Omega)^n$ such that

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \sum_{j=1}^{3} \int_\Omega u_j \frac{\partial \mathbf{u}}{\partial x_j} \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad & \forall \mathbf{v} \in V \\ \operatorname{div} \mathbf{u} = 0 \qquad & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} \qquad & \text{on } \partial\Omega, \end{aligned}$$

where

$$V = \{\mathbf{v} \in H_0^1(\Omega)^n, \operatorname{div} \mathbf{v} = 0\},$$

and we take $W = \{\mathbf{u} \in H^1(\Omega)^n, \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega\}$. The main differences between $W$ and $V$ stem from the boundary conditions and the nature of the constraints. The test functions take zero boundary conditions and are divergence free. The form $a$ in the Navier-Stokes problem comes from a trilinear form $c$ and $a(u, v) = c(u; u, v)$. For existence, it is required that, $\mathbf{f} \in H^{-1}(\Omega)^n, \mathbf{g} \in H^{1/2}(\partial\Omega)^n$ the form $a$ be coercive in the usual sense and $c$ weakly sequentially continuous. For uniqueness, stronger assumptions are needed and they are satisfied if the viscosity is sufficiently large or the body forces sufficiently small, c.f. [13]. We notice that if $W = V$ and $a$ is bilinear and symmetric, the variational problem is equivalent to the minimization of the functional

$$J(v) = \frac{1}{2} a(v, v) - (l, v)$$

over $V$.

From now on, let us assume that $\Omega$ is a polygonal domain and let $\mathcal{T}$ be a conforming triangulation of $\Omega$. In the spline element method, each unknown function $u$ is approximated by a polynomial on each element and associated to a vector of coefficients. Hence the space of test functions $V$ is discretized as

$$V_h = \{c \in \mathbf{R}^N, Rc = 0\},$$

for some integer $N$ with $R$ a suitable matrix. For the space of trial functions, we take

$$W_h = \{c \in \mathbf{R}^N, Rc = G\}$$

for a suitable vector $G$. Here $h$ is a discretization parameter which controls the size of the elements in the triangulation.

The condition $a(u, v) = (l, v)$ for all $v \in V$ translates to

$$K(c)d = L^T d \quad \forall d \in V_h, \text{ that is for all } d \text{ with } Rd = 0$$

for a suitable matrix $K$ which depends on $c$ and $L$ is a vector of coefficients associated to the linear form $l$. If for example $(l, v) = \int_\Omega fv$, then $L = MF$ where $M$ is a mass matrix and $F$ a vector of coefficients associated to the spline interpolant of $f$. In the linear case $K(c)$ can be written $c^T K$.

Introducing a Lagrange multiplier $\lambda$, the functional

$$K(c)d - L^T d + \lambda^T Rd,$$

vanishes identically on $V_h$. The stronger condition

$$K(c) + \lambda^T R = L^T,$$

can be shown to have a unique solution $c$ under the side condition $Rc = G$ in many situations and therefore are the discrete equations to be solved when the uniqueness of a solution to the continuous problem is guaranteed.

By a slight abuse of notation, after linearization by Newton's method for example, this possibly nonlinear equation leads to solving systems of type

$$c^T K + \lambda^T R = L^T.$$

The approximation $c$ of $u \in W$ thus is a solution (or limit of a sequence of solutions)of a system of type

$$\begin{bmatrix} K^T & R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \end{bmatrix} = \begin{bmatrix} L \\ G \end{bmatrix}$$

Although the Lagrange multiplier $\lambda$ may not be unique, it can be shown that the component $c$ of the discrete solution is unique under the assumption that the symmetric part of $K^T$ is positive definite with respect to the side condition $Rc = 0$, that is $c^T K^T c \geq 0$ and $c^T K^T c$ and $Rc = 0$ implies $c = 0$. The component $c$ can be retrieved by a least squares solution of the above system. We refer to [1] for a proof of these results. To avoid systems of large size, a variant of the augmented Lagrangian algorithm is used. We consider the sequence of problems

$$\begin{pmatrix} K^T & R^T \\ R & -\mu M \end{pmatrix} \begin{bmatrix} \mathbf{c}^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} L \\ G - \mu M \lambda^{(l)} \end{bmatrix}, \tag{4}$$

where $\lambda^{(0)}$ is a suitable initial guess for example $\lambda^{(0)} = 0$, $M$ is a suitable matrix and $\mu > 0$ is a small parameter taken in practice in the order of $10^{-5}$. The problem is equivalent to

$$\begin{aligned} (K^T + \frac{1}{\mu} R^T M^{-1} R) c^{(l+1)} + R^T \lambda^{(l)} &= L + \frac{1}{\mu} R^T M^{-1} G \\ \lambda^{(l+1)} &= \lambda^{(l)} + \frac{1}{\mu} M^{-1} (Rc^{(l+1)} - G) \end{aligned}$$

