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ALGEBRAIC PROPERTIES OF k-VALUED LOGICS

Joel Berman

Mathematics Department
University of Illinois at Chicago Circle
Chicago, Illinois 60680

ABSTRACT

The first section of this paper describes various algebraic properties of the 2-valued logical systems. Particular emphasis is placed on the cardinalities of the free objects. Some of these results were obtained by computer enumeration. The next section shows how many of the algebraic properties which hold for 2-element logics, fail for 3-element ones. The approach here is to demonstrate these differences without appealing to the classification of the 2-valued logics as given by E. Post. The final section is concerned with the relationship between the cardinality of the free objects and the finite basis property for polynomial identities. The point of view of the entire paper is that of universal algebra: this motivates many of the questions and provides most of the tools for the solutions, and thus accounts for the notation and terminology used.

If S is a finite set having k elements and if F is a family of operations on S, then F can be thought of as a collection of functions in k-valued logic, or equivalently as a family of switching circuits in k-valued logic. The totality of functions which can be built up using the projection functions and the family F under finite composition form what is called the closed set of k-valued functions generated by F. Another way to view the set S and the family F is as a universal algebra A = <S,F> having universe S with the functions of F as the fundamental operations. If A is such an alebra, then <u>A</u> will denote the variety generated by A. Thus, <u>A</u> can be thought of as all homomorphic images of subalgebras of products of A, or equivalently, by Birkhoff's Theorem, as the class of all algebras of the same type as A that satisfy the same polynomial identities as A.

Let FA(n) and FA(omega) denote the free algebra in the variety <u>A</u> on n and on countably many infinite generators. There is a natural correspondence between FA(omega) and the closed set of k-valued functions generated by F. Likewise, FA(n) corresponds to the collection of n-ary k-valued switching circuits that can be built up from F and the projection functions.

The <u>free spectrum</u> of the variety <u>A</u> is the sequence of cardinalities |FA(0)|, |FA(1)|, |FA(2)|,.... This paper is concerned with how a number of properties of A, and hence of the corresponding family of k-valued logic functions, are determined solely by k and by the free spectrum of <u>A</u>. The approach taken is algebraic, and the methods used often involve recent results from the area of universal algebra.

The first section is a survey of the known classification of varieties generated by 2-element algebras. Particular attention is paid to the free spectra of such varieties. The second section investigates how some properties which hold for varieties <u>A</u> generated by a two element algebra, fail for varieties generated by an algebra of larger cardinality. The final section is concerned with the relationship between the rate of growth of the free spectrum of <u>A</u> and the finite basis property for <u>A</u>.

For background on universal algebra consult G. Grätzer's book [9]. The definitions of polynomial, free algebra, congruence relation etc. may all be found there. In this paper, however, a subdirectly irreducible algebra has at least 2 elements. Also, the name of an algebra and its universe will usually be denoted by the same symbol, e.g. A = <A,F>. The binomial coefficient n choose i is denoted C(n,i). To avoid superscripts on superscripts the notation ** is sometimes used for exponentiation. If $f(x_1,\ldots,x_n)$ is an n-ary function on the set {0,1}, then the dual of f is the function $f'(x_1',\ldots,x_n')$ where ' is the usual complementation operator, i.e. 0' = 1 and 1' = 0.

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Section 1. k=2

All the 2-element algebraic systems have been determined and are explicitly listed in E. Post's monograph [26]. There are 66 inequivalent algebras defined using at most ternary operations, and for each n > 3, there are 8 inequivalent algebras each including some fundamental n-ary operation. If V is any variety, then V is completely determined by FV(omega), the free algebra on a countable number of generators. If A is an algebra, then an element p of FA(omega), the free A algebra on omega generators, may be considered as an n-ary operation on A for some n. Thus, the algebra FA(omega) may be considered as a family of functions, and as such it is closed under composition and contains all the projection functions. Such a class of functions is called closed. So there exists a one-to-one map from varieties generated by 2-element algebras to families of closed functions on a two element set. By eliminating from Post's list those algebras containing only constant functions, and by including only those algebras in which the projection functions are derivable, and by including only one from the dual pairs of classes one gets the 27 algebras and 4 infinite families listed in Table 1.

