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Abstract

CONGRUENCE LATTICES OF FINITE UNIVERSAL ALGEBRAS

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A universal algebra $\mathfrak{A} = \langle A; f_j \rangle_{j \varepsilon J}$ is a set A together with a collection of operations $\{f_j\}_{j \varepsilon J}$ such that f_j is a k-ary operation. If each f_j is unary, then $\mathfrak{A}$ is a multiunary algebra. If $|J| = 1$, then $\mathfrak{A}$ is a unary algebra. If the f_j are partial operations, then $\mathfrak{A}$ is called a partial universal algebra. Let $\mathcal{E}(A)$ denote the lattice of all equivalence relations on A. A congruence relation of $\mathfrak{A}$ is an element $\theta \varepsilon \mathcal{E}(A)$, such that if $\langle x_i, y_i \rangle \varepsilon \theta$ for $1 \leq i \leq k_j$, then $\langle f_j(x_1, x_2, \ldots, x_{k_j}), f_j(y_1, y_2, \ldots, y_{k_j}) \rangle \varepsilon \theta$. If $\mathfrak{A}$ is a partial algebra, then $\theta \varepsilon \mathcal{E}(A)$ is called a strong congruence relation if, whenever $\langle x_i, y_i \rangle \varepsilon \theta$ for $1 \leq i \leq k_j$ and $f_j(x_1, x_2, \ldots, x_{k_j})$ exists, then $f_j(y_1, y_2, \ldots, y_{k_j})$ also exists and $\langle f_j(x_1, x_2, \ldots, x_{k_j}), f_j(y_1, y_2, \ldots, y_{k_j}) \rangle \varepsilon \theta$. The collection of all congruence relations on an algebra form a sublattice $\Theta(\mathfrak{A})$ of $\mathcal{E}(A)$; $\Theta(\mathfrak{A})$ is called the congruence lattice of $\mathfrak{A}$.

Chapter 1 investigates the problem of which finite lattices are the congruence lattices of finite algebras. It is shown that any finite distributive lattice is the congruence lattice of 1) a finite algebra having only two unary operations 2) a finite lattice. Also, the algebras in 1 and 2 have additional virtuous properties. It is also shown that the class of lattices isomorphic to strong congruence lattices of finite algebras is equal to the class of lattices isomorphic to congruence lattices of finite algebras.

Chapter 2 is concerned with the following problem of Pierce: Which sublattices of $\mathcal{E}(S)$ are congruence lattices for some algebra defined on S. The main theorem of this chapter is that if $\mathcal{K}$ is an abstract class of finite lattices such that if $L \varepsilon \mathcal{K}$ and M is a sublattice of L then $M \varepsilon \mathcal{K}$, then the following are equivalent: 1) if L is a sublattice of $\mathcal{E}(A)$ for some finite set A, $A \times A \varepsilon L$, and $\{<a, a> \mid a \varepsilon A\} \varepsilon L$, then $L \cong \Theta(\mathfrak{A})$ for some algebra on A; 2) every lattice in $\mathcal{K}$ is distributive.

It is well known that the study of congruence lattices on arbitrary algebras can be reduced to the study of congruence lattices of multiunary algebras. In Chapter 3 (multiunary) algebras with few operations are examined. It is shown that if $\mathfrak{A}$ is any finite algebra, then there exists: 1) a finite algebra $\mathfrak{A}_1$ with two unary operations and 2) a finite algebra $\mathfrak{A}_2$ with one binary operation such that both $\Theta(\mathfrak{A}_1)$ and $\Theta(\mathfrak{A}_2)$ have unique maximal ideals isomorphic to $\Theta(\mathfrak{A})$.

Chapter 4 is entirely concerned with the congruence lattices of finite unary algebras. Necessary and sufficient conditions are given for a unary algebra to have a congruence lattice that is: 1) distributive 2) upper semimodular 3) lower semimodular 4) modular.

TABLE OF CONTENTS

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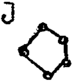
Chapter 0

INTRODUCTION

<u>0.1 Definition</u>: A <u>lattice</u> is a set L, partially ordered under $\leq$, such that if $a, b \varepsilon L$ then a and b, have a greatest lower bound and a least upper bound, denoted by $a \wedge b$ and $a \vee b$ respectively. If $a \leq b$ or $b \leq a$ then a and b are said to be <u>comparable</u>; if not, they are <u>not comparable</u>. The symbol 1 will denote the largest element in L, and 0 will denote the smallest element in L, whenever such elements exist. $[a,b]$ will denote the <u>interval from a to b</u> in L where $[a,b] = \{x \varepsilon L | a \leq x \leq b\}$. The element b is said to <u>cover</u> a, $b \succ a$, if $[a,b] = \{a,b\}$. An <u>atom</u> of L is an element that covers 0. A <u>chain</u> is a set of elements, every two of which are comparable. If this collection is finite and has cardinality $n + 1$, then it is said to have <u>length</u> n. The <u>length of an interval</u> $[a,b]$ is the least upper bound of the lengths of all chains contained in that interval.

<u>0.2 Definition</u>: Certain specific lattices and types of lattices will be studied in some detail. Let $D_n = \{0, 1, x_1, \ldots, x_n\}$ with $x_i \vee x_j = 1$ and $x_i \wedge x_j = 0$ for $i \neq j$. Let N_5 be the lattice with the diagram . A lattice is said to be <u>distributive</u> if it has no sublattice isomorphic to D_3 or N_5. A lattice is said to be <u>modular</u> if it has no sublattice isomorphic to N_5. A lattice of finite length is said to be uppersemimodular if whenever $a \succ c$ and $b \succ c$, $a \neq b$, then $a \vee b \succ a$ and $a \vee b \succ b$. The notion of <u>lower semimodular</u> is defined dually.

0.3 Definition: The collection of all equivalence relations on a set A will be denoted by $\mathcal{E}(A)$. If θ is an equivalence relation, $x,y \in A$, then $x \equiv y(\theta)$ and $\langle x,y\rangle \in \theta$ both signify that x is equivalent to y modulo θ . $[x]\theta$ will denote that equivalence class of θ containing x. For θ and ψ in $\mathcal{E}(A)$ let $\theta \wedge \psi = \{\langle x,y\rangle | \langle x,y\rangle \in \theta$ and $\langle x,y\rangle \in \psi\}$, and let $\theta \vee \psi = \{\langle x,y\rangle |$ there exist elements $z_1,\ldots,z_n \in A$ such that $\langle x_1,z_1\rangle \in \theta$, $\langle z_1,z_2\rangle \in \psi$, $\langle z_2,z_3\rangle \in \theta \ldots \langle z_n,y\rangle \in \theta\}$. Then $\mathcal{E}(A)$ becomes a lattice under these operations. The maximal element is $U_A = A \times A$, and the minimal element is $I_A = \{\langle a,a\rangle | a \in A\}$. If $\theta \in \mathcal{E}(S)$ and θ has equivalence classes $S_1,S_2,\ldots,S_k$, where $S_i = a^i_1,\ldots,a^i_{\ell_i}$, then represent θ by $\theta = a^1_1\ldots a^1_{\ell_1}|a^2_1,\ldots,a^2_{\ell_2}|\ldots|a^k_1,\ldots,a^k_{\ell_k}$. If $S = \{1,2,\ldots,n\}$, then $\mathcal{E}(S)$ is called the partition lattice on n elements and is denoted by π_n.

0.4 Definition: A universal algebra $\mathfrak{A} = \langle A;f_j\rangle_{j\varepsilon J}$ is a set A together with a collection of operations $\{f_j\}_{j\varepsilon J}$ such that f_j is a k_j-ary operation. If J is finite, $\mathfrak{A}$ may be written as $\mathfrak{A} = \langle A;f_1,\ldots,f_n\rangle$. If the $\{f_j\}_{j\varepsilon J}$ are partial operations, then $\mathfrak{A}$ is called a partial universal algebra. A congruence relation of $\mathfrak{A}$ is an element $\theta \in \mathcal{E}(A)$ such that if $x_i \equiv y_i\ (\theta)$ for $1 \leq i \leq k_j$ then $f(x_1,x_2,\ldots x_{k_j}) \equiv f(y_1,y_2,\ldots y_{k_j})\ (\theta)$. If $\mathfrak{A} = \langle A,f_j)_{j\varepsilon J}$ is a partial algebra, an element θ of $\mathcal{E}(A)$ is called a strong congruence relation if, whenever $x_i \equiv y_i$ (θ) for $1 \leq i \leq k_j$ and $f_j(x_1,x_2,\ldots,x_{k_j})$ exists, then $f_j(y_1,y_2,\ldots,y_{k_j})$ also exists and $f_j(x_1,x_2,\ldots,x_{k_j}) \equiv f_j(y_1,y_2,\ldots,y_{k_j})$ (θ). The collection of all congruence relations on an algebra form a sublattice $\mathbf{\Theta}(\mathfrak{A})$ of $\mathcal{E}(A)$; $\mathbf{\Theta}(\mathfrak{A})$ is called the congruence lattice of $\mathfrak{A}$. Similiarly, the collection of all strong congruence relations forms a sublattice of $\mathcal{E}(A)$, denoted by $\mathbf{\Theta}_S(\mathfrak{A})$. If $S \subseteq A$, let $\Theta(S)$ denote the smallest congruence relation on $\mathfrak{A}$ containing S in a congruence class; $\Theta(S)$ is called the congruence relation generated by S. If $S = \{x,y\}$, then write $\Theta(S) = \Theta(x,y)$. Such a congruence relation is called minimal. An equivalence relation Θ is said to be stable under a k-ary operation f if whenever $\langle x_i,y_i\rangle \varepsilon \Theta$, $1 \leq i \leq k$, then $\langle f(x_1,\ldots,x_k) ; f(y_1,\ldots,y_k)\rangle \varepsilon \Theta$.

0.5 Lemma: For $\Theta \varepsilon \mathbf{\Theta}(\mathfrak{A})$, $\Theta = \bigvee\{\Theta(x,y) \mid x \equiv y(\theta)\}$.
proof: Grätzer [6] p.55.

0.6 Definition: A congruence relation Θ on $\mathfrak{A}$ is proper if $\Theta \neq U_A$ and $\Theta \neq I_A$. A congruence class is said to be nontrivial if it has cardinality greater than one, and trivial otherwise.

0.7 Definition: A unary operation on a set A is a function from A to A. A unary algebra is an algebra with only one operation, and this operation is unary.

A <u>multiunary algebra</u> is a universal algebra in which every operation is unary. Unary partial algebras and multiunary algebras are defined similiarly.

<u>0.8 Lemma</u>: Let $\mathfrak{A} = \langle A; F_j \rangle_{j \varepsilon J}$ be a (partial) algebra. Then there exists a multiunary (partial) algebra $\mathfrak{B} = \langle A; G_k \rangle_{k \varepsilon K}$ such that $\Theta(\mathfrak{A}) = \Theta(\mathfrak{B})$ $(\Theta_s(\mathfrak{A}) = \Theta_s(\mathfrak{B}))$.
<u>Proof</u>: This proof is standard and is omitted.

<u>0.9 Definition</u>: Let $\mathfrak{A} = \langle A; f_j \rangle_{j \varepsilon J}$ be a (partial) multiunary algebra. A <u>polynomial</u> for $\mathfrak{A}$ will be any finite composition of the operations f_j.

<u>0.10 Lemma</u>: Let $\mathfrak{A} = \langle A; f_j \rangle_{j \varepsilon J}$ be a (partial) multiunary algebra, the identity function and all constant functions occuring among the f_i. For $a, b \varepsilon A$, let $\theta(a,b)$ be a minimal (strong) congruence relation on $\mathfrak{A}$, and let x and y be arbitrary elements of A. Then $x \equiv y \ (\theta(a,b))$ if and only if there exists $n < \omega$, a sequence $x = z_o, z_1, \ldots, z_n = y$ of elements of A, and a sequence $p_o, \ldots, p_{n-1}$ of polynomials of $\mathfrak{A}$ such that $\{p_i(a), p_i(b)\} = \{z_i, z_{i+1}\}$ for $0 \leq i \leq n-1$.
<u>Proof</u>: Grätzer [6] p.54. Note that the lemma is not true for congruence relations on partial algebras.

<u>0.11 Notation</u>: Let $\Sigma(\mathfrak{A})$ denote the lattice of subalgebras of $\mathfrak{A}$. Let $\emptyset$ denote the empty set. Let $|A| = n < \omega$ mean A has finite cardinality. Let (m,n) denote the greatest common divisor of m and n.

Chapter 1

REPRESENTATIONS OF FINITE LATTICES

In [6] G. Grätzer defines the following classes: $\mathcal{L}_0$: the class of finite lattices; $\mathcal{L}_1$: the class of lattices isomorphic to sublattices of finite partition lattices; $\mathcal{L}_2$: the class of lattices isomorphic to strong congruence lattices of finite partial algebras; and $\mathcal{L}_3$: the class of lattices isomorphic to congruence lattices of finite algebras. Clearly $\mathcal{L}_0 \supseteq \mathcal{L}_1 \supseteq \mathcal{L}_2 \supseteq \mathcal{L}_3$. Grätzer raises the question (problem 13, p.116) as to whether equality or proper inclusion holds in each case. The question of whether $\mathcal{L}_0 = \mathcal{L}_1$ is a special case of the question raised by Birkhoff [1] in 1935: Is every lattice isomorphic to a sublattice of some partition lattice? This question was answered affirmatively by Whitman [14], but in his construction every finite lattice is embedded in a countably infinite lattice. By use of Whitman's techniques and those of Jónsson [11], Hales has recently shown [10] that any finite sublattice of a free lattice can be embedded in the lattice of partitions of a finite set. However, the general problem of whether every finite lattice is isomorphic with a sublattice of some finite partition lattice is still unsolved and apparently quite difficult. For the second inclusion $\mathcal{L}_1 \supseteq \mathcal{L}_2$, it is also unknown whether or not equality holds. Note that the infinite case has been solved by Grätzer and Schmidt [9]: every algebraic lattice is isomorphic to the congruence lattice of some algebra; but again their construction does not preserve finiteness.

In this chapter (see 1.4 and 1.13) it will be shown that $\mathcal{L}_1 = \mathcal{L}_3$ for distributive lattices, and the algebras that occur in the representation will have particularly virtuous properties. Next, in response to a question of Grätzer and Schmidt [8], an investigation will be made of the relationship between the length of a finite distributive lattice D and the length of a lattice L such that $\Theta(L) = D$. (1.15 and 1.16). Finally in 1.19 it will be shown that $\mathcal{L}_2 = \mathcal{L}_3$.

<u>1.1 Definition</u>: An element $a \neq 0$ of a lattice L is said to be <u>join-irreducible</u> when $b, c \in L$ and $a = b \vee c$ implies $a = b$ or $a = c$.

<u>1.2 Definition</u>: If X is a partially ordered set then 2^x will denote the set of all isotone functions from x to the two element lattice $\{0,1\}$.

<u>1.3 Lemma</u>: If D is a finite distributive lattice and X is the partially ordered set of join-irreducible elements of D, then D is isomorphic to 2^x.

<u>Proof</u>. This is a standard result and can be found in Birkhoff [2] (p.59).

<u>1.4 Theorem</u>: Let D be a finite distributive lattice. Then there exists a finite set A and two unary operations f and g on A such that if $\mathfrak{A} = \langle A; f, g \rangle$, then $D \cong \Theta(\mathfrak{A}) \cong \Sigma(\mathfrak{A})$.