Computing $c^{(1)}$ from $\lambda^{(0)}$, one solves

$$(K^T + \frac{1}{\mu} R^T M^{-1} R) c^{(l+1)} = K^T c^{(l)} + \frac{1}{\mu} R^T M^{-1} G, \quad l = 1, 2, \ldots$$

A uniform convergence rate in $\mu$ for this algorithm was shown in [4]. But the algorithm will not get correct results if the condition number of the matrix $K^T + \frac{1}{\mu} R^T M^{-1} R$

deteriorates. As explained in [4], a judicious choice of $M$ can balance the deterioration of the condition number. But the choice of $M$ is delicate. In some cases, a small change of the parameter $\mu$ with $M = I$ is enough. Determining a suitable preconditioner $M$ is an interesting problem still under consideration.

2.2. **Splines on plane triangulations.** We briefly review in this subsection the B-form of polynomials. Precise formulas can be found in [1], [5], [2], [16], [14] and [7]. Let $d \geq 1$ and $r \geq 0$ be two fixed integers. We are going to use the spline spaces

$$S_d^r(\Omega) = \{p \in C^r(\Omega),\ p_{|t} \in P_d,\ \forall t \in \mathcal{T}\},$$

where $P_d$ denotes the space of polynomials of degree $d$ in two variables. Given a non-degenerate triangle $T = \langle v_1, v_2, v_3 \rangle$, it is well known that any point $v = (x, y)$ has a unique set of barycentric coordinates $(b_1, b_2, b_3)$ relative to the triangle $T$, that is

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3, \quad b_1 + b_2 + b_3 = 1.$$

Moreover the Bernstein polynomials,

$$B_{ijk}^d(v) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k, \qquad i + j + k = d,$$

form a basis of $P_d$. Any spline $s$ in $S_d^r$ can then be written uniquely

$$s = \sum_{t \in \mathcal{T}} \sum_{i+j+k=d} c_{ijk}^t B_{ijk}^d.$$

A similar representation holds on boundary edges. Unknown functions are approximated by vector of coefficients $\{c = (c_{ijk}^t), t \in \mathcal{T}\}$, called $B-$nets and the given scalar data are interpolated on a triangle $T = \langle v_1, v_2, v_3 \rangle$ at the domain points $\xi_{ijk} = (iv_1 + jv_2 + kv_3)/d$. and on an edge $\langle v_1, v_2 \rangle$ at the domain points $\xi_{ij} = (iv_1 + jv_2)/d$. Hence a boundary condition of type $u = g$ is discretized as $Bc = G$, for a suitable matrix $B$ where $G$ encodes the coefficients of the interpolant of $g$ on $\partial\Omega$. Moreover, there are matrices $D_1$ and $D_2$ such that if $c$ encodes the $B$-net of $s$, $D_i c$, $i = 1, 2$ encode respectively the $B$-net of $\frac{\partial s}{\partial x_i}$. Thus if $c_1$ and $c_2$ encode respectively the $B$-nets of $u_1$ and $u_2$, then the divergence of the vector with components $u_1$ and $u_2$ has $B$-net $D_1 c_1 + D_2 c_2$ which can be written $Dc$. Finally, the conditions on the coefficient vector $c$ of a spline $s$ which assure that it has global smoothness properties are linear relations which can be encoded as $Hc = 0$. It is straightforward to accommodate the situation where on two adjacent triangles, the spline is represented by polynomials of different degrees $d_1 > d_2$. One simply uses the degree raising formulas which are linear relations connecting the $B-$form of a polynomial of degree $d$ to its $B-$form as a polynomial of degree $d + 1$, c.f. [14].

2.3. **Approximation properties.** In this subsection, we first discuss the approximation properties of scalar functions, then we discuss the approximation properties of divergence-free vector fields. For $s \geq 0$, we denote by $H^s(K)$ the usual Sobolev space on the domain $K$, with norm and semi-norm respectively $\| \cdot \|_{s,K}$ and $| \cdot |_{k,K}$. We will use the notations $\| \cdot \|_s$ and $| \cdot |_k$ when there is no ambiguity about the domain $K$. These notations are extended the usual way to vector valued functions. The space $H_0^s(K)$ is the closure in $H^s(K)$ of the space of $C^\infty$ functions with compact support in $K$.