The list in Table 1 is essentially that of Lyndon [19]. Post's original notation is used however, except that the superscript infinity symbols have been deleted. Also F superM subJ is written FM/J. Here M is always greater than 2.

Post's d original derivation is somewhat difficult to read , due in part to the notation used. C. Platt has redone the classification in [25]. Also [11] contains a proof of Post's result, but I have not seen this monograph. In [17] there is a detailed description of each of the classes. The varieties of monotonic functions are discussed by C. Benzaken in [5]. The varieties in Post's list which are congruence permutable are carefully described by W. Taylor in [31].

The second column of Table 1 contains the description of the algebras. Wherever possible, the common name is given. The notation for the operations is xvy for lattice join, xy for lattice meet (or multiplication), x+y is addition mod 2, and x-y is the dual of implication, i.e. x-y is 0 except in the case that x=1 and y=0. The n-median operation is a function of n variables consisting of the lattice join of n terms, where each term is the lattice meet of all but one of the n variables. Note that the dual of the n-median operation is the lattice join of C(n,2) terms, where each term is the lattice meet of a pair of variables.

The next 5 columns are the exact numerical values of the cardinalities of the free algebras on 0,1,2,3, and 4 generators, while the column headed F(n) is the closed form for the cardinalvity of the free algebra on n free generators, n>0. Most of these entries are well known and follow from easily derived normal forms. The closed forms for the cardinalities of free algebras in congruence permutable varieties are discussed by Taylor in [31]. A closed form for D(n), the cardinality of the free distributive lattice on n free generators is unknown, however see [4], [15], and [16] for further information. This class of functions is often referred to as the class of monotonic Boolean functions. Likewise, S(n) is the number of self-dual monotonic Boolean functions, and a closed form for this quantity is unknown. Values for small n are given in [18] and [28], while from [28] it follows that $D(n-2) \leq S(n) \leq D(n-1)$ for $n \geq 3$. The final 5 columns consist of the P(n) sequences for the varieties. Here P(n) is the number of essentially n-ary polynomials in the given variety. The quantities F(n) and P(n) are related by the formula F(n) = sum over i of C(n,i)P(i) as i goes from 0 to n, and thus P(n) = sum over i of $C(n,i)(-1)^{n-i}F(i)$ as i goes from 0 to n.

Let B(n) denote all of the 2**(2**n) Boolean polynomials. Thus B(n) is the free Boolean algebra on n generators and as such is a distributive lattice of length 2^n. The free algebra on n generators in any of the varieties generated by a two element algebra is a subset of B(n), and it turns out that many of these free algebras can be described in terms of the lattice order of B(n).

For example, an easy induction shows that $FC_3(n)$, the free algebra in the variety of Boolean rings, consists of all polynomials p in B(n) for which $p \leq x_1 v x_2 v \dots v x_n$. The interval $[0, x_1 v x_2 v \dots v x_n]$ is a Boolean lattice with 2**(2**n-1) elements. Likewise, the free n-generated algebra for the variety generated by C_4 consists of all elements in the interval $[x_1 x_2 \dots x_n, x_1 v \dots v x_n]$. Similarly, the free n-generated algebra for dual implication consists of all polynomials p in B(n) for which $p \leq x_i$ for some i, $1 \leq i \leq n$. Denote by G(n) the cardinality of the free dual implication algebra on n free generators. Inclusion-exclusion arguments give G(n) = sum over i of $C(n,i)(-1)^{i-1}$2**(2**(n-i)) where C(n,i) is the binomial coefficient n choose i. However, the P(n) and F(n) sequences for Boolean algebras are such that F(n) - G(n) = P(n), and thus G(n) = F(n) - P(n) as in Table 1.