<u>Proof</u>: Let $|D| = n$ and let Y be the partially ordered set of the join-irreducible elements $\{q_1, q_2, \ldots, q_m\}$ of D. By

lemma 1.3, $D \cong 2^Y$. Let $\{p_1,p_2,\ldots,p_m\}$ be a collection of distinct prime numbers, $p_i > m$ for $1 \leq i \leq m$. Let $C_i = \{x_{i,o},x_{i,1},x_{i,2},\ldots,x_{i,p_i-1}\}$ be a set of p_i distinct elements with $C_i \cap C_k = \emptyset$ for $i \neq k$. Let y be a new element, $y \notin C_i$ for $1 \leq i \leq m$. Define $A = \bigcup_{i=1}^{m} C_i \cup \{y\}$. Now define a unary operation f on A by $f(x_{i,j}) = x_{i,k}$ where $k = j + 1(\text{mod } p_i)$ and $f(y) = y$. Define an operation g on A by

$$g(x_{i,j}) = \begin{cases} y & \text{if } j = 0, \ 1 \leq i \leq m \\ x_{j,i} & \text{if } j \neq 0, \text{ and } q_i > q_j \\ x_{i,j} & \text{if } j \neq 0, \text{ and } q_i \not> q_j \text{ or } j > m \end{cases}$$

and $g(y) = y$.

Let $\mathfrak{A} = \langle A;f,g \rangle$, and consider $\Theta(\mathfrak{A})$. First consider minimal congruence relations $\theta\ (u,v)$ where $u,v \in A$. Without loss of generality, there are three cases: I) $u = y$ and $v \in C_i$ II) $u \in C_i$ and $v \in C_i$ and III) $u \in C_i$ and $v \in C_j$ for $i \neq j$.

Case I) Let $u = y$ and $v = x_{i,j}$. Then iteration of f shows that $\theta\ (C_i \cup \{y\}) \subseteq \theta\ (y,x_{i,j})$. Also, if $q_i > q_k$ in D, then applying g and iterations of f shows that $\theta\ (C_k \cup\ y\ ) \subseteq \theta\ (y,x_{i,j})$. Finally, since $f(y) = g(y) = y$, it follows that $[y]\theta(y,x_{i,j}) = \{y\} \cup \{x_{k,\ell} | 0 \leq \ell \leq p_k-1$ and $q_k \leq q_i\}$, and in fact this set is the only nontrivial congruence class of $\theta\ (y,x_{i,j})$.

Case II) Let $u = x_{i,j}$ and $v = x_{i,k}$ with $j \neq k$. Since p_i is prime and $g(x_i,0) = y$, it follows that $\theta(x_{i,j},x_{i,k}) = \theta(C_i) = \theta(C_i \cup \{y\})$. Hence $\theta(x_{i,j},x_{i,k}) = \theta(y,x_{i,j})$, which reduces to case I.

Case III) Let $u = x_{i,j}$ and $v = x_{k,\ell}$ where $i \neq k$. Then iterating f and using the fact that p_i and p_k are relatively prime, implies $\theta(C_i) \subseteq \theta(x_{i,j},x_{k,\ell})$ and $\theta(C_k) \subseteq \theta(x_{i,j},x_{k,\ell})$. Hence by case II, $\theta(y,x_{i,j})$ and $\theta(y,x_{k,\ell})$ are contained in $\theta(x_{i,j},x_{k,\ell})$; thus $\theta(y,x_{i,j}) \vee \theta(y,x_{k,\ell}) \subseteq \theta(x_{i,j},x_{k,\ell})$. By transitivity equality holds. Using lemma 0.5 and the three cases above, it follows that every element of $\Theta(\mathfrak{A})$ is the join of minimal congruence relations of the form $\theta(y,x_{i,j})$. But these minimal congruence relations have only one nontrivial congruence class, always containing y. Hence, the same thing is true for any join of minimal congruences. Therefore $\Theta(\mathfrak{A})$ can be considered as a sublattice of the lattice of all subsets of A, under the natural correspondence of θ to $[y]\theta$. Hence $\Theta(\mathfrak{A})$ is distributive. Also $\theta(y,x_{i,j})$ is join-irreducible; for if $\theta(y,x_{i,j}) = \psi \vee \phi$, then either $[x_{i,j}]\psi$ or $[x_{i,j}]\phi$ is non-trivial. Therefore ψ or ϕ contains $\theta(y,x_{i,j})$, which implies that ψ or ϕ equals $\theta(y,x_{i,j})$. Since every element of $\Theta(\mathfrak{A})$ is the join of elements of the form $\theta(y,x_{i,j})$, it follows that these elements are the only join-irreducibles. Hence $\Theta(\mathfrak{A}) \cong 2^X$ where X is the partially ordered set of the m distinct congruence relations of the form $\theta(y,x_{i,j})$ for $1 \leq i \leq m$, j arbitrary. But X is isomorphic as a partially ordered set to Y by the correspondence q_i to $\theta(y,x_{i,j})$. Thus $D \cong 2^Y \cong 2^X \cong \Theta(\mathfrak{A})$. Also it is clear that the map p from $\Theta(\mathfrak{A})$ to $\Sigma(\mathfrak{A})$ defined by $p(\theta) = [y]\theta$ is a lattice isomorphism with $p(I_A) = \{y\}$.

1.5 Note: In Chapter 4 it will be shown that many finite distributive lattices are not the congruence lattices of finite algebras with a single unary operation. Hence with regard to minimizing the number of operations on a multiunary algebra, Theorem 1.4 is the best possible result.

1.6 Note: If L is a finite lattice, then $\Theta(L)$ is a finite distributive lattice [5]. Conversely, a theorem of Dilworth states that every finite distributive lattice D is isomorphic to a $\Theta(L)$ for some finite lattice L. The first published proof of this result is in Grätzer and Schmidt [8]. In 1.13 below a different proof of this result will be given, in which the lattice L is easily constructed from the lattice D and indeed contains an isomorphic copy of D as a dual ideal.

1.7 Lemma: If D is a finite distributive lattice, and if p is a join-irreducible element and $p \leq \bigvee_{i=1}^{k} X_i$ for $X_i \in D$, $1 \leq i \leq k$, then $p \leq X_i$ for some i. Every element of D has a strictly unique representation as a join of the join-irreducible elements less than or equal to it.

Proof: Birkhoff [2], p.58.

1.8 Lemma: If a,b are elements of a lattice L and $a \equiv b(\theta)$ for some congruence relation θ on L, then $a \vee b \equiv a \wedge b(\theta)$

and $x \equiv y(\theta)$ for all $x,y \varepsilon [a \wedge b, a \vee b]$.

Proof: Birkhoff [2] p.27.

1.9 Definition: Let a,b,c,d be elements of a lattice L. The elements c,d are said to be weakly projective into the pair of elements a,b written $\overline{a,b} \rightarrow \overline{c,d}$ if for some $\{Z_1, Z_2, \ldots, Z_n\} \subseteq L$ the following is true:

$$(\ldots((((a \cup b) \cup Z_1) \cap Z_2) \cup Z_3) \cap \ldots) \cup Z_n = c \cup d$$

$$(\ldots((((a \cap b) \cup Z_1) \cap Z_2) \cup Z_3) \cap \ldots) \cup Z_n = c \cap d$$

1.10 Lemma: Let $a_i > b_i$ for $i \varepsilon I$ in a lattice L and let $\theta = \vee_{i \varepsilon I} \theta(a_i, b_i)$. Then $c \equiv d(\theta)$ if and only if there exists some finite sequence $c \cup d = Y_o > Y_1 > \ldots > Y_k = c \cap d$ such that for each j, $1 \leq j \leq k$ there exists some i, $i \varepsilon I$, such that $\overline{a_i, b_i} \rightarrow \overline{Y_{j-1}, Y_j}$.

Proof: This result was originally proved in Dilworth [4]. The notation and definition used in the above is taken from Grätzer and Schmidt [7].

1.11 Lemma: If L is a lattice of finite length, then $\theta \varepsilon \Theta(L)$ is join-irreducible if and only if $\theta = \theta(a,b)$ where a covers b in L.

Proof: For $\theta \varepsilon \Theta(L)$, $\theta = \vee\{\theta(a,b) \mid a \equiv b(\theta)$ and $a \succ b\}$. Hence if θ is join-irreducible, then $\theta = \theta(a,b)$ where $a \succ b$. Conversely let $a \succ b$ in L, and suppose $\theta(a,b) = \vee_{i \varepsilon I} \theta(a_i, b_i)$ where $a_i > b_i$ for each $i \varepsilon I$. Then since $a \succ b$, 1.10 implies that $\overline{a_i, b_i} \rightarrow \overline{a,b}$ for some $i \varepsilon I$. But then $a \equiv b(\theta(a_i, b_i))$, so $\theta\ (a,b)$ is join-irreducible.

1.12 Lemma: If D is a distributive lattice and $c > a > b > d$, then $c \equiv d\ (\theta(a,b))$ is impossible.

Proof: Grätzer and Schmidt [7] p.143.

1.13 Theorem: Let D be an arbitrary finite distributive lattice. Then there exists a finite lattice L such that $D \cong \Theta(L)$ and

(i) D is isomorphic to a dual ideal of L;

(ii) The dual of D, $\overline{D}$, is isomorphic to an ideal of L;

(iii) L has length $2n$ if D has length n;

(iv) If $|D| = m$, and D has length n, then $|L| = 2(m+n)-1$.

Proof: Let D be an arbitrary finite distributive lattice; let D_1 be isomorphic to D with $a^1 \varepsilon D_1$ corresponding to $a \varepsilon D$; let D_2 be isomorphic to the dual of D under the correspondence of $a \varepsilon D$ to $a^2 \varepsilon D_2$. Also let D_1 and D_2 be disjoint except for an element z such that $0^1 = 0^2 = z$. Let $D_1 \cup D_2$ be the distributive lattice obtained by having $a^1 \geq z \geq b^2$ for all a and b in D. Let $P = \{p_1, \ldots, p_n\}$ be the join-irreducible elements of D, and adjoin to $D_1 \cup D_2$ the new elements x_i and y_i for $1 \leq i \leq n$, where p_i^1 covers both x_i and y_i and x_i and y_i both cover p_i^2. Let L be the partially ordered set $D_1 \cup D_2 \cup \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\}$. Then L is a lattice with $\vee$ and $\wedge$ defined in the natural fashion. Note that for $a,b \varepsilon L$ the following relationships hold:

$a^1 \vee b^1 = (a \vee b)^1$; $a^2 \vee b^2 = (a \wedge b)^2$; $a^1 \vee b^2 = a^1$;

$a^1 \wedge b^1 = (a \wedge b)^1$; $a^2 \wedge b^2 = (a \vee b)^2$; $a^1 \wedge b^2 = b^2$;

$x_i \vee x_j = x_i \vee y_j = y_i \vee y_j = p_i^1 \vee p_j^1$;

$x_i \wedge x_j = x_i \wedge y_j = y_i \wedge y_j = p_i^2 \wedge p_j^2$;

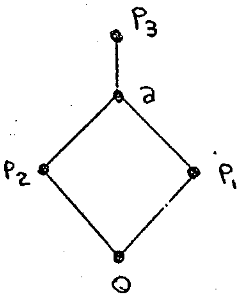
$a^1 \vee x_i = a^1 \vee p_i^1$ and $a^2 \wedge x_i = a^2 \wedge p_i^2$;

$a^1 \wedge x_i = x_i$ if $x_i < a^1$ and $a^1 \wedge x_i = p_i^2$ otherwise;

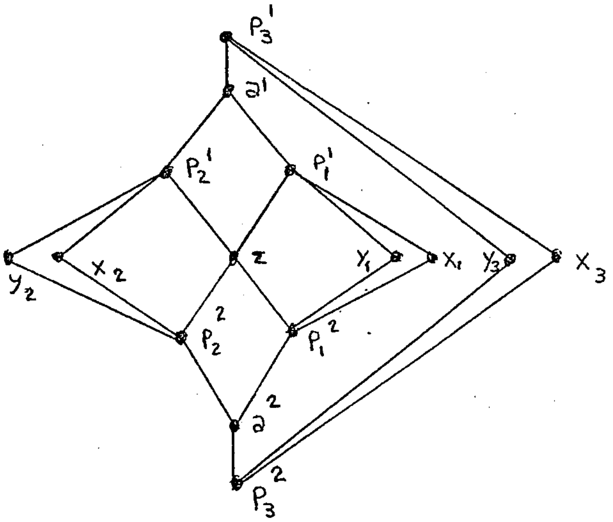
$a^2 \vee x_i = x_i$ if $x_i > a^2$ and $a^2 \vee x_i = p_i^1$ otherwise.

Note that $D_1 \cup D_2$ is a distributive sublattice of L.

Example of construction of L from D:



D



L

Claim: A) $\Theta(L) = \{\theta(a^1,a^2) | a \in D\}$;

B) $a \to \theta(a^1,a^2)$ is an isomorphism from D to $\Theta(L)$.

Proof of A) Consider $\theta(u,v)$ where $u,v \in L$. By lemma 1.8, without loss of generality it can be assumed that $u \geq v$. Also it can be assumed without loss of generality that $u,v \in D_1 \cup D_2$. For if not, and say $u = a^1$, $v = x_i$ and $a^1 > x_i$, then $a^1 \wedge y_i \equiv x_i \wedge y_i\ (\theta(a^1,x_i))$ implies $y_i \equiv p_i^2\ (\theta(a^1,x_i))$, and $a^1 \wedge z \equiv x_i \wedge z\ (\theta(a^1,x_i))$ implies $z \equiv p_i^2\ (\theta(a^1,x_i))$. Hence $y_i \vee z = p_i^1 \equiv p_i^2\ (\theta(a^1,x_i))$. But by lemma 1.8 and the fact that $x_i \in [p_i^2,p_i^1]$, it follows that $\theta(a^1,x_i) = \theta(a^1,p_i^2)$. Consider $\theta(u,v)$ for $u,v \in D_1 \cup D_2$, $u > v$. Without loss of generality take $u > z$. So $u = a^1$ for some $a \in D$. Let p_i^1 be any element of D_1 such that $p_i^1 \leq u$ and $p_i^1 \nleq v$. Then $u \wedge x_i \equiv v \wedge x_i\ (\theta(u,v))$ implies $x_i \equiv v \wedge p_i^2\ (\theta(u,v))$, and $u \wedge y_i \equiv v \wedge y_i\ (\theta(u,v))$ implies $y_i = v \wedge p_i^2\ (\theta(u,v))$. Hence $x_i \equiv y_i$ $(\theta(u,v))$ and $p_i^1 \equiv p_i^2\ (\theta(u,v))$ by 1.8. Let r be the join of all such elements p_i. Then it follows that $r^1 \equiv r^2$ $(\theta(u,v))$. If $v \geq z$ then $r^1 \equiv z\ (\theta(r^1,r^2))$ implies $u = v \vee r^1 \equiv v \vee r^2 = v\ (\theta(r^1,r^2))$, and hence $\theta(u,v) = \theta(r^1,r^2)$ for some $r \in D$. If $v < z$ then $r^1 = u$, and a dual argument shows that for some $s \in D$, $v = s^2$ and $s^1 \equiv s^2\ (\theta(u,v))$. But then $r^1 \vee s^1 \equiv z \equiv r^2 \wedge s^2 (\theta(u,v))$. Since $u,v \in [r^2 \wedge s^2, r^1 \vee s^1]$ it follows that $\theta(u,v) = \theta((r \vee s)^2, (r \vee s)^1)$. Hence every element $\theta(u,v)$ of $\Theta(L)$ is of the form $\theta(a^1,a^2)$ for some $a \in D$. By 0.5 it will be enough to show that $\theta(a^1,a^2) \vee \theta(b^1,b^2) = \theta((a \vee b)^1, (a \vee b)^2)$ in order to prove claim A. But this follows from the fact that $a^1 \equiv z \equiv a^2\ (\theta(a^1,a^2) \vee \theta(b^1,b^2))$ and $b^1 \equiv z \equiv b^2\ (\theta(a^1,a^2) \vee \theta(b^1,b^2))$.