For splines on plane triangulations, it is known that, [16], for $d \geq 3r+2$ and $0 \leq m \leq d$, there exists a linear quasi-interpolation operator $Q$ mapping $L_1(\Omega)$ into the spline space $S_d^r(\triangle)$ and a constant $C$ such that if $f$ is in the Sobolev space $H^{m+1}(\Omega)$,

$$|f - Qf|_k \leq Ch^{m+1-k}|f|_{m+1}, \tag{5}$$

for $0 \leq k \leq m$. If $\Omega$ is convex, the constant $C$ depends only on $d, m$ and on the smallest angle $\theta_\triangle$ in $\triangle$. In the nonconvex case, $C$ depends only on the Lipschitz constant associated with the boundary of $\Omega$.

In three dimensions, the result holds for $d \geq 8r+1$, c.f. [16], although this lower bound can certainly be improved. It is also known c.f. [12] that the full approximation property for spline spaces holds for certain combinations of $d$ and $r$ on special triangulations.

Next, we discuss the approximation properties of divergence free vector fields by continuous divergence free splines. We recall the following result from [13].

**Theorem 2.1.** *A vector field* $\mathbf{f} \in (L^2(\Omega))^2$ *satisfies* $\operatorname{div} \mathbf{f} = 0$ *if and only if there exists a stream function* $\phi$ *in* $H^1(\Omega)$ *such that* $\mathbf{f} = \operatorname{curl} \phi$.

We recall that for $\phi$ in $H^1(\Omega)$

$$\operatorname{curl} \phi = \left( \frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1} \right)$$

It is not clear what would be the regularity of the stream function $\phi$ for $\mathbf{f} \in H^m(\Omega)$, $m \geq 1$, if the boundary of $\Omega$ is not sufficiently smooth. We therefore restrict the analysis to those divergence-free vector fields which are the curl of a stream function. Let

$$V^d = \{ \mathbf{f} \in (H^d(\Omega))^2,\, \mathbf{f} = \operatorname{curl} \phi,\, \phi \in H^{d+1}(\Omega) \}$$

and

$$\mathcal{V}_d = \{ \mathbf{s} \in (S_d^0(\Omega))^2,\, \mathbf{s} = \operatorname{curl} S,\, S \in S_{d+1}^1(\Omega) \}$$

be spaces of divergence-free vector fields. We denote by $\nabla \mathbf{f}$ the matrix of derivatives $\partial f_i / \partial x_j$

**Theorem 2.2.** *For* $\mathbf{f} \in V^d$,

$$\inf_{\mathbf{s} \in \mathcal{V}_d} ||\mathbf{f} - \mathbf{s}||_0 \leq Ch^d |\mathbf{f}|_d \text{ and } \inf_{\mathbf{s} \in \mathcal{V}_d} ||\nabla \mathbf{f} - \nabla \mathbf{s}||_0 \leq Ch^{d-1} |\mathbf{f}|_d,$$

*for* $d \geq 4$.

For a given $\mathbf{f} \in V^d$, let $\phi \in H^{d+1}(\Omega)$ be a stream function satisfying $\operatorname{curl} \phi = \mathbf{f}$. By (5), let $S = Q(\phi)$ in $S_{d+1}^1(\Omega)$ be a spline approximation of $\phi$ such that

$$|\phi - S|_1 \leq Ch^d |\phi|_{d+1},$$

for $d \geq 4$. Then put $\mathbf{s_f} = \operatorname{curl} S$. Then $\mathbf{s_f} \in \mathcal{V}_d$ and

$$\begin{aligned}||\mathbf{f} - \mathbf{s_f}||_0 &= ||\operatorname{curl}\phi - \operatorname{curl} S||_0 \\ &= |\phi - S|_1 \\ &\leq Ch^d|\phi|_{d+1} = Ch^d|\mathbf{f}|_d.\end{aligned}$$

The other assertion is proved similarly.

It is not clear what would be the corresponding result in three dimensions.

## 3. Robustness for a fourth order problem

For simplification in the discussion of the error estimates, we will assume homogeneous boundary conditions, although they are easily imposed in the spline element method. Given $f \in L^2(\Omega)$, take $V = W = H_0^2(\Omega)$, the variational formulation of (1) is: Find $u \in V$ such that

$$\epsilon^2 \int_\Omega \Delta u \Delta v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in W. \tag{6}$$

We discretize the problem (6) using the space of $C^1$ spline spaces $S_d^1(\Omega)$, which yield fully conforming approximations. According to Cea's lemma, the error in the energy norm, $\|u\|_\epsilon^2 = \epsilon^2|u|_2^2 + |u|_1^2$ is bounded by $\inf_{v\in S_d^1}\|u - v\|_\epsilon$.