The variety generated by F_6 has a similar description for its free algebra. It is easily seen that any polynomial p of the free algebra on n generators for this variety has the property that p is a distributive lattice polynomial and $p \leq x_i$ for some free generator x_i of B(n). Moreover, any distributive lattice polynomial which is less than or equal to some such generator is in this free algebra. Also the set of such distributive lattice polynomials in B(n) which are less than or equal to the meet of m free generators has cardinality 1 + D(n-m). So applying inclusion-exclusion and arguing as in the case of F_8 gives the cardinality of the free F_6 algebra on n free generators to be F(n) - P(n) of A_2.

The free algebra on n free generators for the variety generated by F_5 can be seen to be the set of all polynomials in B(n) for which $x_1 x_2 \dots x_n \leq p \leq x_i$ for $1 \leq i \leq n$. A similar inclusion-exclusion type argument gives the entries for this row of Table 1. Note these values are one-half the corresponding values for F_8.

The situation for $F^2{}_8$, $F^2{}_6$, and $F^2{}_5$ is similar, but I know of no closed form for the cardinalites of their free algebras. The free $F^2{}_8$ algebra on n free generators consists of all Boolean polynomials p in B(n) such that $p \leq q$ where q is any element of the free D_2 algebra, i.e. q is a self-dual monotonic polynomial in n variables. The entries for 3 and 4 generators were obtained by a computer enumeration. The free $F^2{}_6$ algebra is the same except that the polynomial p must be monotonic as well. For $F^2{}_5$ the polynomial p can be any Boolean polynomial with $x_1x_2...x_n \leq p \leq q$ with again q being self-dual monotonic.

Finally, the $F^m{}_j$ for $m \geq 3$ and j= 5,6, or 8 are similar: the free algebras on n free generators consist of Boolean polynomials p of the appropriate type such that $p \leq q$ where q is any polynomial in n variables in $F^m{}_7$.

PROBLEM #1. Obtain closed forms for the cardinalities of the free algebras on n free generators for $F^m{}_j$, $m \geq 2$ and j= 6 or 8.

The column headed <u>01</u> in Table 1 lists the algebra which results when the two constant functions 0 and 1 are adjoined to the algebra. Note there are only a finite number of distinct entries, i.e. all of the infinite families reduce to A_1 or C_1.

The next column, headed DIS, lists whether or not the variety is congruence distributive. The importance of this property is due to Jónsson's lemma [13]. For details on the congruence distributivity of some of these varieties see [1], [21]. The column headed PER deals with congruence permutability. Those 2-element algebras which generate varieties that are congruence permutable are precisely the ones with small fine spectrum as described by Taylor [31] (see also Quackenbush [27]).

The column headed S-D indicates those algebras whose closed classes of polynomials are self-dual, i.e. if $p(x_1,...x_n)$ is in the class, then so is $p(x_1',...,x_n')'$. Note the polynomials themselves need not be self-dual in this case.

<u>Section 2.</u> $2 \neq 3$

This section exhibits several properties which hold for algebraic systems on a two element set, but fail for algebraic systems on a larger set. In most cases, a three element algebra suffices. The exact cause of this breakdown from two to three is unclear.

A. The number of closed classes: Post's work [26] shows that there are only countably infinitely many closed classes of functions on a two element set; that is, only countably many inequivalent varieties can be generated by a two element algebra. Moreover, each such closed class can be generated by a finite set of functions. It is known that there are uncountably many closed sets of functions on a 3-element set. This is given in [12] and later in [10]. Also see I. Rosenberg's survey article [29]. The examples are similar: In [10] for example, for each integer n define an n-ary operation f_n on {0,1,2} by $f_n(x_1,...,x_n) = 0$ if at least two distinct variables have values in {0,1}, and $f_n(x_1,...,x_n) = 1$ otherwise. Then distinct sets of these functions produce distinct closed classes.