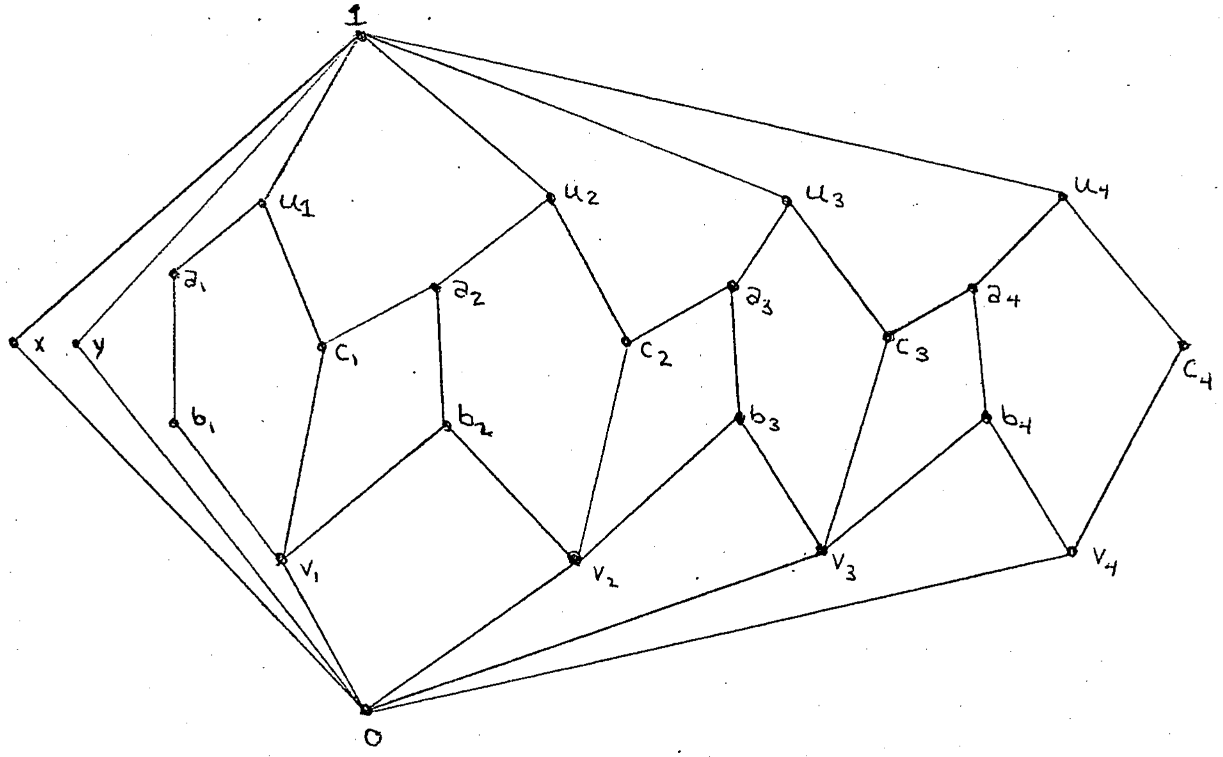
To prove claim B, let $\psi : D \to (L)$ given by $\psi(a) = \theta(a^1,a^2)$. Clearly ψ is onto and isotone. If ψ is one-to-one, it would have isotone inverse, and hence be an isomorphism.

So it is only necessary to show that ψ is one to one. This is equivalent to showing that $[z]\,\theta(a^1,a^2) = [a^2,a^1]$ for arbitrary $a \in D$. Clearly $[a^2,a^1] \subseteq [z]\,\theta(a^1,a^2)$. Assume that equality does not hold, so there exists $b \in D$, $b > a$ such that $[z]\,\theta(a^1,a^2) = [b^2,b^1]$. By 1.10 there exists a finite sequence $b^1 = u_o > u_1 > \dots > u_\ell = b^2$ of $u_j \in L$ such that $\overline{a^1,a^2} \rightarrow \overline{u_{j-1},u_j}$ for $1 \le j \le \ell$. For at least one of the intervals $[u_i,u_{i-1}]$, $[u_i,u_{i-1}] \nsubseteq [a^2,a^1]$. Let $[u_i,u_{i-1}]$ be any such interval. Then there exists a set $\{z_1,z_2,\dots,z_k\}$ such that $(\dots((a^1 \cup z_1) \cap z_2) \cup z_3 \dots \cup z_k) = u_{i-1}$ and $(\dots((a^2 \cup z_1) \cap z_2) \cup z_3 \dots \cup z_k) = u_i$. Let $Q = D_1 \cup D_2 \cup [a^2,a^1]$. Then without loss of generality each z_i can be chosen in Q. For if not, let t be miminal such that $z_t \notin Q$. Suppose $z_t = x_j$ and t is even. Consider $((\dots(a^1 \cup z_1)\dots) \cup z_{t-1}) \cap x_j$. This expression equals either x_j or $((\dots(a^1 \cup z_1)\dots) \cup z_{t-1}) \cap p_j^2$. If the latter holds then z_t can be replaced by p_j^2 without affecting anything. So assume that the expression equals x_j. Then $x_j < p_j^1 \le (\dots(a^1 \cup z_1)\dots) \cup z_{t-1}$, since x_j is both meet and join-irreducible and t is minimal. If $p_j^1 \le z_{t-1}$, then $x_j \le (\dots(a^2 \cup z_1)\dots) \cup z_{t-1}$, contradicting the fact that $u_{i-1} \neq u_i$. So 1.7 gives $p_j^1 \le (\dots(a^1 \cup z_1)\dots) \cap z_{t-2}$. Thus $p_j^1 \le z_{t-2}$, and $p_j^1 \le (\dots(a^1 \cup z_1)\dots) \cup z_{t-3}$. But if $p_j^1 \le z_{t-3}$, then $u_i = u_{i-1}$ as before, and it follows that $p_j^1 \le (\dots(a^1 \cup z_1)\dots) \cap z_{t-4}$, which implies $p_j^1 \le z_{t-4}$. Continuing in this way it follows that $p^1_j \leq z_\ell$ for $1 < \ell < t$ and ℓ an even integer. Hence $p^1_j < z_2$ and $p^1_j \leq a^1 \vee z_1$. But $p^1_j \nleq a^1$, since $x_j \notin Q$. Hence $p^1_j \leq z_1$. Therefore, $p^1_j < (\ldots((a^2 \cup z_1) \cap z_1) \cup z_3 \ldots \cup z_{t-1}$ and $x_j = (\ldots(a^2 \cup z_1) \cap z_2 \ldots z_{t-1}) \cap x_j$, contradicting the fact that $u_i \neq u_{i-1}$. A similiar argument holds for $z_t = y_1$, and a dual argument applies for t odd if x_i is replaced by p^1_j . Hence each z_t can be chosen to be in Q . Moreover when considering $\theta(a^1,a^2) \in \Theta(D_1 \cup D_2)$, each z_t can be considered to be in $D_1 \cup D_2$. In fact, if any $z_t = x_i$ or y_i for $a^2 < x_i, y_i < a^1$, then without loss of generality x_i or y_i can be replaced by an appropriate p^1_i or p^2_i . But then $b^1 = b^2 \; \theta(a^1,a^2)$ in $\Theta(D_1 \cup D_2)$, which contradicts 1.12 and hence proves claim B . Claims A and B together imply that $D \cong \Theta(L)$; i, ii, iii follow from the construction; and iv follows from the fact that the length of a finite distributive lattice is equal to the number of its join-irreducible elements (Birkhoff [2], p. 58).

<u>1.14 Remark</u>: Result 1.13 raises the following natural question: if D is a finite distributive lattice of length n , then what is the minimal length of a lattice L such that $D \cong \Theta(L)$? In 1.13 this length is shown to be at most $2n$ and in [8] Grätzer and Schmidt show it is at most $2n-1$. Theorems 1.15 and 1.16 below show that the minimal length for L depends heavily on the structure of D .

1.15 Theorem: Let D be a chain of length n. Then there exists a finite lattice L such that $D \cong \Theta(L)$ and L has length 5.

Proof: Let $D = \{z, p_1, \ldots, p_{n-1}, e\}$ where $z < p_1 < p_2 \ldots < p_{n-1} < e$. Let $L = \{0, 1, x, y\} \cup \bigcup_{i=1}^{n-2} \{u_i, a_i, b_i, c_i, v_i\}$, where $0 \prec x \prec 1$, $0 \prec y \prec 1$; $0 \prec v_i \prec b_i \prec a_i \prec u_i \prec 1$, $v_i \prec c_i \prec u_i$, for $1 \leq i \leq n-2$; and $c_i \prec a_{i+1}$, $v_i \prec b_{i+1}$ for $1 \leq i \leq n-3$. Then $\{0, 1, x, y, t\}$, for $t \in L \setminus \{0, 1, x, y\}$, is isomorphic to D_3; $\{u_i, c_i, a_i, b_i, v_i\}$ and $\{1, u_i, u_{i+1}, a_{i+1}, c_i\}$ are both isomorphic to N_5; $\{c_i, v_i, a_{i+1}, b_{i+1}\}$ and $\{v_i, b_{i+1}, v_{i+1}, 0\}$ are both isomorphic to 2^2. It follows easily that L is a lattice. Below is a covering diagram for L when $n = 6$:



Consider $\theta(r,s) \in \Theta(L)$ where $s \prec r$. If $r = 1$ or $s = 0$, then $\theta(r,s) = U_L$, since D_3 has no proper congruence relations. If $r = u_i$ and $s = c_i$, $1 \leq i \leq n-3$, then $u_i \vee u_{i+1} \equiv c_i \vee u_{i+1}$ $(\theta(r,s))$ implies $1 \equiv u_{i+1}$ $(\theta(r,s))$, so $\theta(r,s) = U_L$. In the case $r = u_{n-2}$, $s = c_{n-2}$, $u_{n-2} \wedge b_{n-2} \equiv c_{n-2} \wedge b_{n-2}$ $(\theta(r,s))$, so $b_{n-2} \equiv v_{n-2}$ $(\theta(r,s))$ and $0 \equiv v_{n-3}$ $(\theta(r,s))$. So again $\theta(r,s) = U_L$. If $r = u_i$ and $s = a_i$, then $\theta(u_i, a_i)$ has as its congruence classes the intervals $[b_j,u_j]$, $[v_j,c_j]$ for $1 \leq j \leq i$ and $[b_{i+1},a_{i+1}]$. If $r = a_i$ and $s = b_i$, then $\theta(a_i,b_i) = \theta(u_{i-1},a_{i-1})$. Similiarly $\theta(c_i,v_i) = \theta(u_i,a_i)$. Finally if $r = b_i$, $i > 1$, and $s = v_i$, then $v_{i-1} \wedge b_i = v_{i-1} \wedge v_i$ $(\theta(b_i,v_i))$ implies $v_{i-1} \equiv 0$ $(\theta(b_i,v_i))$, so $\theta(b_i,v_i) = U_L$. Similiarly for $s = v_{i-1}$. If $r = b_1$, then $s = v_1$. But $b_1 \vee u_2 = v_1 \vee u_2$ $(\theta(b_1,v_1))$ gives $1 \equiv u_2(\theta(b_1,v_1))$, so $\theta(b_1,v_1) = U_L$. Hence, the congruence relations on $\Theta(L)$ are $\{ I_L, \theta(a_1,b_1), \theta(u_1,a_1), \ldots, \theta(u_{n-2},a_{n-2}), U_L \}$, and these form a chain of length n.

<u>1.16 Theorem</u>: Let L be any lattice such that $\Theta(L) \cong 2^n$ where n is finite. Then L has length at least n.

<u>Proof</u>: Suppose L has length $m < \omega$. Then in L there is a chain $0 = z_0 \prec z_1 \prec z_2 \prec \ldots \prec z_m = 1$, and by 1.11 each $\theta(z_i, z_{i+1})$ for $0 \leq i \leq m-1$ is join-irreducible. By transitivity $\theta(0,1) = \bigvee_{0 \leq i \leq m-1} \theta(z_i, z_{i+1})$. But also since $\Theta(L) \cong 2^n$, $\theta(0,1)$ is a unique join of n join-irreducible

elements. Hence at least n of the $\theta(z_i, z_{i+1})$ are distinct. Thus $m \geq n$.

1.17 Remark: In connection with 1.16 it should be noted that if L is a finite lattice that is modular or relatively complemented, then $\Theta(L) \cong 2^n$. See Dilworth [4] or Crawley [3] for proofs.

1.18 Theorem: Let $\mathfrak{A}$ be any finite partial multiunary algebra and $\Theta_s(\mathfrak{A})$ the lattice of strong congruence relations on $\mathfrak{A}$. Then there exists a finite multiunary algebra $\bar{\mathfrak{A}}$ such that $\Theta_s(\mathfrak{A}) = \Theta(\bar{\mathfrak{A}})$.

Proof: Let $\mathfrak{A} = \langle A; f_1, \ldots, f_n \rangle$ be any finite partial multiunary algebra. Without loss of generality, assume that the identity function and all constant functions occur among the f_i . A polynomial of length k will denote an operation of the form $f_{i_1} f_{i_2} \ldots f_{i_k}$ where $f_{i_1} f_{i_2} \ldots f_{i_k}(x) = f_{i_1}(f_{i_2}(\ldots(f_{i_k}(x))\ldots))$. A polynomial $f_{i_1} f_{i_2} \ldots f_{i_k}$ is said to anihilate an element $x \varepsilon A$ if $f_{i_1} f_{i_2} \ldots f_{i_k}(x)$ does not exist but $f_{i_2} \ldots f_{i_k}(x)$ does exist. By convention, if $f_i(x)$ does not exist, then f_i anihilates x . Let $a(x) = \{p \mid p$ is a polynomial and p anihilates $x\}$. Let $A(x) = \{y \mid y \varepsilon A$ and $a(x) = a(y)\}$. Let $A_0 = \{x \mid x \varepsilon A$ and $a(x) = \phi\}$. Now index the $A(x)$ arbitrarily to obtain n subsets $A_1, \ldots, A_n$ that form a partition of A: $A = A_0 \cup A_1 \cup \ \cup A_n$, where $A_i \cap A_j = \phi$ for $i \neq j$. In the event that $n = 0$ then $\mathfrak{A}$ is itself an algebra and the proof is complete so assume $n \geq 1$. Under these definitions, it follows that if $\theta \varepsilon \Theta_s(\mathfrak{A})$ and

$x \equiv y(\theta)$, then x and y must be in the same A_i for some i, $0 \leq i \leq n$. If not, then say there exists a polynomial $f_{i_1} \dots f_{i_k}$ that anihilates x but does not anihilate y. So either $f_{i_1} \dots f_{i_k}(y)$ does exist and $f_{i_1} \dots f_{i_k}(x)$ does not exist or else $f_{i_2} \dots f_{i_k}(y)$ does not exist and $f_{i_2} \dots f_{i_k}(x)$ does exist. However, both cases violate the hypothesis that θ is a strong congruence relation.

Let $\bar{A} = A_0 \times A_1 \times \dots \times A_n$, and define a function of $\Theta_s(\mathfrak{A})$ to $\mathcal{E}(\bar{A})$ by letting $\theta \in \Theta_s(\mathfrak{A})$ correspond to $\bar{\theta} \in \mathcal{E}(\bar{A})$, where $\langle x_0, x_1, \dots, x_n \rangle \equiv \langle y_0, y_1, \dots, y_n \rangle \; (\bar{\theta})$ if and only if $x_i \equiv y_i \; (\theta)$ for $0 \leq i \leq n$. This function is obviously well defined.

<u>Claim</u>: $\theta \to \bar{\theta}$ is a lattice isomorphism of $\Theta_s(\mathfrak{A})$ into $\mathcal{E}(\bar{A})$. To verify this claim let θ and φ be arbitrary elements of $\Theta_s(\mathfrak{A})$, and let $\bar{x} = \langle x_0, x_1, \dots, x_n \rangle$ and $\bar{y} = \langle y_0, y_1, \dots, y_n \rangle$ be typical elements of $\bar{A}$. $\underline{\overline{\theta \wedge \varphi} = \bar{\theta} \wedge \bar{\varphi}}$: $\bar{x} \equiv \bar{y}(\bar{\theta} \wedge \bar{\varphi}) \leftrightarrow \bar{x} \equiv \bar{y}(\theta)$ and $\bar{x} \equiv \bar{y}(\bar{\varphi}) \leftrightarrow x_i = y_i(\theta)$ and $x_i = y_i(\varphi)$ for $0 \leq i \leq n \leftrightarrow x_i = y_i(\theta \wedge \varphi)$ for $0 \leq i \leq n \leftrightarrow \bar{x} \equiv \bar{y} \; (\overline{\theta \wedge \varphi})$.