Moreover, for $u \in H^{d+1}(\Omega)$, for $d \geq 5$,

$$\begin{aligned}\inf_{v\in S_d^1}\|u - v\|_\epsilon^2 &\leq \epsilon^2 h^{2(d-1)}|u|_{d+1}^2 + h^{2d}|u|_{d+1}^2, \\ &= h^{2(d-1)}(h^2 + \epsilon^2)|u|_{d+1}^2.\end{aligned}$$

It follows that for smooth solutions, we have a $h^{d-1}$ convergence rate which improves to $h^d$ as $\epsilon \to 0$ for $d \geq 5$. Clearly, this convergence rate would not hold for $\epsilon$ dependent solutions. We notice the remark at the end of [19] which seems to suggest that the best convergence rate uniformly in $\epsilon$ is $\sqrt{h}$. The numerical investigation of the boundary layers turns out to be difficult since the linear systems become more ill-conditioned. We illustrate the robustness of the spline element method using polynomials of low degrees 3, 4 and 5, Tables 1, 2 and 3.

The computational domain is the unit square $[0, 1]^2$ which is first divided into squares of side length $h$. Then each square is divided into two triangles by the diagonal with negative slope. We compute the relative error in the energy norm, $\frac{||u_h^I - u_h||_\epsilon}{||u_h^I||_\epsilon}$ where $u_h^I$ is the spline interpolant of the exact solution. We include the results for the case $\epsilon = 0$ for the Poisson equation using $C^1$ cubic splines and for the Poisson equation using only $C^0$ continuous functions. The parameter $\mu$ in the variant of the augmented Lagrangian algorithm was chosen to be $0.99\ 10^{-3}$. Numerical results with a * were computed with $\mu = 10^{-5}$. The test function in Tables 1, 2 and 3 is

$$u(x, y) = (\sin(\pi x)\sin(\pi y))^2.$$

For $d = 3$, the convergence rate oscillates between 1 and 2 but definitely gets close to 2 as $\epsilon \to 0$. For the Poisson and biharmonic equations using $C^1$ splines, the

| $\epsilon/h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ |
|---|---|---|---|---|
| $2^0$ | $2.1265\ 10^{-1}$ | $1.0271\ 10^{-1}$ | $1.9464\ 10^{-2}$ | $3.3889\ 10^{-3}$ |
| $2^{-2}$ | $1.8980\ 10^{-1}$ | $9.4724\ 10^{-2}$ | $4.5625\ 10^{-2}$ | $8.5138\ 10^{-3}$ |
| $2^{-4}$ | $9.6005\ 10^{-2}$ | $4.5406\ 10^{-2}$ | $2.2429\ 10^{-2}$ | $1.0786\ 10^{-2}$ |
| $2^{-6}$ | $4.0374\ 10^{-2}$ | $1.4142\ 10^{-2}$ | $6.3389\ 10^{-3}$ | $3.0774\ 10^{-3}$ |
| $2^{-8}$ | $3.2036\ 10^{-2}$ | $7.5016\ 10^{-3}$ | $2.2452\ 10^{-3}$ | $8.6681\ 10^{-4}$ |
| $2^{-10}$ | $3.1423\ 10^{-2}$ | $6.8540\ 10^{-3}$ | $1.6684\ 10^{-3}$ | $4.4247\ 10^{-4}$ |
| Poisson ($C^1$) | $2.3071\ 10^{-2}$ | $5.2191\ 10^{-3}$ | $1.2678\ 10^{-3}$ | $3.1412\ 10^{-4}$ |
| Poisson | $1.9685\ 10^{-3}$ | $2.6203\ 10^{-4}$ | $3.3516\ 10^{-5}$ | $4.2260\ 10^{-6}$ |
| Biharmonic | $2.1450\ 10^{-1}$ | $1.0367\ 10^{-1}$ | $1.965410^{-2}$ | $4.3614\ 10^{-3}$ |