PROBLEM #2. Provide a direct proof that there exist only countably many inequivalent varieties generated by a two element set. Here "direct" means avoiding an enumeration of all of the classes as done by Post.

PROBLEM #3. How pathological is the case of k = 3? Can some large chunk of the varieties generated by the 3-element algebras be classified in some way? I. Rosenberg, in [29], also mentions this problem.

B. Finite basis property: An algebra A is finitely based if there is a finite set S of polynomial equations such that each equation in S holds in A, and every polynomial equation which holds in A is a logical consequence of the equations S. For details see [9, p.351 and 385]. It is known that every 2-element algebraic system is finitely based. Lyndon proved this in [19] using Post's classification [26]. In [3], I gave a different proof, avoiding Post's list. V. L. Murskii [23] has exhibited a 3-element groupoid that is not finitely based.

PROBLEM #4. What other 3-element groupoids, if any, are not finitely based? ↖ Ježek has one.

In [24], R. E. Park presents a 3-element groupoid and asks whether or not it is finitely based (he conjectured it was finitely based). In [33] this groupoid was shown to have definable principal congruences (see D below) and therefore it is finitely based. Another potential non-finitely based 3-element groupoid is the groupoid of Grzegorczyk which is a subgroupoid of a 4-element groupoid that is known to be non-finitely based. The multiplication table is given below. For details on this see [14].

	0	1	2
0	0	0	1
1	0	1	1
2	1	2	0

C. Residsual finiteness: Every variety generated by a 2-element algebra has only a finite number of subdirectly irreducible algebras, and each of these subdirectly irreducible algebras is finite. W. Taylor observed this in [30] by examining the members of Post's list. In [3] I gave a direct proof; in particular, each such variety has at most three subdirectly irreducible members, and any such subdirectly irreducible has at most three elements. In [30], Taylor mentions some 3-element algebras that generate varieties that contain arbitrarily large subdirectly irreducible algebras.

D. Definable principal congruence relations: A variety is said to have <u>definable principal congruences</u> if there is some first order formula which determines principal congruence relations for all algebras in the variety. For details on this concept see [2]. An examination of Post's list would show that all the varieties generated by a 2-element algebra have definable principal congruences, although some of the defining formulas are probably quite large. As with the property of being finitely based and the property of having a bound on the size of subdirectly irreducible algebras, a proof can be given directly, using some results from universal algebra instead of Post's catalog. To this end, let A = <{0,1}, F> where F is some family of operations. If all members of F are at most essentially unary, then since the variety <u>A</u> is locally finite, the results of [2] apply to show <u>A</u> has definable principal congruences. Likewise if A is a semilatice with nullary or unary operations, then again <u>A</u> is known to have definable principal congruences. By lemma 1 of [3], it follows that the only remaining cases are such that <u>A</u> is congruence distributive or congruence permutable. If <u>A</u> is congruence distributive, then by Jónsson's lemma [13], A is the only subdirectly irreducible in <u>A</u>. So all the subdirectly irreducible members of the variety <u>A</u> are described by some first order sentence and have the congruence extension property [9, p. 395]. So by a result of A. Day [8], or more generally by the work of B. Davey [7], the variety <u>A</u> also has the congruence extension property. But then by [2] <u>A</u> has definable principal congruences. If on the other hand, <u>A</u> is congruence permutable, then it is known that every finite algebra in the variety $\underline{A}$ is the direct product of 2-element algebras. Thus, McKenzie's theorem 5 in [20] applies (the overriding restriction of [20] to finite type is not needed here).

There exist algebras of cardinality greater than 2 that do not have definable principal congruences. S. Burris in [6] exhibited a 4-element algebra without definable principal congruences. In [20] McKenzie showed that any nondistributive lattice generates a variety that does not have definable principal congruences. W. Taylor [32] has constructed a three element semigroup that does not have definable principle congruences. If, say, its universe is {1,2,3}, then all products are 1 except for 3*3 which is equal to 2.