$\underline{\overline{\theta \vee \varphi} = \bar{\theta} \vee \bar{\varphi}}$: Clearly $\overline{\theta \vee \varphi} \geq \bar{\theta} \vee \bar{\varphi}$, so the opposite inequality must be demonstrated. If $\bar{x} \equiv \bar{y} \; (\overline{\theta \vee \varphi})$ then $x_j \equiv y_j \; (\theta \vee \varphi)$, and hence there exists $z_j^1, \dots z_j^{\ell_j}$ such that $x_j \equiv z_j^1 \; (\theta)$, $z_j^1 \equiv z_j^2 \; (\varphi)$, $z_j^2 \equiv z_j^3 \; (\theta) \dots \; z_j^{\ell_j} \equiv y_j \; (\varphi)$. Let $\ell = \max\limits_{0 \leq j < n} \{\ell_j\}$ and for those j such that $\ell_j < \ell$ define

$z_j^{\ell_j} = z_j^{\ell_j+1} = \ldots = z^{\ell}$. Now define $\bar{z}^{\,i} = \langle z_0^i, z_1^i, \ldots, z_n^i \rangle$ for $1 \leq i \leq \ell$. Then $\bar{x} \equiv \bar{z}^1(\theta)$, $\bar{z}^1 \equiv \bar{z}^2(\varphi)$, $\bar{z}^2 \equiv z^3(\theta)$, $\ldots$, $\bar{z}^{\ell} \equiv \bar{y}\,(\varphi)$, and hence $\bar{x} \equiv \bar{y}\,(\bar{\theta} \vee \bar{\varphi})$.

$\underline{\theta \neq \varphi \text{ implies } \bar{\theta} \neq \bar{\varphi}}$: If $\theta \neq \varphi$, then, without loss of generality, there exists some i and elements x_i and y_i in A_i such that $x_i \equiv y_i(\theta)$ and $x_i \not\equiv y_i(\varphi)$. Then for arbitrary $z_j \varepsilon A_j$, when $j \neq i$, $\langle z_0, z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots z_n \rangle \equiv \langle z_0, z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_n \rangle\,(\bar{\theta})$ $\langle z_0, z_1, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n \rangle \not\equiv \langle z_0, z_1, \ldots, z_{i-1}, y_i, z_{i+1}, \ldots, z_n \rangle\,(\bar{\varphi})$. Hence the function $\theta \rightarrow \bar{\theta}$ is a lattice isomorphism. Denote the image of $\Theta_s(\mathfrak{A})$ by $\overline{\Theta_s(\mathfrak{A})}$.

Define the following unary operations on A : for each $x \,\varepsilon\, A_i \subseteq A$ define $p_x(\langle z_0, z_1, \ldots, z_n \rangle) = \langle z_0, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n \rangle$. For each i, $1 \leq i \leq m$ define $h_i(\langle z_0, z_1, \ldots, z_n \rangle) = \langle f_i(z_0), z_1, \ldots, z_n \rangle$. For each A_i, $1 \leq i \leq n$ and each f_j such that f_j is defined on A_i, define $g_{i,j}(\langle z_0, z_1, \ldots, z_n \rangle) = \langle z_0, \ldots, z_{k-1}, f_j(z_i), z_{k+1}, \ldots, z_n \rangle$ where $f_j(z_i) \,\varepsilon\, A_k$. Note that p_x is a well defined operation as is h_i, since $z_0 \,\varepsilon\, A_0$ implies $f_i(z_0) \,\varepsilon\, A_0$. If $A_0 = \emptyset$ then $\{h_i\} = \emptyset$. To show $g_{i,j}$ is well defined, let $u, v \,\varepsilon\, A_i$, $i > 0$ and let $f_{i_1} \ldots f_{i_\ell}$ be any anihilating polynomial for $f_j(u)$. It then follows that $f_{i_1} \ldots f_{i_\ell} f_j$ is an anihilating polynomial for u and hence for v . So $f_{i_1} \ldots f_{i_\ell}$ is an anihilating polynomial for $f_j(v)$. So $a(f_j(u)) \subseteq a(f_j(v))$. Similiarly $a(f_j(v)) \subseteq a(f_j(u))$, and thus $a(f_j(v)) = a(f_j(u))$. Thus $f_j(v)$ and $f_j(u)$ are in A_k for some k , so $g_{i,j}$ is well defined.

Let $\overline{\mathfrak{A}} = \langle \bar{A}, F \rangle$, where $F = \{p_x \mid x \in A\} \cup \{h_i \mid 1 \leq i \leq m\} \cup \{g_{i,j} \mid f_j$ exists for $x \in A_i, 1 \leq i \leq n\}$.

Claim $\Theta(\overline{\mathfrak{A}}) = \overline{\Theta_s(\mathfrak{A})}$: A direct verification shows that each $\bar{\theta} \in \overline{\Theta_s(\mathfrak{A})}$ is a congruence relation on $\overline{\mathfrak{A}}$ and hence $\overline{\Theta_s(\mathfrak{A})} \subseteq \Theta(\overline{\mathfrak{A}})$. To prove the opposite inclusion, let $\bar{y} = \langle y_o, y_1, \dots, y_n \rangle$ and $\bar{z} = \langle z_o, z_1, \dots, z_n \rangle$ be arbitrary elements of $\bar{A}$ and let $\theta(\bar{y}, \bar{z})$ be the minimal congruence relation generated by $\bar{y}$ and $\bar{z}$ in $\overline{\mathfrak{A}}$. Let $\theta(y_i, z_i)$ for $0 \leq i \leq n$ be the minimal strong congruence relation generated by y_i and z_i in $\mathfrak{A}$. It is enough to show that $\theta(\bar{y}, \bar{z}) = \overline{\theta(y_o, z_o)} \vee \overline{\theta(y_1, z_1)} \vee \dots \vee \overline{\theta(y_n, z_n)}$.

$$
\begin{aligned}
\text{But } \langle y_o, y_1, \dots, y_n \rangle &\equiv \langle y_o, y_1, \dots, y_{n-1}, z_n \rangle \quad (\overline{\theta(y_n, z_n)}) \\
&\equiv \langle y_o, y_1, \dots, z_{n-1}, z_n \rangle \quad (\overline{\theta(y_{n-1}, z_{n-1})}) \\
&\quad \cdot \\
&\quad \cdot \\
&\quad \cdot \\
&\equiv \langle z_o, z_1, \dots, z_{n-1}, z_n \rangle \quad (\overline{\theta(y_o, z_o)}),
\end{aligned}
$$

so $\theta(\bar{y}, \bar{z}) \leq \overline{\theta(y_o, z_o)} \vee \overline{\theta(y_1, z_1)} \vee \dots \vee \overline{\theta(y_n, z_n)}$. To prove the opposite inequality, it will be sufficient to show $\theta(\bar{y}, \bar{z}) \geq \overline{\theta(y_i, z_i)}$ for $0 \leq i \leq n$. Let $\bar{u} = \langle u_o, u_1, \dots, u_n \rangle$ and $\bar{v} = \langle v_o, v_1, \dots, v_n \rangle$ be arbitrary elements of $\bar{A}$ such that $\bar{u} \equiv \bar{v}\ (\overline{\theta(y_i, z_i)})$. Note that if $u_j \equiv v_j\ (\theta(y_i, z_i))$, then by 0. 10 there exist elements $u_j = x_o^j, x_1^j, \dots, x_{s_j}^j = v_j$ and polynomials $q_o^j, \dots, q_{s_j - 1}^j$ such that $\{q_k^j(y_i), q_k^j(z_i)\} = \{x_k^j, x_{k+1}^j\}$ for $0 \leq k \leq s_j - 1$. For each of these polynomials

q_k^j , let $\bar{q}_k^j$ be the corresponding polynomial in $\bar{\mathfrak{A}}$, i.e. $f_t = h_t$ if $i = 0$ and $\bar{f}_t = g_{i,t}$ for $i > 0$. Successively applying p_{y_t}, $t \neq i$ to $\bar{y} \equiv \bar{z}$ $(\Theta(\bar{y},\bar{z}))$ gives $\langle y_o, y_1, \ldots, y_i, y_{i+1}, \ldots, y_n \rangle \equiv \langle y_o, y_1, \ldots, z_i, y_{i+1}, \ldots, y_n \rangle$ $(\Theta(\bar{y}, \bar{z}))$. Applying the polynomials $\bar{q}_k^j$ for $0 \leq k \leq s_{j-1}$ gives $\langle y_o, y_1, \ldots, u_j, y_{j+1}, \ldots, y_n \rangle \equiv \langle y_o, y_1, \ldots, x_1^j, \ldots, y_n \rangle$ $(\Theta(\bar{y}, \bar{z}))$

$\langle y_o, y_1 \ldots x_1^j, y_{j+1}, \ldots, y_n \rangle \equiv \langle y_o, y_1, \ldots, x_2^j, \ldots, y_n \rangle$ $(\theta(\bar{y}, \bar{z}))$

$\langle y_o, y_1 \ldots x_{s_{j-1}}^j , y_{j+1} \ldots y_n \rangle \equiv \langle y_o, y_1, \ldots, v_j, y_{j+1}, \ldots, y_n \rangle$ $(\Theta(\bar{y}, \bar{z}))$. Then by transitivity it follows that $\langle y_o, \ldots, u_j, y_{j+1}, \ldots, y_n \rangle \equiv \langle y_o, \ldots, v_j, y_{j+1}, \ldots, y_n \rangle$ $(\Theta(\bar{y}, \bar{z}))$. The above holds for all j , $0 \leq j \leq n$. In particular it follows that

(1) $\langle y_o, y_1, \ldots, y_{n-2}, y_{n-1}, u_n \rangle \equiv \langle y_o, y_1, \ldots, y_{n-2}, y_{n-1}, v_n \rangle$ $(\theta(\bar{y}, \bar{z}))$ and

(2) $\langle y_o, y_1, \ldots, y_{n-2}, u_{n-1}, y_n \rangle \equiv \langle y_o, y_1, \ldots, y_{n-2}, v_{n-1}, y_n \rangle$ $(\theta(\bar{y}, \bar{z}))$. Applying $p_{u_{n-1}}$ to (1) and p_{v_n} to (2), and using transitivity, gives $\langle y_o, y_1, \ldots, y_{n-2}, u_{n-1}, u_n \rangle \equiv \langle y_o, y_1, \ldots, y_{n-2}, v_{n-1}, v_n \rangle$ $(\Theta(\bar{y}, \bar{z}))$. Continuing in this manner, it follows that $\bar{u} = \bar{v}$ $(\Theta(\bar{y}, \bar{z}))$, and therefore $\Theta(\bar{y}, \bar{z}) \geq \overline{\Theta(y_i, z_i)}$ as desired.

<u>1.19 Theorem:</u> The class $\mathcal{L}_2$ of lattices isomorphic to strong congruence lattices of finite partial algebras is equal to the class $\mathcal{L}_3$ of lattices isomorphic to congruence lattices of finite algebras.

<u>Proof</u>: This follows directly from 0.8 and 1.18.

<u>Chapter 2</u>

DISTRIBUTIVE LATTICES OF EQUIVALENCE RELATIONS

In [13] Pierce poses the problem of characterizing those sublattices L of $\mathcal{E}(A)$ such that there is a finitary algebra $\mathfrak{A} = \langle A; f_j \rangle_{j \varepsilon J}$ such that $\Theta(\mathfrak{A}) = L$. This section is concerned with this problem when A is finite.

<u>2.1 Definition</u>: A sublattice L of $\mathcal{E}(A)$ is said to be <u>spanning</u> if the unit element of L exists and is U_A and the zero element of L exists and is I_A.

<u>2.2 Remark</u>: If L is a sublattice of $\mathcal{E}(A)$ such that L is the congruence lattice for some algebra defined on A, then clearly L must be a spanning sublattice. In [13, p. 58] Pierce shows that if $|A| = p^2$ for p an odd prime, then $\mathcal{E}(A)$ has a spanning sublattice which is not the congruence lattice of any algebra on A.

<u>2.3 Example</u>: Let $A = \{1,2,3,4\}$. Let L be the sublattice of $\mathcal{E}(A)$ with elements $\{I_A, U_A, 12|34, 13|24, 14|23\}$ and let M be the sublattice of $\mathcal{E}(A)$ with elements $\{I_A, U_A, 12|34, 13|24, 14|2|3\}$ Then $L \cong M \cong D_3$, and both L and M are spanning sublattices of $\mathcal{E}(A)$. It can be shown that L is the congruence lattice of an algebra defined on A, while M cannot be the congruence lattice of any algebra defined on A. This example illustrates the fact that whether or not a

spanning sublattice L of $\mathcal{E}(A)$ is congruence lattice for some algebra defined on A may depend on the way L is embedded in $\mathcal{E}(A)$. However if L is distributive this is not the case as the following theorem shows.

<u>2.4 Theorem</u>: Let $|A| = n < \omega$ and let D be a distributive spanning sublattice of $\mathcal{E}(A)$. Then D is the congruence lattice of some multiunary algebra defined on A.

<u>Proof</u>: Let $P = \{p_1, p_2, \ldots, p_k\}$ be the set of join-irreducible elements of D. For $p_i \in P$ define $\bar{p}_i = \bigvee\{p \in P \mid p \not> p_i\}$. If $p_i \leq p$ for all $p \in P$, then $\bar{p}_i = I_A$. Note that $\bar{p}_i \in D$. Also $\bar{p}_i > p_i$ is impossible; for if $p_i = p_{i_1} \vee \ldots \vee p_{i_m} \geq p_i$, then by 1.7 and the hypothesis that D is distributive, there exists some p_{j_ℓ} such that $p_{j_\ell} \geq p_i$. However, $p_{j_\ell} \not\geq p_i$ because of the definition of $\bar{p}_i$.

For arbitrary $p_j \in P$ and arbitrary $\langle x,y \rangle \in p_j$ and arbitrary $u \in A$, define the unary function ${}^{j}f_u^{x,y}$ on A by ${}^{j}f_u^{x,y}(v) = x$ if $\langle u,v \rangle \in \bar{p}_j$ and ${}^{j}f_u^{x,y}(v) = y$ if $\langle u,v \rangle \notin \bar{p}_j$. Let $\mathfrak{A}$ be the multiunary algebra defined on A by all of these unary operations.

Claim: $\Theta(\mathfrak{A}) = D$. Let $d \in D$ and ${}^{j}f_u^{x,y}$ be an operation in $\mathfrak{A}$. If $\langle x,y \rangle \in d$ then d is stable under ${}^{j}f_u^{x,y}$ since the image of ${}^{j}f_u^{x,y}$ is contained in $\{x,y\}$. If $\langle x,y \rangle \notin d$ then $p_j \not\leq d$, and $d \leq \bar{p}_j$. So if $\langle a,b \rangle \in d$ and

${}^{j}f_u^{x,y}(a) = x$, then by definition, $<u,a> \in \bar{p}_j$. But $<a,b> \in d$ implies $<a,b> \in \bar{p}_j$, and by transitivity $<u,b> \in \bar{p}_j$. Thus ${}^{j}f_u^{x,y}(b) = x$ also. So again d is stable under ${}^{j}f_u^{x,y}$. It follows that $D \subseteq \Theta(\mathfrak{A})$. Let $\theta \in \Theta(\mathfrak{A})$, and suppose $\theta \notin D$. Define $R = \{p \in P \mid p \nleq \theta\}$. $R \neq \phi$ because $\theta \neq U_A = \bigvee_{1 \leq i \leq k} p_i$. For $p_i \in R$ note that $p_i > \theta$. Otherwise $\bar{p}_i \ngeq \theta$, and there would exist $<a,b> \in \theta$ such that $<a,b> \notin \bar{p}_i$. Then, for all $<x,y> \in p_i$, ${}^{i}f_a^{x,y}(a) = x$ and ${}^{i}f_a^{x,y}(b) = y$, so that $<x,y> \in \theta$. Hence $\theta > p_i$, contradicting the hypothesis that $p_i \in R$. Let $r = \bigcap_{p_i \in R} \bar{p}_i$. Then $r > \theta$ and in particular $r \neq I_A$. Let $r = p_{i_1} \vee \dots \vee p_{i_k}$ where $\{p_{i_1}, \dots, p_{i_k}\} \subseteq P$. None of the p_{i_ℓ} , $1 \leq \ell \leq k$ can be in R , since otherwise $\bar{p}_{i_\ell} \geq r \geq p_{i_\ell}$, which is impossible as was previously noted. So each p_{i_ℓ} is less than θ . This implies that $r < \theta < r$, which is a contradiction. Thus, $\theta \notin D$ is impossible and the theorem is proved.