Table 1. Fourth order equation using polynomials of degree 3

| $\epsilon/h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ |
|---|---|---|---|---|
| $2^0$ | $2.4378\ 10^{-2}$ | $5.8674\ 10^{-3}$ | $1.0959\ 10^{-3}$ | $2.7988\ 10^{-4}*$ |
| $2^{-2}$ | $2.1598\ 10^{-2}$ | $5.1899\ 10^{-3}$ | $1.2694\ 10^{-3}$ | $2.3759\ 10^{-4}*$ |
| $2^{-4}$ | $1.0534\ 10^{-2}$ | $2.4702\ 10^{-3}$ | $6.0576\ 10^{-4}$ | $9.4533\ 10^{-5}$ |
| $2^{-6}$ | $4.0397\ 10^{-3}$ | $7.6437\ 10^{-4}$ | $1.7134\ 10^{-4}$ | $4.1365\ 10^{-5}$ |
| $2^{-8}$ | $2.9540\ 10^{-3}$ | $3.9977\ 10^{-4}$ | $6.1923\ 10^{-5}$ | $1.1852\ 10^{-5}$ |
| $2^{-10}$ | $2.8664\ 10^{-3}$ | $3.6170\ 10^{-4}$ | $4.6177\ 10^{-5}$ | $6.2184\ 10^{-6}$ |
| Poisson ($C^1$) | $1.9956\ 10^{-3}$ | $2.571210^{-4}$ | $3.2459\ 10^{-5}$ | $4.5161\ 10^{-6}$ |
| Poisson | $2.4134\ 10^{-4}$ | $1.5286\ 10^{-5}$ | $9.5869\ 10^{-7}$ | $6.2866\ 10^{-10}$ |
| Biharmonic | $2.4605\ 10^{-2}$ | $5.9116\ 10^{-3}$ | $1.2668\ 10^{-3}$ | $3.0533\ 10^{-4}*$ |

Table 2. Fourth order equation using polynomials of degree 4

convergence rate is 2, while the convergence rate is 3 for the Poisson equation using $C^0$ piecewise functions.

For $d = 4$, the results are more consistent with convergence rates increasing from 2 to 3. The convergence rate is 3 for the Poisson equation using $C^1$ functions but 4 when $C^0$ functions are used. The convergence rate of the biharmonic equation remained 3.

For $d = 5$, the convergence rates are consistent with the known approximation properties of the spline spaces. It increases from 4 to 5.

Clearly, for $d < 5$, we have suboptimal convergence rate but the optimal convergence rate should be restored on a triangulation where the spline space has full approximation properties.

## 4. Robustness for Darcy-Stokes flow

We assume homogeneous boundary conditions to simplify the discussion of the error estimates. We will also assume that $g = 0$ since for $g \in L^2(\Omega)$, there is $w \in H^1(\Omega)^2$ such that $\operatorname{div} w = g$.

| $\epsilon/h$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ | $2^{-6}$ |
|---|---|---|---|---|
| $2^{0}$ | $1.2589\ 10^{-3}$ | $7.5266\ 10^{-5}$ | $5.0278\ 10^{-5}$ | $3.6544\ 10^{-3}+$ |
| $2^{-2}$ | $1.1136\ 10^{-3}$ | $6.6545\ 10^{-5}$ | $4.0541\ 10^{-6}$ | $1.9367\ 10^{-6}$ |
| $2^{-4}$ | $5.3098\ 10^{-4}$ | $3.1458\ 10^{-5}$ | $1.9126\ 10^{-6}$ | $1.1867\ 10^{-7}$ |
| $2^{-6}$ | $1.6784\ 10^{-4}$ | $8.9118\ 10^{-6}$ | $5.2641\ 10^{-7}$ | $3.2259\ 10^{-8}$ |
| $2^{-8}$ | $9.3491\ 10^{-5}$ | $3.3119\ 10^{-6}$ | $1.5078\ 10^{-7}$ | $8.3996\ 10^{-9}$ |
| $2^{-10}$ | $8.6210\ 10^{-5}$ | $2.5592\ 10^{-6}$ | $8.1835\ 10^{-8}$ | $3.0649\ 10^{-9}$ |
| Poisson (S) | $2.3672\ 10^{-5}$ | $7.5557\ 10^{-7}$ | $2.3764\ 10^{-8}$ | $8.1405\ 10^{-10}$ |
| Poisson | $7.0434\ 10^{-5}$ | $2.2509\ 10^{-6}$ | $7.0976\ 10^{-8}$ | $2.2292\ 10^{-9}$ |
| Biharmonic | $1.2708\ 10^{-3}$ | $7.5979\ 10^{-5}$ | $1.1226\ 10^{-4}+$ | $1.0374\ 10^{-2}+$ |

TABLE 3. Fourth order equation using polynomials of degree 5

Given $f \in L^2(\Omega)$, let $W = V = \{\mathbf{u} \in H_0^1(\Omega)^2, \operatorname{div}\mathbf{u} = 0\}$. We have the variational formulation: Find $\mathbf{u} \in W$ such that

$$\epsilon^2 \int_\Omega \nabla\mathbf{u}\cdot\nabla\mathbf{v} + \int_\Omega \mathbf{u}\cdot\mathbf{v} = \int_\Omega \mathbf{f}\cdot\mathbf{v} \quad \forall \mathbf{v} \in V. \tag{7}$$