PROBLEM #5. Among the 3-element groupoids, how prevalent are those with definable principal congruences? Do they have any special characterization?

E. Adjoining constants: If A is any algebra, and if each element of A is a nullary constant, then A is said to have a <u>full set of constants</u>. So, in this case, the free algebra for $\underline{A}$ with 0 free generators is isomorphic to A and every algebraic function on A is a polynomial.

An examination of Post's list shows that there are only a finite number of algebras which have a full set of constants. An explicit proof of this fact, not appealing to Post's catalog, now follows.

THEOREM 2.1. Let A be an algebra having universe {0,1} and suppose A contains both 0 and 1 as nullary constants. Then A is one of the following: a) a set with 0 and 1; b) complementation with 0 and 1; c) semilattice with 0 and 1; d) distributive lattice with 0 and 1; e) complemented Boolean group; f) Boolean algebra.

Proof: If A has no nonconstant nonprojection operations, then case a) holds. Otherwise, let $f(x_1,\ldots,x_n)$ be any nonconstant nonprojection operation of A with $n \geq 1$. The domain of f is $D = \{0,1\}^n$ which may be considered as a lattice in the usual manner. Let $\underline{0}$ and $\underline{1}$ be the n-tuples of all 0's and 1's respectively. First suppose f is monotonic: so $n \geq 2$, and $a \leq b$ implies $f(a) \leq f(b)$, and $f(\underline{0}) = 0$ and $f(\underline{1}) = 1$. Let a be in D and suppose a is minimal with the property that $f(a) = 1$. If two or more coordinates of a have value 1, say a_1 and a_2, then let f^* be the binary operation derived from f by replacing x_i by the constant a_i, for all $i > 2$. Then f^* is a meet semilattice operation. If there are two such minimal members of D, say a and b for which $f(a) = f(b) = 1$, then construct a binary polynomial from f in two variables y and z as follows: for those coordinates i for which $a_i = b_i$ replace the variable x_i by the constant a_i; for those i for which the value of a_i is 1 and b_i is 0, replace the variable x_i by y; for those i for which $a_i = 0$ and $b_i = 1$, replace x_i by z. The resulting function is $y \vee z$. Thus for a given monotonic nonconstant function f:

(i) If there is only one minimal a in D for which $f(a) = 1$, then f is the lattice meet of k variables, $2 \leq k \leq n$.

(ii) If every minimal a in D for which $f(a) = 1$ is an atom, then f is the lattice join of k variables, $2 \leq k \leq n$.

(iii) If there are at least two such minimal members of D for which f has value 1, and at least one of these is not an atom, then both meet and join are derivable from f; and since f is montonic, f can be generated from the meet and join operations.

Therefore if all fundamental operations of A are monotonic, then A is either a semilattice with 0 and 1 or is a lattice with 0 and 1.

If some operation of A is not monotonic, then there exist elements a and b in D such that a and b differ only in say the i-th coordinate, $a < b$, and $f(a) = 1$ and $f(b) = 0$. Freezing all but this coordinate gives complementation. If all the operations of A are at most unary, then this gives case (b); if some other function is monotonic and depends on more than one variable, then A has a semilattice operation and A is a Boolean algebra. So suppose f is not monotonic and depends on n variables, $n \geq 2$. If there exist elements a and b in D such that $f(a) = f(b) = f(ab) \neq f(avb)$ (or dually), then by making the appropriate identifications, a semilattice operation can be obtained. Likewise, if there exist a and b such that $f(a) = f(ab) = f(avb) \neq f(b)$, then implication or dual implication are polynomials for A. Any of these operations together with complementation give a Boolean algebra. Finally, suppose for some $a < b$, with b covering a, $f(a) = f(b)$ holds. Without loss of generality, $a_1 = 0$, $b_1 = 1$ and $a_i = b_i$ for all $i \geq 2$. Consider any c and d for which $c_1 = 0$ and $d_1 = 1$ and $c_i = d_i$ for $i \geq 2$. Then there exists a sequence of transpositions from the pair {a,b} to the pair {c,d}. Moreover, if {r,s} is any covering pair in this sequence of transpositions, then $f(r) = f(s)$ since otherwise a semilattice or implication or dual implication would be derivable. But since this holds for all such c and d, it follows that f does not depend on the variable x_1. So if b covers a, then $f(a) \neq f(b)$. So starting from $\underline{0}$, alternate levels in D have alternate values. Thus $f(x_1,\ldots,x_n) = x_1+\ldots+x_n$ or $f(x_1,\ldots,x_n) = 1+x_1+\ldots+x_n$. The replacement of n-2 of these variables by 0 yields the operations $x+y$ or $x+y+1$. Thus A is a complemented Boolean group.