2.5 Remark: An examination of the proof of 2.4 shows that distributivity was only used to guarantee that $\bar{p}_i \ngeq p_i$. However it can be shown that for finite lattices this condition is equivalent to distributivity.

If L is a spanning sublattice of $\mathcal{E}(A)$ the functions ${}^{i}f_u^{x,y}$ defined in the proof of 2.4 will give an algebra $\mathfrak{A}$ over A such that L is a sublattice of $\Theta(\mathfrak{A})$.

Let $Q = \{ p_i \varepsilon L | p_i$ is join-irreducible in L and $\bar{p}_i \not\geq p_i \}$. Call an element r of L a Q-type element if r can be written as the join of elements in Q. Note that the join of Q-type elements is a Q-type element. For any element $r \varepsilon L$ such that r is not a Q-type element define $q(r)$ to be the join of all Q-type elements of L less than r.

<u>2.6 Corollary</u>: If L is a spanning sublattice of $\mathcal{E}(A)$, then $L = \Theta(\mathfrak{A})$ for some algebra $\mathfrak{A}$ over A provided $q(r) < \theta < r$ implies $\theta \varepsilon L$ for all $r \varepsilon L$ and $\theta \varepsilon \mathcal{E}(A)$.

<u>Proof</u>: Let $\mathfrak{A}$ be the multiunary algebra constructed in the proof of 2.4. Then L is a sublattice of $\Theta(\mathfrak{A})$. If $\theta \varepsilon \Theta(\mathfrak{A})$ and $\theta \not\varepsilon L$, then, as in the proof of 2.4, $\theta < r$, where $r = \bigcap_{p_i \varepsilon R} \bar{p}_i$, $R = \{ p \varepsilon P | p \not\leq \theta \}$. If $p \leq r$ and $p \varepsilon R$, then $\bar{p} \geq r \geq p$. Hence, $p \not\varepsilon Q$. Thus $q(r) < \theta < r$, and by hypothesis $\theta \varepsilon L$, a contradiction.

<u>2.7 Example</u>: As an application of 2.6, note that if $r \varepsilon L$ is not a Q-type element but $q(r)$ is covered by r in $\mathcal{E}(A)$, then L is $\Theta(\mathfrak{A})$, where $\mathfrak{A}$ is defined as in 2.4 and 2.6. In particular, if L is a spanning sublattice of $\mathcal{E}(A)$, $L \cong N_5$, and the element corresponding to $a \varepsilon N_5$ covers (in $\mathcal{E}(A)$) the element corresponding to $b \varepsilon N_5$, then L is the congruence lattice of some algebra defined on A. However, if $A = \{1, 2, \ldots, 9\}$, and L is the sublattice of $\mathcal{E}(A)$ with elements $\{I_A, U_A$, 1|2|3|45|67|89, 123|45|67|89, 158|26|34|79$\}$, then $L \cong N_5$, but $L \neq \Theta(\mathfrak{A})$ for any multiunary algebra $\mathfrak{A}$ on A. For if $L = \Theta(\mathfrak{A})$, then $\theta(1,3) =$ 123|45|67|89; and since 1|2|3|45|67|89 is a congruence relation,

it follows that for some polynomial f, $f(\{1,3\}) = \{2,i\}$, where $i = 1$ or 3. Examination of the different possibilities for f shows this is impossible.

<u>2.8 Theorem</u>: Let $\mathcal{K}$ be an abstract class of finite lattices such that if $L \in \mathcal{K}$ and M is a sublattice of L then $M \in \mathcal{K}$. Then the following are equivalent:

(i) if L is a spanning sublattice of $\mathcal{E}(A)$ for some finite set A and $L \in \mathcal{K}$, then $L = \Theta(\mathfrak{A})$ for some algebra $\mathfrak{A}$ on A.

(ii) every lattice in $\mathcal{K}$ is distributive.

<u>Proof</u>: This theorem follows from 2.4, 2.3, 2.7, and the fact that a lattice L is distributive if and only if L contains no sublattice isomorphic to N_5 or D_3.

Chapter 3

MULTIUNARY ALGEBRAS

It was shown in 0.8 that the study of the class of congruence lattices of algebras can be reduced to the study of congruence lattices of multiunary algebras. Moreover, finite algebras have only finitely many unary operations. This section is concerned with the congruence lattices of finite multiunary algebras having few operations.

3.1 Theorem: (McKenzie) Let $|A| = n < \omega$, and let $\mathfrak{A} = \langle A; f_1, \ldots, f_n \rangle$ be a multiunary algebra. If $B = A^{n+m+1}$, then there exist four unary operations g_1, g_2, g_3, g_4 on B such that $\Theta(\mathfrak{A}) \cong \Theta(\mathfrak{B})$, where $\mathfrak{B} = \langle B; g_1, g_2, g_3, g_4 \rangle$.

Proof: See McKenzie [12] p.12.

3.2 Example: One might hope to improve McKenzie's result by obtaining a theorem such as the following: If $\mathfrak{A} = \langle A; f_1, \ldots, f_n \rangle$ is a finite multiunary algebra, then there exists a multiunary algebra $\mathfrak{B} = \langle A; g_1, \ldots, g_\ell \rangle$ with $2 \leq \ell < m$ such that $\Theta(\mathfrak{A}) = \Theta(\mathfrak{B})$. A simple example shows this is impossible. Let $S = \{1, 2, \ldots, 2n\}$ and define on S the unary operations f_j, $1 \leq j \leq n-1$, by $f_j(1) = 1 + 2j$, $f_j(2) = 2 + 2j$, $f_j(k) = k$ for $2 < k \leq 2n$. Let $\mathfrak{A} = \langle S; f_1, \ldots, f_{n-1} \rangle$. Then if $\mathfrak{B}$ is any multiunary algebra defined on S such that $\Theta(\mathfrak{A}) = \Theta(\mathfrak{B})$,

$\mathfrak{B}$ must have at least $n-1$ unary operations. This assertion is a consequence of the fact that any polynomial formed from the $f_1,\dots,f_{n-1}$ applied to $\{1,2\}$ must have as its image a set of the form $\{1+2j,\ 2+2j\}$ $1 \leq j \leq n-1$. Hence, $\theta(1,2)$ has as its congruence classes $\{1,2\}$, $\{3,4\}$,..., $\{2n-1,2n\}$. However, any $\theta(i,i+1) \varepsilon \Theta(\mathfrak{A})$, $i = 2k+1,\ 1 \leq k \leq n-1$, has only one non trivial congruence class $\{i,\ i+1\}$. Therefore, if g is any unary operation from $\mathfrak{B}$, $g(\{i,i+1\}) \subseteq \{i,i+1\}$ for $i = 2k+1,\ 1 \leq k \leq n-1$. Consequently, a multinuary algebra $\mathfrak{B}$ must have $n-1$ operations in order that $\theta(1,2) \varepsilon \Theta(\mathfrak{B})$ equals $\theta(1,2) \varepsilon \Theta(\mathfrak{A})$.

<u>3.3 Theorem</u>: Let $\mathfrak{A} = \langle A; f_1,\dots,f_q \rangle$ be a finite multiunary algebra with $A = \{a_1,\dots,a_n\}$. Then there exists a finite algebra $\mathfrak{B} = \langle B;\ g,h \rangle$ such that $\Theta(\mathfrak{B})$ has a unique maximal ideal isomorphic to $\Theta(\mathfrak{A})$.

<u>Proof</u>: By 3.1 it can be assumed that $q = 4$, although this is not crucial for the following proof. Let $m+1$ be a prime, $m > 5n$. Let $B = \{a_1,\dots,a_n\} \cup \{b_1,\dots,b_n\} \cup \{c_1,\dots,c_n\} \cup \{d_1,\dots,d_n\} \cup \{e_1,\dots,e_n\} \cup \{p_o,p_1,\dots,p_m\}$, where the six sets composing B are pairwise disjoint. Define g and h on B as follows:

$$\begin{array}{c|cccccc}
 & a_i & b_i & c_i & d_i & e_i & p_0p_1p_2\cdots p_np_{n+1}\cdots p_{2n+1}\cdots \\ \hline
g & f_1(a_i) & f_2(a_i) & f_3(a_i) & f_4(a_i) & p_0 & p_0a_1a_2\cdots a_nb_1\cdots \quad c_1\cdots \\
h & b_i & c_i & d_i & e_i & a_i & p_1p_2p_3\cdots p_{n+1}p_{n+2}\cdots p_{2n+2}\cdots
\end{array}$$

$$\begin{array}{c|cccc}
 & p_{3n+1}\cdots & p_{4n+1}\cdots & p_{5n} & p_{5n+1}\cdots p_m \\ \hline
g & d_1\cdots & e_1\cdots & e_n & p_{5n+1}\cdots p_m \\
h & p_{3n+2}\cdots & p_{4n+2}\cdots & p_{5n+1} & p_{5n+2}\cdots p_0
\end{array}$$

.

Let $\mathfrak{B} = \langle B; g, h \rangle$. Consider $\theta \varepsilon \Theta(\mathfrak{B})$ and $x \equiv y(\theta)$, $x \neq y$. Several cases arise:

Case I: $x = p_o$ and $y = z_i$ where $z \varepsilon \{a,b,c,d,e\}$. Without loss of generality let $z_i = a_i$. Since $m + 1$ is prime and $m + 1 > 5$, iterations of h will give $a_i \equiv p_o \equiv p_1 \equiv \cdots \equiv p_m(\theta)$; then application of g shows $\theta = U_B$.

Case II: $x = p_j$ and $y = z_i, z_i$ as in case 1. Applying h $m - j + 1$ times reduces this to case I .

Case III: $x = p_j$ and $y = p_i$. Enough iterations of h show that $p_o \equiv p_1 \equiv p_2 \equiv \cdots \equiv p_m (\theta)$ and an application of g gives $\theta = U_B$.

Case IV: $x = z_i$ and $y = w_j$ where $w, z \varepsilon \{a,b,c,d,e\}$, $w \neq z$. Without loss of generality let $x = a_i$ and $y = b_j$. Three iterations of h give $d_i \equiv e_j(\theta)$, and then an application of g reduces this to case I.

Case V: The final case is when $x = z_i$ and $y = z_j$. Successive applications of h show that $\langle a_i, a_j \rangle$, $\langle b_i, b_j \rangle$, $\langle c_i, c_j \rangle$, $\langle d_i, d_j \rangle$ and $\langle e_i, e_j \rangle$ are all in θ. Hence there exists a natural isomorphism of $\Theta(\mathfrak{A})$ into $\Theta(\mathfrak{B})$ with $\theta(a_i, a_j) \varepsilon \Theta(\mathfrak{A})$ corresponding to $\theta(a_i, a_j) \cup I_B$ in $\Theta(\mathfrak{B})$. Moreover, the various cases considered show that if $\theta \varepsilon \Theta(\mathfrak{B})$, then either θ is the image of some element in $\Theta(\mathfrak{A})$ under the above map or else $\theta = U_B$. Hence $\Theta(\mathfrak{B}) = \{U_B\} \cup \Theta(\mathfrak{A})$, as desired.

<u>3.4 Theorem</u>: Let $\mathfrak{A} = \langle A; f_1, \dots, f_q \rangle$ be a finite multiunary algebra with $A = \{a_1, \dots, a_n\}$. Then there exists a finite algebra $\mathfrak{B} = \langle B; \oplus \rangle$ where $\oplus$ is a binary operation on B. such that $\Theta(\mathfrak{B})$ has a unique maximal ideal isomorphic to $\Theta(\mathfrak{A})$.

<u>Proof</u>: Let $B = A \cup \{p_1, p_2, \dots p_{q-1}\}$ where $A \cap \{p_1, p_2, \dots, p_{q-1}\} = \phi$. Define $\oplus$ on B by the following table:

$\oplus$	a_1	a_2	$\cdots$	a_n	p_1	p_2	p_3	$\cdots$	p_{q-1}
a_1	$f_q(a_1)$	$f_q(a_2)$	$\cdots$	$f_q(a_n)$	a_1	a_1	a_1	$\cdots$	a_1
a_2	$f_q(a_1)$	$f_q(a_2)$	$\cdots$	$f_q(a_n)$	a_2	a_2	a_2	$\cdots$	a_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
a_n	$f_q(a_1)$	$f_q(a_2)$	$\cdots$	$f_q(a_n)$	a_n	a_n	a_n	$\cdots$	a_n
p_1	$f_1(a_1)$	$f_1(a_2)$	$\cdots$	$f_1(a_n)$	p_1	a_1	a_1	$\cdots$	a_1
p_2	$f_2(a_1)$	$f_2(a_2)$	$\cdots$	$f_2(a_n)$	p_2	p_1	a_1	$\cdots$	a_1
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
p_{q-1}	$f_{q-1}(a_1)$	$f_{q-1}(a_2)$	$\cdots$	$f_{q-1}(a_n)$	p_{q-1}	a_1	a_1	$\cdots$	p_1

An examination of the table shows that if $\theta \ \varepsilon \textcircled{H}(\mathfrak{A})$ then $\theta \cup I_B \ \varepsilon \ \textcircled{H}(\mathfrak{B})$. If $<p_1, a_i> \varepsilon \theta \varepsilon \textcircled{H}(\mathfrak{B})$, then left addition by all of the a_i, $1 \leq i \leq n$ and p_i, $1 \leq i \leq q - 1$, shows that all of B is congruent to p_1, so $\theta = U_B$. If $<p_j, a_i> \varepsilon \ \theta \ \varepsilon \ \textcircled{H}(\mathfrak{B})$, $2 \leq j \leq q - 1$, then left addition by p_j gives $<p_1, f_j(a_i)> \varepsilon \ \theta$, and again $\theta = U_B$. Finally, if $<p_i, p_j> \varepsilon \theta \varepsilon \textcircled{H}(\mathfrak{B})$, $1 \leq i < j \leq q - 1$, then right addition by p_j gives $<a_1, p_j> \varepsilon \ \theta$, and hence $\theta = U_B$. This exhausts all possibilities for the minimal congruence relations of $\mathfrak{B}$, and shows that $\textcircled{H}(\mathfrak{B}) = \{\theta \cup I_B | \theta \ \varepsilon \ \textcircled{H}(\mathfrak{A})\} \cup \{U_B\}$. The theorem follows easily.

Chapter 4

UNARY ALGEBRAS

This section will be an investigation of the congruence lattices of finite unary algebras. This will be done by examining certain interval sublattices of the congruence lattice. It will be shown that various number theoretic properties of the unary operation determine the congruence lattice of the algebra.

4.1 Notation: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be the set of orbits of f, i.e. $C_i = \{ x_o, x_1, \ldots, x_{m_i-1} \}$, where $f(x_o) = x_{m_i-1}, \ldots,$ $f(x_2) = x_1$, $f(x_1) = x_o$. Note that $C_i \cap C_j = \phi$ for $i \neq j$. Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be such that $C_i \subseteq B_i$ for $1 \leq i \leq n$ and $B_i = \{ x \varepsilon A \mid f^k(x) \varepsilon C_i$ for some integer $k \}$. The set $\mathcal{B}$ forms a partition of A. If f is a permutation then $C_i = B_i$ for all i, $1 \leq i \leq n$. Let Z_n denote the cyclic group of n elements; thus $\Sigma(Z_n) \cong \Theta(Z_n) \cong$ lattice of positive integral divisors of n. Also let $\mathcal{K} = \{K_o, K_1, \ldots, K_\ell\}$, where $K_o \subset K_1 \subset \ldots \subset K_\ell$, $K_o = C_1 \cup C_2 \cup \ldots \cup C_n$, and $K_j = \{x \mid f(x) \varepsilon K_{j-1}\}$ for

$1 \leq j \leq \ell + 1$. If n is a positive integer let $\tau(n)$ denote the number of positive integral divisors of n and let $\sigma(n)$ denote the sum of the positive integral divisors of n .