We notice that in the singular limit, the variational problem is not well posed on $V$ but on the space

$$\tilde{V} = \{\mathbf{u} \in L^2(\Omega)^2, \operatorname{div}\mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n} = 0 \,\text{on}\, \partial\Omega\},$$

but that the energy norm $|||.|||_\epsilon$ defined by

$$|||\mathbf{v}|||_\epsilon^2 = ||\mathbf{v}||_0^2 + \epsilon^2||\nabla\mathbf{v}||_0^2,$$

converges to the norm of $L^2(\Omega)^2$.

We discretize the problem (7) using the space of spline vectors

$$V_h = \{\mathbf{s} \in (S_d^0(\Omega))^2, \operatorname{div}\mathbf{s} = 0, \mathbf{s} = 0 \,\text{on}\, \partial\Omega\}.$$

For the case $\epsilon = 0$, it is possible to use

$$\tilde{V_h} = \{\mathbf{s} \in P_d(T)^2, T \in \mathcal{T}, \mathbf{s}\cdot\mathbf{n} \text{ is continuous across edges}, \operatorname{div}\mathbf{s} = 0,\ \mathbf{s}\cdot\mathbf{n} = 0 \,\text{on}\, \partial\Omega\},$$

since $\tilde{V} \subset H(\operatorname{div}) = \{\mathbf{u} \in L^2(\Omega)^2, \operatorname{div}\mathbf{u} \in L^2(\Omega)\}$.

For $\mathbf{u} \in V^d \cap V$, the error in the energy norm is bounded by $\inf_{\mathbf{s}\in W_h}|||\mathbf{u}-\mathbf{s}|||_\epsilon$, for $d \geq 4$. However, we have

$$\{\mathbf{s} \in \mathcal{V}^d, \mathbf{s} = 0 \,\text{on}\, \partial\Omega\} \subset V_h.$$

Hence, for $d \geq 4$, the error in the energy norm cannot be worse that

$$\inf_{\substack{\mathbf{s}\in\mathcal{V}^d \\ \mathbf{s}=0 \,\text{on}\, \partial\Omega}} |||\mathbf{u}-\mathbf{s}|||_\epsilon \leq (\epsilon h^{d-1} + h^d)|\mathbf{u}|_d$$

using Theorem (2.2). This shows that the convergence rate is at least $h^{d-1}$ for $\epsilon$ non zero and $h^d$ in the singular limit for $d \geq 4$.

We do not know what are the approximation properties of the space $\tilde{V_h}$. Next we discuss the approximation of the pressure.

| $\epsilon/h$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ | $2^{-5}$ |
|---|---|---|---|---|
| $2^{-0}$ | $1.0061\ 10^{-1}$ | $2.3718\ 10^{-2}$ | $5.8212\ 10^{-3}$ | $1.4467\ 10^{-3}$ |
| $2^{-2}$ | $8.9696\ 10^{-2}$ | $2.1014\ 10^{-2}$ | $5.1490\ 10^{-3}$ | $1.2791\ 10^{-3}$ |
| $2^{-4}$ | $4.6793\ 10^{-2}$ | $1.0240\ 10^{-2}$ | $2.4505\ 10^{-3}$ | $6.0662\ 10^{-4}$ |
| $2^{-6}$ | $2.3921\ 10^{-2}$ | $3.8922\ 10^{-3}$ | $7.5706\ 10^{-4}$ | $1.8105\ 10^{-4}$ |
| $2^{-8}$ | $2.0934\ 10^{-2}$ | $2.8315\ 10^{-3}$ | $3.9537\ 10^{-4}$ | $1.3380\ 10^{-4}$ |
| 0.00 | $2.0708\ 10^{-2}$ | $2.7403\ 10^{-3}$ | $3.5598\ 10^{-4}$ | $5.8378\ 10^{-5}$* |
| Darcy | $4.3368\ 10^{-3}$ | $8.6596\ 10^{-4}$ | $1.7278\ 10^{-5}$* | $1.9652\ 10^{-2}$+ |

Table 4. Darcy-Stokes using polynomials of degree 3

Recall that we have assumed $g = 0$. Formally taking the divergence and the normal component of the first equation in (2), we obtain

$$-\Delta p = \operatorname{div} \mathbf{f}$$
$$\frac{\partial p}{\partial \mathbf{n}} = (-\mathbf{f} + \mathbf{u} - \epsilon^2 \Delta \mathbf{u}) \cdot \mathbf{n}.$$

We use the spline space $S^0_{d+1}(\Omega)$ to approximate the pressure when polynomials of degree $d$ are used for the velocity because of the second equation above.