What is the situation for 3-element algebras that have a full set of constants? The example in [12], but not [10], still provides an uncountable family of varieties. This again illustrates the change when going from 2-element algebras to 3-element algebras.

PROBLEM #6. An algebra A is said to be functionally complete if the algebra A^*, obtained from A by adjoining all constants of A as operations, is primal, i.e. for all n, every function from A^n to A is a polynomial of A^*. If A is finite, is of finite type, and is functionally complete, is A finitely based?

No, McKenzie has a groupoid.

Section 3. The growth of free spectra

Let A be any finite algebra of cardinality k. Then the cardinality of FA(n), the free algebra on n free generators for the variety $\underline{A}$, is at most k**(k**n). In this section it is shown that if the free spectrum of $\underline{A}$, as a function of n, grows very slowly or if it grows very rapidly, then A is finitely based.

LEMMA 3.1. Let $\underline{V}$ be any variety of algebras. The following are equivalent for an integer s:

(i) The cardinality of FV(n) is bounded above by a polynomial in n of degree s.

(ii) The number of essentially n-ary polynomials for $\underline{V}$, P(n), has value 0 for all $n > s$.

Proof: Note that FV(n) has cardinality equal to the sum of the products C(n,i)P(i) where i ranges from 0 to n. So if $P(i) = 0$ for all $i > s$, then $|FV(n)|$ is bounded by the sum of C(n,i)P(i) as i ranges from 0 to s. This sum, in turn, is bounded by the quantity p (s+1) C(n,s) for sufficiently large n, where p is the maximum of the P(i), $0 \leq i \leq s$. But C(n,s), as a function of n, is an s-th degree polynomial. Conversely, if $P(i) \neq 0$ for arbitrarily large i, then the cardinality of FV(n) is greater than C(n,i), and thus C(n,i)P(i) is greater than n^i.

Theorem 3.2. Let $\underline{V}$ be any variety of finite type such that the free spectrum of $\underline{V}$ is bounded by some polynomial function of n. Then $\underline{V}$ has a finite basis for its identities.