4.2 Lemma: Let $\mathfrak{A} = \langle A, f \rangle$ be as in 4.1. Then $[I_A, \theta(C_i)] \cong \Sigma(Z_{m_i})$, where $m_i = |C_i|$.

Proof: Since $C_i \in \Sigma(\mathfrak{A})$, define $\mathcal{C}_i$ to be the unary algebra $\langle C_i, f|C_i \rangle$. Then $[I_A, \theta(C_i)] \cong \Theta(\mathcal{C}_i)$. Let g be a function from Z_{m_i} to Z_{m_i} defined by $g(x) = x+1 \pmod{m_i}$. Then $\langle Z_{m_i}, g \rangle \cong \mathcal{C}_i$, so $\Theta(\mathcal{C}_i) \cong \Theta(\langle Z_{m_i}, g \rangle)$. However $g^k(x) = x+k = k+x$, so that $\Theta(Z_{m_i}, g) \cong \Theta(Z_{m_i}) = \Sigma(Z_{m_i})$.

4.3 Lemma: Let $\mathfrak{A} = \langle A, f \rangle$ be as in 4.1. Then $[I_A, \theta(C_1) \vee \ldots \vee \theta(C_n)] \cong \Sigma(Z_k)$ for some integer k .

Proof: Let $\theta \in [I_A, \theta(C_1) \vee \ldots \vee \theta(C_n)]$. Then $\langle x, y \rangle \in \theta$ implies $\{x, y\} \subseteq C_i$ for some i . So $\theta \in [I_A, \theta(C_1)] \vee \ldots \vee [I_A, \theta(C_n)]$. It follows that $[I_A, \theta(C_1) \vee \ldots \vee \theta(C_n)] \cong [I_A, \theta(C_1)] \times \ldots \times [I_A, \theta(C_n)]$. Hence by 4.2, $[I_A, \theta(C_1) \vee \ldots \vee \theta(C_n)]$ is isomorphic to $\Sigma(Z_{m_1}) \times \ldots \times \Sigma(Z_{m_n})$.

In general, if $p = p_1^{e_1} p_2^{e_2} \ldots p_t^{e_t}$ where the p_i are distinct primes, and if $q = q_1^{e_1} q_2^{e_2} \ldots q_t^{e_t}$ where the q_i are distinct primes, then $\Sigma(Z_p) \cong \Sigma(Z_q)$. From this it follows that $\Sigma(Z_{m_1}) \times \ldots \times \Sigma(Z_{m_n}) \cong \Sigma(Z_{\ell_1}) \ldots \Sigma(Z_{\ell_n})$, where the ℓ_i, $1 \leq i \leq n$, are all relatively prime. But if $K = \ell_1 \cdot \ell_2 \cdots \ell_n$, then $\Sigma(Z_{\ell_1}) \times \ldots \times \Sigma(Z_{\ell_n}) \cong \Sigma(Z_k)$.

<u>4.4 Lemma</u>: If $D \subseteq \mathcal{C} = \{C_1, \ldots, C_n\}$, $|D| = \ell$, then the interval $[\bigvee_{C_i \varepsilon D} \theta(C_i), \theta(\bigcup_{C_i \varepsilon D} C_i)]$ is isomorphic to π_ℓ, and $\sum_{D \subseteq \mathcal{C}} |[\bigvee_{C_i \varepsilon D} \theta(C_i), \theta(\bigcup_{C_i \varepsilon D} C_i)]| \leq |\pi_{n+1}|$.

<u>Proof</u>: Let $D \subseteq \mathcal{C}$ and $|D| = \ell$. Define a function α from $[\bigvee_{C_i \varepsilon D} \theta(C_i), \theta(\bigcup_{C_i \varepsilon D} C_i)]$ to π_ℓ such that if

$\psi \varepsilon [\bigvee_{C_i \varepsilon D} \theta(C_i), \theta(\bigcup_{C_i \varepsilon D} C_i)]$, $<i, j> \varepsilon\ \alpha(\psi)$ if and only if $\psi \geq \theta(C_i \cup C_j)$. It is easily verified that $\alpha(\psi)$ is an element of π_ℓ, and that α is a lattice isomorphism onto π_ℓ. The second claim in the lemma follows from the recursion $|\pi_{n+1}| = \Sigma_{o \leq k \leq n} \binom{n}{k} |\pi_k|$.

<u>4.5 Lemma</u>: For $C_i \neq C_j$ and $|C_i| = m_i$ and $|C_j| = m_j$, the interval $[I_A, \theta(C_i \cup C_j)]$ is distributive if

$(m_i, m_j) = 1$. If $(m_i, m_j) > 1$, then this interval is neither upper nor lower semimodular. Also

$$|[I_A, \Theta(C_i \cup C_j)]| = \tau(m_i)\tau(m_j) + \sigma((m_i, m_j)) .$$

Proof: For $(m_i, m_j) = 1$, if $x \varepsilon C_i$ and $y \varepsilon C_j$ then $\Theta(x, y) = \Theta(C_1 \cup C_2)$. So the interval $[I_A, \Theta(C_i \cup C_j)]$ is just $[I_A, \Theta(C_i) \vee \Theta(C_j)]$ with $\Theta(C_i \cup C_j)$ adjoined, which is distributive by 4.3.

Let $C_i = \{x_o, x_1, \dots, x_{m_i-1}\}$ and $C_j = \{y_o, y_1, \dots, y_{m_j-1}\}$. If $(m_i, m_j) = r$ and $d|r$ then the congruence relation $\psi = \Theta(x_o, x_d, y_j, y_{j+d})$ for $0 \leq j < d$ has d congruence classes and satisfies $\psi \leq \Theta(C_i \cup C_j)$ and $\psi \not\leq \Theta(C_i) \vee \Theta(C_j)$. Moreover, if ψ is any congruence relation such that $\psi \leq \Theta(C_i \cup C_j)$ and $\psi \not\leq \Theta(C_i) \vee \Theta(C_j)$, then ψ must be of the form $\Theta(x_o, x_d, y_j, y_{j+d})$ for suitable $d|(m_i, m_j)$ and $0 \leq j < d$. This assertion follows from the fact that ψ must be the join of congruences of the form $\Theta(x_o, x_d)$, $\Theta(y_o, y_d)$ and $\Theta(x_o, y_j)$. Note that if $(m_i, m_j) = 1$ then any such ψ is $\Theta(C_i \cup C_j)$. By 4.3 $|[I_A, \Theta(C_i) \vee \Theta(C_j)]| = \tau(m_i) \cdot \tau(m_j)$. For each d such that $d|(m_i, m_j)$, there exist d congruence relations in

$[I_A, \theta(C_i \cup C_j)]$ but not in $[I_A, \theta(C_i) \vee \theta(C_j)]$. Hence there are a total of $\sigma((m_i, m_j))$ congruences of this type. Therefore,

$|[I_A, \theta(C_i \cup C_j)]| = \tau(m_i)\tau(m_j) + \sigma((m_i, m_j))$.

It remains to show that the interval $[I_A, \theta(C_i \cup C_j)]$ is neither upper nor lower semimodular when $(m_i, m_j) = r > 1$. Let p be any prime such that $p|r$. Let $\tau = \theta(x_o, x_p, y_o, y_p)$. Since p is prime, $\tau \prec \theta(C_i \cup C_j)$. Also $\tau \wedge (\theta(C_i) \vee \theta(C_j)) \prec \tau$, since $\tau \wedge (\theta(C_i) \vee \theta(C_j)) = \theta(x_o, x_p) \vee \theta(y_o, y_p)$. By 4.4 $\theta(C_i) \vee \theta(C_j) \prec \theta(C_i \cup C_j)$. Let $\sigma = \theta(x_o, x_p) \vee \theta(C_j)$. Then again since p is prime, $\tau \wedge (\theta(C_i) \vee \theta(C_j)) \prec \sigma \prec \theta(C_i) \vee \theta(C_j)$. Hence the five congruence relations $\theta(C_i \cup C_j)$, τ , $\theta(C_i) \vee \theta(C_j)$, σ and $\tau \wedge (\theta(C_i) \vee \theta(C_j))$ form a sublattice of $[I_A, \theta(C_i \cup C_j)]$ that is isomorphic to N_5 . In this sublattice, $\leq$ can be replaced by $\prec$. Hence $[I_A, \theta(C_i \cup C_j)]$ is neither upper nor lower semimodular when $(m_i, m_j) > 1$.

4.6 Lemma: Let K_j and K_{j+1} be as in 4.1, with $0 \leq j \leq \ell - 1$, then $[\theta(K_j), \theta(K_{j+1})] \cong \mathcal{E}(S_j)$ where $|S_j| = |K_{j+1} \setminus K_j| + 1$.

Proof: Let $K_{j+1} \setminus K_j = \{x_1, \ldots, x_{n_j}\}$ and let $S_j = \{\{K_j\}, x_1, \ldots, x_{n_j}\}$. The correspondence between $[\Theta(K_j), \Theta(K_{j+1})]$ and $\mathcal{E}(S_j)$ is then clear.

4.7 Theorem: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra. Then there exists a chain of elements of $\Theta(\mathfrak{A})$
$I_A \leq \Theta(C_1) \vee \ldots \vee \Theta(C_n) \leq \Theta(C_1 \cup \ldots \cup C_n) = \Theta(K_0) < \Theta(K_1) < \ldots < \Theta(K_\ell) = U_A$ such that $[I_A, \Theta(C_1) \vee \ldots \vee \Theta(C_n)] \cong \Sigma(Z_k)$ for some k, and all other interval sublattices of this chain are isomorphic to π_t for appropriate integers t .

Proof: 4.3, 4.4 and 4.6.

4.8 Theorem: Let D be a distributive lattice. Then D is isomorphic to the congruence lattice of some finite unary algebra it and only if $D \cong \Sigma(Z_q)$ for some non negative integer q or $D \cong (\Sigma(Z_q) \cup \{u\}) \times \Sigma(Z_{p^\ell})$ where $u > H$ for all $H \in \Sigma(Z_q)$, p is any prime, and ℓ is a non negative integer.

Proof: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra with $\Theta(\mathfrak{A})$ distributive. Note that $\mathcal{E}(S)$ is not distributive for $|S| > 2$. By 4.4, $1 \leq |\mathcal{C}| \leq 2$. By 4.6, at most one of the $C_i \in \mathcal{C}$

is such that $C_i \neq B_i$, and if such a C_i exists, then $B_i \setminus C_i$ must be a set $\{z_1, z_2, \ldots, z_\ell\}$ where $f(z_\ell) = z_{\ell-1}, \ldots, f(z_2) = z_1$ and $f(z_1) \in C_i$. Also by 4.5 $(|C_1|, |C_2|) = 1$, provided $|C_1|\,|C_2| \neq 0$. So without loss of generality, let $C_1 = \{x_0, x_1, \ldots, x_{r-1}\}$ and $C_2 = \{y_0, y_1, \ldots, y_{s-1}\}$, with the possiblility that $C_2 = \emptyset$. Then $(r, s) = 1$ if $rs \neq 0$, $f(x_i) = x_{i-1(\mathrm{mod}\ r)}$ $f(y_i) = y_{i-1(\mathrm{mod}\ s)}$, and $f(z_1) = x_0$. Let $L = [I_A, \theta(C_1 \cup C_2)]$ and let $M = [I_A, \theta(z_\ell, x_{\ell(\mathrm{mod}\ r)})]$. In the event that $C_2 = \emptyset$, then $L = [I_A, \theta(C_1)]$; and if $\ell = 0$, then $M = \{I_A\}$. Note that $M = \{I_A\} \cup \{\theta(z_i, x_{i(\mathrm{mod}\ r)}) \mid 1 \leq i \leq \ell\}$. Plainly, $L \wedge M = I_A$ and $L \vee M = U_A$. If $\theta \in \Theta(\mathfrak{A})$, then $\theta = \theta \wedge U_A = \theta \wedge (\theta(C_1 \cup C_2) \vee (\theta(z_\ell, x_{\ell(\mathrm{mod}\ r)}))) = (\theta \wedge \theta(C_1 \cup C_2)) \vee (\theta \wedge (\theta(z_\ell, x_{\ell(\mathrm{mod}\ r)})))$ by distributivity. So every element of $\Theta(\mathfrak{A})$ is the unique join of elements from L and M, that is $\Theta(\mathfrak{A}) = L \times M$. Since $|\mathcal{C}| \leq 2$ and $(r, s) = 1$, it follows that

$$L = [I_A, \theta(C_1 \cup C_2)] = [I_A, \theta(C_1) \vee \theta(C_2)] \cup [\theta(C_1) \vee \theta(C_2), \theta(C_1 \cup C_2)].$$

So either $L \cong \Sigma(Z_k)$ or $L \cong \Sigma(Z_k) \cup \{u\}$, where $u > H$ for every $H \in \Sigma(Z_k)$. Also $M \cong \Sigma(Z_{p^\ell})$ where p is any prime and ℓ is the length of the chain $[I_A, \theta(z_\ell, x_{\ell(\mathrm{mod}\ r)})]$. Therefore, either $\Theta(\mathfrak{A}) \cong \Sigma(Z_k) \times \Sigma(Z_{p^\ell}) \cong \Sigma(Z_q)$, where q is an appropriate integer, or $\Theta(\mathfrak{O}) \cong (\Sigma(Z_k) \cup \{u\}) \times \Sigma(z_{p^\ell})$.

Conversely given any non negative integer q one can easily construct a unary algebra having $\Sigma(Z_q)$ as its congruence lattice. Or given integers k, ℓ and any prime p, one can construct a unary algebra having congruence lattice $(\Sigma(Z_k) \cup \{u\}) \times \Sigma(Z_{p^\ell})$ defined as follows:

$B_1 = \{x_o, z_1, \ldots, z_\ell\}$ and $B_2 = C_2 = \{y_o, y_1, \ldots, y_{k-1}\}$

where $f(z_i) = z_{i-1}$, $f(z_1) = x_o$, $f(x_o) = x_o$, and $f(y_i) = y_{i-1} \pmod{k}$.

<u>4.9 Corollary</u>: The class of lattices isomorphic to congruence lattices of finite unary algebras properly contains the class of lattices isomorphic to congruence lattices of finite unary algebras where the unary operation is a permutation.

<u>Proof</u>: Let L be a distributive lattice that is isomorphic to the congruence lattice of a finite unary algebra whose unary operation is a permutation. Then L is either isomorphic to $\Sigma(Z_q)$ for some integer q, or to $\Sigma(Z_q) \cup \{u\}$, where u is a new maximal element. Therefore L is self dual, or its maximal element is join-irreducible. However, the distributive lattice $(\Sigma(Z_q) \cup \{u\}) \times \Sigma(Z_{p^\ell})$, for p a

prime q not a prime power, and $\ell > 2$, is the congruence lattice of some finite unary algebra, but this lattice is neither self dual, nor is its maximal element join-irreducible.

4.10 Corollary: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra with $\Theta(\mathfrak{A})$ distributive. If $|C_1| = r > 0$, $|C_2| = s \geq 0$, $|B_1 \setminus C_1| = t \geq 0$, and $B_2 = C_2$, then the cardinality of $\Theta(\mathfrak{A})$ is $(\tau(s)\tau(r) + \delta)(t+1)$ where $\delta = 1$ if $s > 0$, $\delta = 0$ if $s = 0$, and $\tau(0)$ is defined to be 1.

Proof: 4.8 and 4.2.