The computational domain is the same as in the previous section. We compute the relative error of the velocity in the energy norm, $\frac{|||\mathbf{u_h^I} - \mathbf{u_h}|||_\epsilon}{|||\mathbf{u_h^I}|||_\epsilon}$ where $\mathbf{u_h^I}$ is the spline interpolant of the exact solution. We include the results for the case $\epsilon = 0$ which correspond to approximations of the Darcy equation using continuous splines and Dirichlet boundary conditions. Under Darcy below, we list the numerical results when the space $\tilde{W}_h$ is used. The relative error for the scalar pressure equation is computed as in the previous section. The test functions are

$$\mathbf{u} = \operatorname{curl}\big(\sin(\pi x)^2 \sin(\pi y)^2\big), \qquad \text{and } p = \sin(\pi x).$$

It would be interesting to prove convergence rates independent of $\epsilon$ for boundary layers similar to the ones in [18] and [19]. For d=3, we observe that the convergence rate for the velocity increases from 1.8 to 2.7, while for the Darcy equation, requiring only continuity of the normal derivative, the convergence rate is approximately 3.5. For d=4, the convergence rate increases from 4 to 4.5. A "+" in the tables indicate that we were not able to solve successfully the discrete system of equations. Some type of preconditioning for the linear systems need to be used. This is a topic which is still under investigation.

## References

[1] Awanou, G. (2003). Energy methods in 3D spline approximations, Ph.d. Dissertation, University of Georgia, Athens, Georgia.

[2] Awanou, G. and Lai, M.J., (2005). Trivariate spline approximations of 3D Navier-Stokes equations. Math. Comp. 74, pp. 585–601.

[3] Awanou, G. and Lai, M.J., (2004). Trivariate spline approximations of divergence-free vector fields. Preprint.

| $\epsilon/h$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ |
|---|---|---|---|
| $2^{-0}$ | 7.9689 | 3.1380 | $8.9906\ 10^{-1}$ |
| $2^{-2}$ | $5.0369\ 10^{-1}$ | $1.9471\ 10^{-1}$ | $5.6019\ 10^{-2}$ |
| $2^{-4}$ | $3.3121\ 10^{-2}$ | $1.1486\ 10^{-2}$ | $3.4025\ 10^{-3}$ |
| $2^{-6}$ | $2.9191\ 10^{-3}$ | $6.8701\ 10^{-4}$ | $1.9454\ 10^{-4}$ |
| $2^{-8}$ | $1.5185\ 10^{-3}$ | $9.0396\ 10^{-5}$ | $1.1731\ 10^{-5}$ |
| 0.00 | $1.4690\ 10^{-3}$ | $6.9076\ 10^{-5}$ | $2.7063\ 10^{-6}$ |
| Darcy | $1.4690\ 10^{-3}$ | $6.9076\ 10^{-5}$ | $2.7063\ 10^{-6}$ |

TABLE 5. Pressure using polynomials of degree 4

| $\epsilon/h$ | $2^{-2}$ | $2^{-3}$ | $2^{-4}$ |
|---|---|---|---|
| $2^{-0}$ | $1.7486\ 10^{-2}$ | $1.0465\ 10^{-3}$ | $6.0180\ 10^{-5}$ |
| $2^{-2}$ | $1.5541\ 10^{-2}$ | $9.2597\ 10^{-4}$ | $5.3232\ 10^{-5}$ |
| $2^{-4}$ | $7.8996\ 10^{-3}$ | $4.4325\ 10^{-4}$ | $2.5327\ 10^{-5}$ |
| $2^{-6}$ | $3.6643\ 10^{-3}$ | $1.4529\ 10^{-4}*$ | $2.0257\ 10^{-5}$ |
| $2^{-8}$ | $3.0271\ 10^{-3}$ | $8.5252\ 10^{-5}$ | $8.3301\ 10^{-6}*$ |
| 0.00 | $2.9751\ 10^{-3}$ | $1.7913\ 10^{-3}$ | $1.7288\ 10^{-3}+$ |
| Darcy | $6.4139\ 10^{-4}$ | $2.6876\ 10^{-3}+$ | $1.2037\ 10^{-4}+$ |