Proof: If m is an arbitrary integer, then let $C=\{c_i | 1 \leq i \leq |FV(m)|\}$ be a fixed enumeration of some representation of $FV(m)$. Let $E(m)$ be a listing of all the operation tables of $FV(m)$, i.e. $E(m)$ consists of all equations of the form $f(d_1,\ldots,d_n) = d$, where f is a fundamental n-ary operation of $\underline{V}$ and $d_1,\ldots,d_n,d$ are all in C. A standard induction shows $E(m)$ is a basis for all identities of $\underline{V}$ which involve at most m variables. Let t be the maximal number of arguments of any fundamental operation of the variety $\underline{V}$. By lemma 2.1 there is some integer s such that $P(n) = 0$ for all $n > s$. Claim $E(st)$ is a basis for $\underline{V}$. It suffices to show that if $p = q$ holds in $\underline{V}$ then $p = q$ is derivable from $E(st)$. Let $c_i(x_1,\ldots,x_s)$ be a list of all elements of $FV(s)$, $1 \leq i \leq |FV(s)|$. It suffices to show that if $p(x_1,\ldots,x_n)$ is any polynomial in the type of $\underline{V}$, then there exists some i such that $p(x_1,\ldots,x_n) = c_i(y_1,\ldots,y_s)$ is derivable from $E(st)$, where $\{y_1,\ldots,y_s\}$ is some subset of the set $\{x_1,\ldots,x_n\}$. Note that since $st \geq 2s$, the equations $E(st)$ and transitivity may be used to show two such c_i with possibly different sets of variables are equal. So suppose $p(x_1,\ldots,x_n) = f(p_1,\ldots,p_m)(x_1,\ldots,x_n)$ where each p_i is derivably equal to some $c_{i'}$ using $E(st)$. Each such $c_{i'}$ involves at most s variables and $m \leq t$, so $f(c_{1'},\ldots,c_{n'})$ involves at most st variables and thus using $E(st)$, the polynomial $p(x_1,\ldots,x_n)$ can be shown to be equal to some c_k.

Note that if $\underline{V}$ has only unary operations, then $P(n) = 0$ for all $n > 1$. Thus, the following known result is a consequence of lemma 3.1 and theorem 3.2.

Corollary 3.3. Let $\underline{V}$ be any variety of unary algebras of finite similarity type. Then $\underline{V}$ is finitely based.

Let A be a finite algebra of cardinality k. As previously mentioned, the cardinality of $FA(n)$ is at most $k**(k**n)$. The next result shows that if this cardinality is $O(k**(k**n))$, then A is finitely based. This confirms a conjecture of G. McNulty.

Theorem 3.4. Let A be an algebra of cardinality k for which there is some positive element c such that the cardinality of $FA(n)$ is $\geq c(k**(k**n))$ for all n. Then A is equivalent to some finitely based algebra.

Proof: Let $A = \{1,2,\ldots,k\}$. A function $p(x_1,\ldots,x_n)$ in $FA(n)$ is called <u>quasi sheffer</u> if there exists some subset S of $\{1,2,\ldots,k\}$ such that any m-ary function f on A for which $f(x,\ldots,x) = x$ for all x in S, can be built from p under composition. By Murski [23], it suffices to show that for some n, there exists a quasi-Sheffer function in $FA(n)$. But by lemmma 10 of that paper, if $q(n)$ denotes the number of quasi-Sheffer functions with n variables on a set with k elements, then the limit as n goes to infinity of the quantity $q(n)/(k**(k**n))$ is 1. So for some n large enough, it must be that $q(n) > c(k**(k**n))$, and so $FA(n)$ must contain some quasi-Sheffer function.

PROBLEM #7. Can the bounds in Theorems 3.2 and 3.4 be sharpened?