4.11 Lemma: Let $\mathfrak{A} = \langle A, f \rangle$, $|A|$ finite, f unary with orbits $\mathcal{C} = \{c_1, c_2, \ldots, c_n\}$. Then the atoms of $\Theta(\mathfrak{A})$ are of the following three types:

(i) the atoms of $[I_A, \theta(C_i)]$ where $|C_i| \geq 2$

(ii) $\{\theta(x, y) \mid f(x) = f(y)$ and $x \neq y\}$

(iii) $\{\theta(x, y) \mid x \in C_i, y \in C_j, i \neq j, |C_i| = |C_j|\}$.

The number of atoms of type i is $u_1 + \ldots + u_n$, where u_i is the number of primes dividing $|C_i|$ for $1 \leq i \leq n$. The number of atoms of type ii is greater than or equal to the number of orbits C_i such that $B_i \neq C_i$. The number of atoms of type iii is greater than or equal to $\binom{t}{2}$ where t is the number of orbits of cardinality one.

Proof: Clearly the elements of each type listed are atoms of $\Theta(\mathfrak{A})$, and the bounds on the number of each type are obvious. It remains to show that every atom of $\Theta(\mathfrak{A})$ is of the above form. Let α be an atom of $\Theta(\mathfrak{A})$, $\langle x,y\rangle \in \alpha$, $x \neq y$. If $f(\{x, y\}) = \{x, y\}$ or $f(\{x, y\}) = \{z\}$ then α must be of type ii or iii or else $\{x, y\}$ is an orbit of cardinality 2, in which case α is of type i. If $f(\{x, y\}) = \{u, v\}$ where $u \neq v$, $\{u, v\} \neq \{x, y\}$, then $\alpha = \theta(x, y) = \theta(u, v)$. Thus x, y, u, v are in the union of all orbits. If x and y are in the same orbit, then α is of type i. If x and y are in different orbits, say C_1 and C_2, then $|C_1| = |C_2|$, for if not $I_A < (\theta(C_1) \vee \theta(C_2)) \wedge \alpha < \alpha$.

4.12 Remark: It is well known that a universal algebra is subdirectly irreducible if and only if $\Theta(\mathfrak{A})$ has exactly one atom. Using this characterization of subdirect irreducibility and the previous lemma it possible to prove the following theorem due to McKenzie [12] p.18.

4.13 Theorem (McKenzie) Let $\mathfrak{A} = \langle A, f\rangle$ be a finite unary algebra. Then $\mathfrak{A}$ is subdirectly irreducible if and only if $\mathfrak{A}$ is one the following algebras:

(a) $A = C_1$, $|C_1| = p^\ell$ $\ell \geq 0$, p a prime.

(b) $A = C_1 \cup C_2$, $|C_1| = p^\ell$ $\ell \geq 0$, p a prime, $|C_2| = 1$.

(c) $A = B_1 = \{x_\ell, x_{\ell-1}, \ldots, x_1, x_0\}$ where $f(x_\ell) = x_{\ell-1}, \ldots, f(x_2) = x_1$ $f(x_1) = x_0$ and $f(x_0) = x_0$.

Moreover, $\Theta(\mathfrak{A})$ is a chain in all cases.

Proof: For each of the types a, b and c of unary algebras $\mathfrak{A}$, $\Theta(\mathfrak{A})$ is a chain and hence $\mathfrak{A}$ is subdirectly irreducible by 4.12. To obtain the converse let $\mathfrak{A} = \langle A, f \rangle$ be subdirectly irreducible. By 4.12 it must have a single atom α. If α is of type i in 4.11, and α is an atom of C_1 say, then $|C_1| = p^\ell$ for $\ell \geq 1$. Moreover $|C_i| = 1$ for $i > 1$ since there is no other atom of type i. Because there are no atoms of type iii or ii, it follows that either $\mathcal{C} = \{C_1\}$ and $B_1 = C_1$, or $\mathcal{C} = \{C_1, C_2\}$ and $B_1 = C_1$ and $B_2 = C_2$. Hence $\mathfrak{A}$ is of type a or b. If the atom α is of type ii, then there exists a unique B_i such that $B_i \neq C_i$. Since there are no atoms of type i $|C_j| = 1$ for all $C_j \in \mathcal{C}$. If $|\mathcal{C}| > 1$, then there would be an atom of type iii. Thus $\mathfrak{A}$ is an algebra of the kind described in c. Finally if the atom α is of type iii, then all orbits must have cardinality 1 and $A = C_i \cup C_j$ is an algebra of type b with $\ell = 0$.

4.14 Theorem: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra. Then $\Theta(\mathfrak{A})$ is upper semimodular if and only if $(|C_i|, |C_j|) = 1$ for all orbits C_i and C_j .

Proof: By 4.5 the orbits of $\mathfrak{A}$ must have relatively prime cardinalities if $\Theta(\mathfrak{A})$ is to be upper semimodular. To prove the converse suppose that ψ , ϕ and τ in $\Theta(\mathfrak{A})$ are such that $\psi \succ \tau$ $\phi \succ \tau$. It must be shown that $\psi \vee \phi$ covers both ψ and ϕ . Let $\bar{\mathfrak{A}} = \mathfrak{A}/\tau$. Let $\bar{C}_i$ be the image of the orbit C_i in $\mathfrak{A}$. Then $\bar{C}_i$ is an orbit of $\bar{\mathfrak{A}}$. Moreover every orbit of $\bar{\mathfrak{A}}$ is of this form since $(|C_i|, |C_j|) = 1$ for all $1 \le i, j \le n$. Also $|\bar{C}_i| \mid |C_i|$ for each i, $1 \le i \le n$. Hence the orbits of $\bar{\mathfrak{A}}$ also have relatively prime cardinalities.

It is well known that the interval $[\tau, U_A]$ in $\Theta(\mathfrak{A})$ is isomorphic to $\Theta(\mathfrak{A})$ (Gratzer [6], p. 61). Let $\bar{\varphi}$ and $\bar{\phi}$ be the images of ψ and ϕ under this isomorphism, then $\bar{\psi}$ and $\bar{\phi}$ are atoms in $\Theta(\bar{\mathfrak{A}})$. Since $(|C_i|,|C_j|) = 1$ any atoms in $\Theta(\mathfrak{A})$ which are of type iii described in 4.11 will have only one non trivial congruence class and this class will have cardinality 2 .

It will therefore be sufficient to show $\bar{\phi} \vee \bar{\psi} \succ \bar{\phi}$ and $\bar{\phi} \vee \bar{\psi} \succ \bar{\psi}$ in $\Theta(\bar{\mathfrak{A}})$. Note that if $\bar{\psi}$ and $\bar{\phi}$ are such that their non trivial congruence classes are disjoint, then indeed $\bar{\phi} \vee \bar{\psi} \succ \bar{\phi}$ and $\bar{\phi} \vee \bar{\psi} \succ \bar{\psi}$. Since $\bar{\phi}$ and $\bar{\psi}$ are atoms, and by 4.11 there are only three different types of atoms, there are only six different cases to consider. If $\bar\psi$ and $\bar\phi$ are both of type i, then $\bar\psi \vee \bar\phi \ \varepsilon \ [I_A, \theta(\bar C_i) \vee \theta(\bar C_j)]$; by 4.3 this interval is distributive, and therefore upper semimodular. If $\bar\psi$ is of type i and $\bar\phi$ is of type ii then, by the above observation concerning disjoint non trivial congruence classes, it can be assumed that $\bar\phi = \theta(\bar x, \bar y)$ where $f(\bar x) = f(\bar y) \ \varepsilon \ \bar C_i$, $\bar x \notin \bar C_i$ and $\bar y \ \varepsilon \ \bar C_i$. Then $\bar\phi \vee \bar\psi$ has as its congruence classes those of $\bar\psi$ with $\bar x$ adjoined to one of them. Clearly $\bar\phi \vee \bar\psi \succ \bar\phi$, and also $\bar\psi \vee \bar\phi \succ \bar\psi$, since $\bar\phi$ is an atom. The remaining four cases can be handled similiarly, using the note on disjoint congruence classes and the restriction on the atoms of type iii in $\Theta(\bar{\mathfrak{A}})$.

<u>4.15 Corollary</u>: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra and let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be the orbits of f. Denote by $S(b, k)$ the number of members of the partition lattice on k elements in which each equivalence class has cardinality at least b, and let $S(b, k) = 0$ for $b > k > 0$, $S(b, 0) = 1$. Then $\Theta(\mathfrak{A})$ is upper semimodular if and only if

$$|[I_A, \theta(C_1 \cup \ldots \cup C_n)]| = \sum_{T \subseteq \{1,2,\ldots,n\}} S(2, n-|T|) \cdot \prod_{j \varepsilon T} (|C_j|).$$

<u>Proof</u>: Let $\Theta(\mathfrak{A})$ be upper semimodular. By 4.14, $(|C_i|, |C_j|) = 1$ for $1 \leq i,j \leq n$. Hence if $<x,y> \varepsilon \theta \varepsilon \Theta(\mathfrak{A})$,

$x \varepsilon C_i$, and $y \varepsilon C_j$, $i \neq j$, then $\theta(C_i \cup C_j) \subseteq \theta$. So θ generates an equivalence relation $\bar{\theta}$ on $\{1, 2, \ldots, n\}$ by $\langle i, j \rangle \varepsilon \bar{\theta}$ if and only if $\theta(C_i \cup C_j) \subseteq \theta$ or $i = j$. Let $T_\theta = \{i \varepsilon \{1, 2, \ldots, n\} \mid \{i\}$ is a congruence class of $\bar{\theta}\}$. Let $\sigma_T = \{\theta \varepsilon [I_A, \theta(C_1 \cup \ldots \cup C_n)] \mid T_\theta = T\}$. Then the interval $[I_A, \theta(C_1 \cup \ldots \cup C_n)]$ is the disjoint union of all of the σ_T as T ranges over the subsets of $\{1, 2, \ldots, n\}$. Denote by $\pi_n^{2, T}$ the collection of partitions of the set $\{1, 2, \ldots, n\} \setminus T$ in which each block has cardinality at most two. Then there exists a one to one correspondence between σ_T and $\pi_n^{2, T} \times \bigvee_{j \varepsilon T} \theta(C_j)$ obtained by letting $\theta \varepsilon \sigma_T$ correspond to $(\bar{\theta} \wedge U_T c$, $\theta \wedge \bigvee_{j \varepsilon T} \theta(C_j))$, where $T^c = \{1, 2, \ldots, n\} \setminus T$. Hence $|\sigma_T L| = |\pi_n^{2,T} \times \bigvee_{j \varepsilon T} \theta(C_j)| = S(2, n - |T|) \cdot \prod_{j \varepsilon T} \tau(|C_j|)$. Summing over all $T \subseteq \{1, 2, \ldots, n\}$ the conclusion follows.

Conversely, note that in any unary algebra, the cardinality of the interval $[I_A, \theta(C_1 \cup \ldots \cup C_n)]$ is at least $\Sigma_{T \subseteq \{1, 2, \ldots, n\}} S(2, n - |T|) \cdot \prod_{j \varepsilon T} \tau(|C_j|)$. It will be greater than this sum if there exists C_i and C_j such that $(|C_i|, |C_j|) > 1$. Applying 4.14, the corollary is proved.

<u>4.16 Lemma</u>: Let T be a finite lattice, $L = [0, \lambda]$ and $M = [0, \mu]$ be two principal ideals of T and suppose that

$\theta = (\theta\wedge\lambda)\vee(\theta\wedge\mu)$ for every $\theta \in T$, and if $\theta = \varphi\vee\psi$ for $\varphi \leq \lambda$ and $\psi \leq \mu$, then $\varphi = \theta\wedge\lambda$ and $\psi = \theta\wedge\mu$. Then $T \cong L\times M$.

Proof: The function taking $\theta = (\theta\wedge\lambda)\vee(\theta\wedge\mu)$ into $(\theta\wedge\lambda, \theta\wedge\mu) \in L\times M$ is easily shown to be the desired lattice isomorphism.

4.17 Lemma: If L and M are lower or upper semimodular lattices then $L\times M$ is also lower or upper semimodular.

Proof: Birkhoff [2] p. 83.

4.18 Theorem: Let $\mathfrak{A} = \langle A, f\rangle$ be a finite unary algebra with $\mathcal{C}$, B_i, and C_i defined as in 4.1. Then $\Theta(\mathfrak{A})$ is lower semimodular if and only if all of the following conditions hold:

(i) $|\mathcal{C}| \leq 3$

(ii) $(|C_i|, |C_j|) = 1$ for $i \neq j$

(iii) $B_i \neq C_i$ for at most one of the C_i

(iv) If $B_i \neq C_i$ then B_i has one of the following forms:

a) $B_i = \{x_o, w_1, \ldots, w_s, u_1, \ldots, u_p, v_1, \ldots, v_q\}$

b) $B_i = \{x_o, x_1, \ldots, x_{r-1}, w_1, \ldots, w_s\}$ or

c) $B_i = \{x_o, x_1, \dots, x_{r-1}, w_1, \dots, w_s, u, v\}$ where $f(x_i) = x_{i-1(\text{mod } r)}$, $f(w_i) = w_{i-1}$, $f(w_1) = x_o$, $f(v_i) = v_{i-1}$, $f(u_i) = u_{i-1}$, $f(u_1) = f(v_1) = f(u) = f(v) = w_s$.

Proof: Let $\Theta(\mathfrak{A})$ be lower semimodular. Then condition ii must hold by virtue of 4.5. Condition i follows from 4.4 and the fact that π_k is not lower semimodular for $k > 3$, Birkhoff [2], p. 16. If iii did not hold, there would be $x \in B_i \setminus C_i$, $f(x) \in C_i$ and $y \in B_j \setminus C_j$, $f(y) \in C_j$, $i \neq j$. Then $\Theta(C_i \cup C_j \cup \{x\} \cup \{y\})$ covers both $\psi = \Theta(C_i \cup \{x\}) \vee \Theta(C_j \cup \{y\})$ and $\varnothing = \Theta(C_i \cup C_j) \vee \Theta(x, y)$. However $\psi \wedge \varnothing = \Theta(C_i) \vee \Theta(C_j)$ which is not covered by $\varnothing$, contradicting lower semimodularity. So iii holds. If $B_i \neq C_i$, $C_i = \{x_o, x_1, \dots, x_{r-1}\}$, $r \geq 1$, with $f(x_i) = x_{i-1(\text{mod } r)}$, and if, contrary to iv , there exist y and z in $B_i \setminus C_i$ such that $f(z) = f(x_1) = x_o$ and $f(y) = f(x_{k+1}) = x_k$, where $k \neq 0$, then $\Theta(y, z, x_1, x_{k+1})$ covers both $\varnothing = \Theta(y, z)$ and $\psi = \Theta(z, x_1) \vee \Theta(y, x_{k+1})$. However, $\varnothing \wedge \psi = I_A$ which is covered by neither $\varnothing$ nor ψ . This violation of lower semimodularity shows that $k = 0$. Now assume, contrary to iv, that there exist two pairs, not containing x_o , $\{x, y\}$ and $\{u, v\}$, such that $f(x) = f(y)$

and $f(u) = f(v)$. By 4.6, $|K_{j+1} \setminus K_j| \leq 2$ for all $j \geq 0$. Without loss of generality, assume that $|K_{j+1} \setminus K_j| = 1$ for $j < s$, $K_{s+1} \setminus K_s = \{u, v\}$, $\{x , y\} \subseteq K_{t+1}$, where $t > s$, and $f^{\ell}(x) = f^{\ell}(y) = v$, $\ell = t - s$. Note that $f^j(x) \neq u$ for all integers j . Then $\theta(x, y, f(x), u)$ covers both $\psi = \theta(u, y) \vee \theta(x, f(x))$ and $\emptyset = \theta(x , y) \vee \theta(u, f(x), f^2(x))$, but $\emptyset \wedge \psi = \theta(f(x), f^2(x))$ is not covered by $\emptyset$, violating lower semimodularity. Hence if $C_i = \{x_o\}$ then iv - a holds. If $C_i = \{x_o, x_1, \ldots, x_{r-1}\}$ for $r > 1$, and $|K_{j+1} \setminus K_j| \leq 1$ for some $j > 0$ then iv - b holds. If $|C_i| > 1$, and $|K_j \setminus K_{j-1}| = 2$ for some integer j , then $K_j = K_{j+t}$ for all $t \geq 1$. In fact, assume not. Let $f(u) = f(v)$, $u \neq v$, $u, v \notin C_i$, and suppose z is such that $f(z) = v$. Also, $f^j(u) = x_o = f^j(x_t)$ where $t \equiv j \pmod{r}$. Let $\tau \equiv \theta(u, v, z)$, $\psi = \theta(u, z)$, and $\emptyset = \theta(u, x_t) \vee \theta(z, x_{t+1}) \vee \gamma$, where $\gamma \prec \theta(C_i)$. Then $\tau \succ \psi$, since ψ has exactly two congruence classes, and τ has only one. Also $\tau \succ \emptyset$, since every congruence class in $\emptyset$ contains some element of C_i , and hence the fusion of any two classes would give a congruence containing $\theta(C_i)$ and therefore $\theta(u, z, v)$. But $\emptyset \wedge \psi$ has u and z as singleton classes and v and x_t in a congruence class of cardinality 2 . Thus $\emptyset > \theta(z, x_{t+1}) \vee \gamma > \theta(v, x_t) \vee \gamma$, so lower semimodularity is violated. So if $|C_i| > 1$, and $|K_{j+1} \setminus K_j| = 2$ for some j, then iv - c holds.