TABLE 6. Darcy-Stokes using polynomials of degree 4

| $\epsilon/h$ | $2^{-2}$ | $2^{-3}$ |
|---|---|---|
| $2^{-0}$ | 7.5061 | $9.0088\ 10^{-1}$ |
| $2^{-2}$ | $4.7198\ 10^{-1}$ | $5.6434\ 10^{-2}$ |
| $2^{-4}$ | $2.9984\ 10^{-2}$ | $3.5811\ 10^{-3}$ |
| $2^{-6}$ | $2.5980\ 10^{-3}$ | $2.2229\ 10^{-4}$ |
| $2^{-8}$ | $1.1138\ 10^{-3}$ | $1.7635\ 10^{-5}$ |
| 0.00 | $1.0343\ 10^{-3}$ | $7.4925\ 10^{-6}$ |
| Darcy | $1.0343\ 10^{-3}$ | $7.4925\ 10^{-6}$ |

TABLE 7. Pressure using polynomials of degree 5

[4] Awanou, G. and Lai, M.J., (2005). On convergence rate of the augmented Lagrangian algorithm for nonsymmetric saddle point problems. Appl. Numer. Math. 54, no. 2, pp 122–134.

[5] Awanou, G., Lai M.J. and Wenston, P., (2005). The multivariate spline method for scattered data fitting and numerical solutions of partial differential equations. In Wavelets and Splines, Nashboro Press, pp. 24–74.

[6] Baradmize V. and Lai, M.J. (2005). Spherical spline solution to a PDE on sphere. In Wavelets and Splines, Nashboro Press, pp. 75–92

[7] de Boor, C. (1987). B-form basics. In Farin, G. (ed) Geometric Modeling, SIAM Publications, Philadelphia, pp. 131–148.

[8] Brenner, S.C. and Sung, L.Y., (2005). $C^0$ interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. Journal of Scientific Computing, 22-23, Issue 1-3, pp 83–118.

[9] Burman E. and Hansbo, P., (2005). Stabilized Crouzeix-Raviart element for the Darcy-Stokes problem, Numer. Meth. PDEs 21, pp. 986–997.

[10] Chen, S.C., Zhao, Y.C. and Shi, D.Y., (2005). Non $C^0$ nonconforming elements for elliptic fourth order singular perturbation problem. J. Comput. Math, **23**, 2, pp 185–198.

[11] Danisch G. and Starke, G. First-order system least squares for Darcy-Stokes flow, to appear in SIAM Journal on Numerical Analysis.

[12] Dyer, B.W. and Hong D., (2003). Algorithm for optimal triangulations in scattered data representation and implementation. Journal of computational analysis and applications, 5 N°3, pp 25–43.

[13] Girault V. and Raviart P.A. (1986). Finite element method for Navier-Stokes equations, Springer-Verlag.

[14] Hu, X.L., Han D.F. and Lai, M.J. (2007). Bivariate splines of various degrees for numerical solution of partial differential equations, accepted for publication in SIAM journal of scientific computing.

[15] Lai, M.J. and Schumaker, L., (1998). Approximation power of bivariate splines, Advances in Comput. Math., 8, pp. 1–29.

[16] Lai, M.J. and Schumaker, L. (2007). Spline functions over triangulations, to be published by Cambridge University Press.

[17] Lai, M.J. and Wenston, P. (2004). Bivariate splines for fluid flows, Computer & Fluids 33, pp. 1047–1073.

[18] Mardal, K.A., Tai, X.C. and Winther, R., (2002). A robust finite element method for Darcy-Stokes flow, SIAM J. Numer. Anal., 40, pp. 1605--1631.

[19] Nilssen, T.K., Tai X.C. and Winther, R. (2001). A robust nonconforming $H^2$ Element, Math. Comp. 70, no. 234, pp. 489–505.

[20] Tai, X.C. and Winther, R. (2006). A discrete de Rham complex with enhanced smoothness, Calcolo 43, pp 287–306.

[21] M. Wang and X. Meng. A Robust Finite Element Method for a 3-D Elliptic Singular Perturbation Problem. J. Comp. Math., 25 (2007), pp. 631–644.

[22] Ming Wang, Zhong-Ci Shi and Jinchao Xu. Some n-Rectangle Nonconforming Elements for Fourth Order Elliptic Equations. J. Comp. Math., 25 (2007), pp. 408–420.

[23] Wang, M., Xu, J. and Hu, Y.C., (2006). Modified Morley element method for a fourth order elliptic singular perturbation problem. J. Comput. Math., 24, 2, pp. 113–120.

Northern Illinois University, Department of Mathematical Sciences, Dekalb, IL, 60115

*E-mail address*: awanou@math.niu.edu

*URL*: http://www.math.niu.edu/~awanou