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POST	DESCRIPTION	0	1	2	3	4	F(N)	01	DIS	S-2	PER	P(N), N=0	1	2	3	4
R4	COMPLEMENT: X'	0	2	4	6	8	2N	09	NO	YES	NO	0	2	0	0	0
R13	COMPLEMENT WITH CONSTANTS: X',0,1	2	4	6	8	10	2N + 2	09	NO	YES	NO	2	2	0	0	0
S2	SEMILATTICE: XVY	0	1	3	7	15	2**N - 1	S6	NO	NO	NO	0	1	1	1	1
S5	SEMILATTICE WITH 0: XVY, 0	1	2	4	8	16	2**N	S6	NO	NO	NO	1	1	1	1	1
S4	SEMILATTICE WITH 1: XVY, 1	1	2	4	8	16	2**N	S6	NO	NO	NO	1	1	1	1	1
S6	SEMILATTICE WITH 0,1: XVY,0,1	2	3	5	9	17	2**N + 1	S6	NO	NO	NO	2	1	1	1	1
A4	DISTRIBUTIVE LATTICE: XY,XVY	0	1	4	18	166	D(N)	A1	YES	YES	NO	0	1	2	9	114
A2	DIST. LATTICE WITH 0: XY,XVY,0	1	2	5	19	167	1 + D(N)	A1	YES	NO	NO	1	1	2	9	114
A1	DIST. LATTICE WITH 0,1: XY,XVY,0,1	2	3	6	20	168	2 + D(N)	A1	YES	YES	NO	2	1	2	9	114
L3	BOOLEAN GROUP: X+Y	1	2	4	8	16	2**N	L1	NO	NO	YES	1	1	1	1	1
L1	COMP. BOOLEAN GROUP: X+Y, X'	2	4	8	16	32	2**(N+1)	L1	NO	YES	YES	2	2	2	2	2
C3	BOOLEAN RING: X+Y, XY	1	2	8	128	2**15	2**(2**N - 1)	C1	YES	NO	YES	1	1	5	109	32297
C1	BOOLEAN ALGEBRA	2	4	16	256	2**16	2**(2**N))	C1	YES	YES	YES	2	2	10	218	64594
L4	BOOLEAN 3-GROUP: X+Y+Z	0	1	2	4	8	2**(N-1)	L1	NO	YES	YES	0	1	0	1	0
L5	COMP. BOOLEAN 3-GROUP: X+Y+Z, X'	0	2	4	8	16	2**N	L1	NO	YES	YES	0	2	0	2	0
F8	DUAL IMPLICATION: X-Y	1	2	6	38	942	F(N) - P(N) OF C1	C1	YES	NO	NO	1	1	3	25	819
F6	X(YVZ)	0	1	3	10	53	F(N) - P(N) OF A2	A1	YES	NO	NO	0	1	1	4	27
F7	X(YVZ), 0	1	2	4	11	54	F(N) - P(N) OF A1	A1	YES	NO	NO	1	1	1	4	27
F5	XYZ V XY'Z'	0	1	3	19	471	F(N) - P(N) OF C3	C1	YES	NO	NO	0	1	1	13	409
C4	BOOLEAN LATTICE: XYZ V XY'Z',XVY	0	1	4	64	2**14	2**(2**N-2)	C1	YES	YES	YES	0	1	2	55	16148
D2	MEDIAN: XY V XZ V YZ	0	1	2	4	12	S(N)	A1	YES	YES	NO	0	1	0	1	4
D1	BOOLEAN 3-RING: XY V XZ V YZ,X+Y+Z	0	1	2	8	128	2**(2**(N-1)-1)	C1	YES	YES	YES	0	1	0	5	104
D3	BOOLEAN 3-ALG.: XY V XZ V YZ,X'	0	2	4	16	256	2**(2**(N-1))	C1	YES	YES	YES	0	2	0	10	208
F2/8	MEDIAN, X-Y	1	2	6	40	1376	???	C1	YES	NO	NO	1	1	3	27	1245
FM/8	M-MEDIAN, X-Y	1	2	6	38		???	C1	YES	NO	NO	1	1	3	25	
F2/6	MEDIAN, X(YVZ)	0	1	3	11	80	???	A1	YES	NO	NO	0	1	1	5	50
FM/6	M-MEDIAN	0	1	3	10		???	A1	YES	NO	NO	0	1	1	4	
F2/7	MEDIAN, X(YVZ), 0	1	2	4	12	81	1 + F(N) OF F2/6	A1	YES	NO	NO	1	1	1	5	50
FM/7	M-MEDIAN, 0	1	2	4	11		1 + F(N) OF FM/6	A1	YES	NO	NO	1	1	1	4	
F2/5	MEDIAN, XYZ V XY'Z'	0	1	3	20	688	1/2 OF F(N) OF F2/8	C1	YES	NO	NO	0	1	1	14	622
FM/5	M-MEDIAN, XYZ V XY'Z'	0	1	3	19		1/2 OF F(N) OF FM/8	C1	YES	NO	NO	0	1	1	13	

TABLE 1