Conversely, assume that $\mathfrak{A} = \langle A, f \rangle$ satisfies conditions i - iv. For convenience, let x_t denote $x_{t(\text{mod } r)}$, $u_o = v_o = w_s$, $w_o = x_o$, and choose notation so that $B_i \neq C_i$ implies $i = 1$. Let $L = [I_A, \theta(C_1 \cup C_2 \cup C_3)]$ and $M = [I_A, \beta]$ where $\beta = \theta(B_1)$ in case iv-a, $\beta = \theta(w_s, x_s)$ in case iv-b, and $\beta = \theta(u, v, x_{s+1})$ in case iv-c. Hence $L \cap M = \{I_A\}$. The remainder of the proof will consist of showing first that L and M are ideals satisfying the hypothesis of 4.16 (and thus $\Theta(\mathfrak{A}) = L \times M$), and then proving that L and M are both lower semimodular.

For any $\lambda \varepsilon L$ and $\mu \varepsilon M$, every non trivial congruence class of λ is contained in $C_1 \cup C_2 \cup C_3$, and each class of μ contains at most one element of $C_1 \cup C_2 \cup C_3$. If $\theta = \lambda \vee \mu$ and T is a congruence class of θ, then $T \cap (C_1 \cup C_2 \cup C_3)$ is a class of λ, so $\theta \wedge \theta(C_1 \cup C_2 \cup C_3) = \lambda$. Moreover if $\theta = \lambda_1 \vee \mu_1 = \lambda_2 \vee \mu_2$, $\lambda_i \varepsilon L$, $\mu_i \varepsilon M$, then $\lambda_1 = \lambda_2$, since the congruence classes of θ determine those of λ_1 and λ_2. If $\theta = \lambda \vee \mu$, $\lambda \varepsilon L$, $\mu \varepsilon M$, and $\mathfrak{A}$ safisfies condition iv-a, then μ must equal $\theta \wedge \theta(B_1)$. If $\mathfrak{A}$ satisfies condition iv-b, and if $\theta \wedge \beta$ has w_t

in a non trivial congruence class, with t maximal, then $\mu = \theta \wedge \beta \in M$; otherwise w_t would not occur with t maximal. A similiar argument for the case that $\mathfrak{A}$ satisfies iv-c also shows $\mu = \theta \wedge \beta$. Hence whenever $\theta \in \Theta(\mathfrak{A})$ is such that $\theta = \lambda \vee \mu$, $\lambda \in L$, $\mu \in M$, then $\lambda = \theta \wedge \theta(C_1 \cup C_2 \cup C_3)$ and $\mu = \theta \wedge \beta$.

It will now be shown that for every $\theta \in \Theta(\mathfrak{A})$, $\theta = (\theta \wedge \theta(C_1 \cup C_2 \cup C_3)) \vee (\theta \wedge \beta)$. By 0.5, the fact that L and M are ideals, and the previous paragraph, it will be sufficient to show this for minimal congruence relations $\theta = \theta(a, b)$, $a, b \in A$. Note that $\theta \geq (\theta \wedge \theta(C_1 \cup C_2 \cup C_3)) \vee (\theta \wedge \beta)$. If $a, b \in C_1 \cup C_2 \cup C_3$ then $\theta(a, b) \leq (C_1 \cup C_2 \cup C_3)$, so $\theta(a, b) = (\theta(a, b) \wedge \theta(C_1 \cup C_2 \cup C_3)) \vee (\theta(a, b) \wedge \beta)$. Suppose that $a \in C_2 \cup C_3$, say $a \in C_2$, $b \in B_1 \setminus C_1$, and $f^k(b) = x_o$, so that $b \equiv x_k(\beta)$. Since the cardinalities of C_1 and C_2 are relatively prime, $\theta(a, b) \supseteq \theta(C_1 \cup C_2)$, and it follows that $\theta(a, b) \wedge \beta = \theta(b, x_k)$ and

$\theta(a, b) = (\theta(a, b) \wedge \theta(C_1 \cup C_2 \cup C_3)) \vee \theta(b, x_k)$. If $a = x_i \in C_1$ and $b \in B_1 \setminus C_1$, $b \equiv x_k(\beta)$, then $f^k(a) \equiv f^k(b)(\theta(a, b))$ implies $x_{i-k} \equiv x_o(\theta(a, b)$, so $f^{r-k}(x_{i-k}) = x_i \equiv f^{r-k}(x_o) = x_k(\theta(a, b))$. Therefore, $b \equiv a = x_i \equiv x_k(\theta(a, b))$, and $\theta(a, b) \geq \theta(a, x_k) \vee \theta(b, x_k) \geq \theta(a, b)$. So again

$\theta(a, b) = (\theta(a, b) \wedge \theta(C_1 \cup C_2 \cup C_3)) \vee (\theta(a, b) \wedge \beta)$. Finally, suppose that $a, b \in B_1 \setminus C_1$, $a \equiv x_j(\beta)$ and $b \equiv x_i(\beta)$.

If $r = 1$, or $i = j$, then $\theta(a, b) \leq \beta$. Therefore, assume that $r > 1$ and $j > i$. It follows that $f^{j-i}(a) = b$. Then $a \equiv b$ $(\theta(a, b))$ implies $f^i(a) \equiv f^i(b) = x_o$ $(\theta(a, b))$, so that by the previous case, $\theta(a, b) \geq \theta(f^i(a), x_o) = \theta(f^i(a), x_{j-i}) \vee \theta(x_o, x_{j-1})$. Note that $a \equiv b = f^{j-1}(a)$ $(\theta(a, b))$, and therefore $a \equiv f^{(k+1)(j=i)}(a)$ $(\theta(a, b))$ for all k. Choose k so that $0 \leq i - k(j - i) < j - i$. Applying f $(j - i) - (i - k(j - i))$ times to the congruence $f^i(a) \equiv x_{j-i}$ $(\theta(a, b))$ gives $a \equiv f^{(k+1)(j-i)}(a) \equiv x_{i-k(j-i)}$ $(\theta(a, b))$. But $x_o \equiv x_{j-i}$ $(\theta(a, b))$ and therefore $x_o \equiv x_{-k(j-i)}$ $(\theta(a, b))$ and $x_{i-k(j-i)} \equiv x_i \equiv x_j$ $(\theta(a, b))$. Consequently $\theta(a, b) \geq \theta(x_i, x_j) \vee \theta(a, x_j) \geq \theta(a, b)$, so $\theta(x_i, x_j) = \theta(a, b) \wedge \theta(C_1 \cup C_2 \cup C_3)$ and $\theta(a, x_j) = \theta(a, b) \wedge \beta$. Therefore by 4.17, $\Theta(\mathfrak{A}) = L \times M$.

By 4.18 it remains only to show that L and M are both lower semimodular. In $L = [I_A, \theta(C_1 \cup C_2 \cup C_3)]$, the subinterval $[I_A, \theta(C_1) \vee \theta(C_2) \vee \theta(C_3)]$ is distributive by 4.3. Hence this interval is lower semimodular. Therefore it is only necessary to consider pairs of elements covered by an element $\tau = \theta(C_1 \cup C_2 \cup C_3)$, or by an element $\tau = \theta(C_1 \cup C_2) \vee \gamma$ where $\gamma \leq \theta(C_3)$. In the first case, τ covers only the elements

$\theta(C_1 \cup C_3) \vee \theta(C_2)$, $\theta(C_1 \cup C_2) \vee \theta(C_3)$, and $\theta(C_2 \cup C_3) \vee \theta(C_1)$. Since each pair of these elements intersects in $\theta(C_1) \vee \theta(C_2) \vee \theta(C_3)$, lower semimodularity holds in this case. In the second case, the elements covered by τ are $\theta(C_1 \cup C_2) \vee \delta$ where γ covers δ and $\theta(C_1) \vee \theta(C_2) \vee \gamma$. An examination of the possible intersections shows lower semimodularity holds in this case. Hence L is lower semimodular. Now consider M . If $\mathfrak{A}$ satisfies condition iv-b, then M is a chain, and therefore lower semimodular. If satisfies condition iv-c, then $M \cong \{0, 1, \ldots s, 0', 1', \ldots, s', x, y, z\}$, where $0 \prec 1 \prec \ldots \prec s \prec x$, $s \prec y$, $i \prec i'$ for $1 \leq i \leq s$, $0 \prec 0' \prec 1' \prec \ldots \prec s'$, $x \prec z$, $y \prec z$ and $s' \prec z$. In this case M is actually modular, and therefore lower semimodular. Finally consider the case when $\mathfrak{A}$ satisfies condition iv-a . Let $\tau \leq \beta$. If the congruence class of τ containing x_o is just $\{x_o\}$, then $\tau = \theta(u_i, v_i)$ for some $i \geq 0$. The interval $[I_A, \tau]$ is then a chain. Therefore $\tau \succ \phi$, $\tau \succ \psi$, and $\phi \neq \psi$ cannot occur. If $\tau = \theta(x_o, w_t)$, or $\tau = \theta(x_o, v_t)$, or $\tau = \theta(x_o, u_t)$, then again $[I_A, \tau]$ is a chain and the lower semimodular condition again holds vacuously for elements covered by τ . Finally consider the case in which $\tau = \tau_1 \vee \tau_2$, where τ_1 is of the form $\theta(u_i, v_j)$ and τ_2 has one of the forms $\theta(x_o, w_k)$, $\theta(x_o, v_k)$, or $\theta(x_o, u_k)$. Then τ only covers elements of the form $\gamma_1 \vee \tau_2$ or $\tau_1 \vee \gamma_2$, where $\gamma_i \prec \tau_i$ for $i = 1$ or 2, or, when τ has only one non trivial congruence class $\tau = \theta(u_i, v_j, x_o)$, in which case τ covers the elements of the form $\theta(u_i, v_j) \vee \theta(u_{i-1}, x_o)$, $\theta(u_{i-1}, v_j, x_o)$, or $\theta(u_i, v_{j-1}, x_o)$. A check of the intersections of these elements covered by τ shows that lower semimodularity is satisfied.

<u>4.19 Corollary</u>: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra. The following are equivalent:

(i) $\Theta(\mathfrak{A})$ is lower semimodular

(ii) $\Theta(\mathfrak{A})$ is modular

(iii) Conditions i, ii, iii, and iv of 4.18 hold for $\mathfrak{A}$.

<u>Proof</u>: By 4.18, i is equivalent to iii. It is well known that ii implies i (Birkhoff [2], p. 14). It follows from 4.14 and 4.18 that lower semimodularity implies upper semimodularity. Since $\Theta(\mathfrak{A})$ is finite, both semimodular conditions together give modularity, so i implies ii.

<u>4.20 Corollary</u>: Let $\mathfrak{A} = \langle A, f \rangle$ be a finite unary algebra such that $\Theta(\mathfrak{A})$ is modular. Then $\Theta(\mathfrak{A}) \cong L \times M$ where

L and M are the ideals defined in the proof of 4.18.

Proof: This follows from 4.19 and the proof of 4.18.

4.21 Definition: If $\mathfrak{A}$ is an algebra and ψ and $\emptyset$ are elements of $\Theta(\mathfrak{A})$, then ψ and $\emptyset$ are said to permute if $\psi \circ \emptyset = \emptyset \circ \psi$, where $\psi \circ \emptyset = \{<a, b> \mid$ for some x, $<a, x> \varepsilon \psi$ and $<x, b> \varepsilon \emptyset\}$, and $\emptyset \circ \psi$ is similiarly defined. In this case $\psi \circ \emptyset = \psi \vee \emptyset$. It is known that if all the congruence relations of $\mathfrak{A}$ permute with each other, then $\Theta(\mathfrak{A})$ is modular. Also, the congruence relations of any group are permutable. Much work has been done to determine which algebras have permutable congruence relations. The following theorem characterizes the finite unary algebras with permuting congruence relations.

4.22 Theorem: Let $\mathfrak{A} = <A, f>$ be a finite unary algebra. Then the congruences of $\mathfrak{A}$ are permutable if and only if A satisfies one of the following conditions:

(i) $A = C_1 \cup C_2$ where C_1 and C_2 are orbits of relatively prime cardinality

(ii) $A = C_1$ where C_1 is an orbit

(iii) $A = \{x_o, w_1, w_2, \ldots, w_s\}$ with $f(w_i) = w_{i-1}$, $f(w_1) = x_o$, $f(x_o) = x_o$.

Proof: If C_1, C_2, and C_3 are three distinct orbits with $b \in C_1$, $a \in C_3$, and if $\psi = \theta(C_1 \cup C_2)$ and $\phi = \theta(C_2 \cup C_3)$, then $\langle a, b \rangle \in \phi \circ \psi$, but $\langle a, b \rangle \notin \psi \circ \phi$. So if the congruences of $\mathfrak{A}$ are permutable, then f can have at most two orbits. These must have relatively prime cardinalities, since $\Theta(\mathfrak{A})$ is modular. Suppose that $u \neq v$ and $f(u) = f(v) = x$, where $x \notin \{u, v\}$. Let $\psi = \theta(u, v)$ and $\phi = \theta(u, x)$. Plainly $\langle x, v \rangle \in \phi \circ \psi$, but $\langle x, v \rangle \notin \psi \circ \phi$. Therefore, if $|C_i| > 1$ then $B_i = C_i$, and if $|C_i| = 1$, then $|B_i \cap (K_{j+1} \setminus K_j)| \leq 1$, for all $j \geq 0$. Finally, if $C_1 = \{x_0\}$, $a \neq x_0$, $f(a) = x_0$, and $b \in C_2$, then let $\phi = \theta(x_0, a)$ and $\psi = (C_2 \cup \{x_0\})$. Plainly, $\langle a, b \rangle \in \phi \circ \psi$, but $\langle a, b \rangle \notin \psi \circ \phi$. Therefore, if $|\cup C_i| > 1$, then $A = \cup C_i$. Hence, must satisfy one of the conditions i, ii, or iii.

Conversely, suppose that $A = C_1 \cup C_2$ where $|C_1| = p$ and $|C_2| = q$ with $(p, q) = 1$. Then $[I_A, \theta(C_1) \vee \theta(C_2)]$ is essentially identical with $\Theta(Z_p \times Z_q)$. Since the congruences of a group are permutable, it follows that any two congruence relations in this interval permute. By the proof of 4.5, $\Theta(\mathfrak{A}) = [I_A, \theta(C_1) \vee \theta(C_2)] \cup \{U_A\}$. Therefore $\mathfrak{A}$ has permutable congruence relations. If $A = C_1$, then a similiar argument shows $\mathfrak{A}$ has permutable congruence relations. Finally, if $A = \{x_0, w_1, \ldots, w_s\}$, $\Theta(\mathfrak{A})$ is a chain, and in this case, congruences obviously permute.

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