W. J. Blok
J. Rebagliato

Algebraic Semantics for Deductive Systems

Abstract. The notion of an *algebraic semantics of a deductive system* was proposed in [3], and a preliminary study was begun. The focus of [3] was the definition and investigation of *algebraizable deductive systems*, i.e., the deductive systems that possess an *equivalent* algebraic semantics. The present paper explores the more general property of possessing an algebraic semantics. While a deductive system can have at most one equivalent algebraic semantics, it may have numerous different algebraic semantics. All of these give rise to an algebraic completeness theorem for the deductive system, but their algebraic properties, unlike those of equivalent algebraic semantics, need not reflect the metalogical properties of the deductive system. Many deductive systems that don't have an equivalent algebraic semantics do possess an algebraic semantics; examples of these phenomena are provided. It is shown that all extensions of a deductive system that possesses an algebraic semantics themselves possess an algebraic semantics. Necessary conditions for the existence of an algebraic semantics are given, and an example of a protoalgebraic deductive system that does not have an algebraic semantics is provided. The mono-unary deductive systems possessing an algebraic semantics are characterized. Finally, weak conditions on a deductive system are formulated that guarantee the existence of an algebraic semantics. These conditions are used to show that various classes of non-algebraizable deductive systems of modal logic, relevance logic and linear logic do possess an algebraic semantics.

Keywords: deductive system, equational consequence, matrix semantics, algebraic semantics, protoalgebraic logic, mono-unary algebras, modal logic, intuitionistic logic, linear logic. AMS subject classification: 03B22 (03B47, 03G25, 08C15, 08A60).

Introduction

One of the issues addressed in abstract algebraic logic is the question what exactly constitutes an "algebraic semantics" of a logic. A precise definition was proposed in [3], and a preliminary study of the notion was begun. The present paper explores the notion further.

Following [3], by a logic we shall mean a consequence relation $\vdash$, thought of as a binary relation between sets of formulas and individual formulas; a logic in this sense is often referred to as a *deductive system*. Broadly speaking, a class K of algebras is an *algebraic semantics* of a deductive system $\mathcal{S}$ if the

Presented by **Guest Editors**; *Received* January 31, 2002

Studia Logica **73**: 1–29, 2003.

consequence relation $\vdash_{\mathcal{S}}$ can be interpreted in $\models_{\mathsf{K}}$ by means of an equational translation; the interpretation will then give rise to a strong completeness theorem for $\mathcal{S}$ with respect to the class K. The focus of [3] was a stronger notion: K is an *equivalent algebraic semantics* of $\mathcal{S}$ if it is an algebraic semantics of $\mathcal{S}$, and in addition there is an inverse translation of $\models_{\mathsf{K}}$ in $\vdash_{\mathcal{S}}$. A deductive system is *algebraizable* if it possesses an equivalent algebraic semantics. Certain metalogical properties of algebraizable deductive systems—such as that of possessing a deduction theorem, for example—correspond to algebraic properties of their equivalent algebraic semantics; see [5] for an extensive discussion. While an algebraic semantics of a deductive system provides an algebraic completeness theorem for the system, no such corresponds exists in general between metalogical properties of the deductive system and the algebraic properties of an algebraic semantics.

One of the main results of [3] asserts that an equivalent algebraic semantics of a deductive system, if it exists, is essentially unique. By contrast, a deductive system may have many different algebraic semantics, even with respect to the same translation. Some examples of this phenomenon were given in [3]. In Section 2.1 we add a particularly natural illustration, by showing that not only is the variety of Boolean algebras an algebraic semantics of classical propositional logic—in fact, of course, an equivalent one—but so is the variety of Heyting algebras, both using the same translation, albeit not the 'standard' one. We next turn to the question whether the property of possessing an algebraic semantics is preserved under the formation of extensions. In [8] this had been shown to be the case provided the extension is an axiomatic one; in Theorem 2.15 we show that it holds in general.

While there are various satisfying characterizations of algebraizable deductive systems, the property of possessing an algebraic semantics seems hard to capture. In Section 2.3 we establish some necessary conditions, strengthening a result obtained in [3]. Using these we are able to give an example of a well-behaved (in fact, protoalgebraic) and non-trivial deductive system that has no algebraic semantics whatsoever (Theorem 2.19). The same conditions enable us to provide a complete characterization of the deductive systems possessing an algebraic semantics in the special case in which the language consists of just one unary connective (Section 2.4).

In Section 3 we provide sufficient conditions for the existence of an algebraic semantics. It is well-known that every deductive system has a complete *matrix* semantics, and often there is a lot of leeway in the choice of the matrix semantics. We mention a typical application of the main result of the paper (Theorem 3.1). If a deductive system has a matrix semantics consisting of matrices the underlying algebras of which universally satisfy a certain type of equation in one variable, then the deductive system has an algebraic semantics with the translation given by that equation. For example, if $\mathcal{S}$ is a deductive system over a language including say the connective $\wedge$, and if $\mathcal{S}$ possesses a matrix semantics consisting of matrices the underlying algebras of which satisfy the identity $(p \wedge p) \approx p$, then $\mathcal{S}$ has an algebraic semantics. It follows that all quasi-classical modal logics, although generally not algebraizable, do possess an algebraic semantics (Corollary 3.5). On the same grounds we see that the system **E** of entailment, which fails to be algebraizable as well, possesses an algebraic semantics, and hence so do all of its extensions.

Although Theorem 3.1 covers a wide array of logics, it leaves some important examples out. The implicational fragment of intuitionistic linear logic—itself algebraizable due to the presence of the lattice operations—is known not to be algebraizable, but the theorem is not applicable. We conclude the paper by providing mild conditions on a binary connective $\rightarrow$ that, if satisfied, ensure that the deductive system under consideration have an algebraic semantics.

1. Preliminaries

In this section we review some basic facts concerning deductive systems, matrices and equational logic. For further details we refer the reader to [3].

1.1. Deductive systems

A *propositional language* $\mathcal{L}$ is a set of propositional connectives, each of given finite arity; in an algebraic context its elements are often referred to as *fundamental operations*. Given a propositional language $\mathcal{L}$, $\mathrm{Fm}_{\mathcal{L}}$ denotes the set of propositional $\mathcal{L}$-*formulas* (also called $\mathcal{L}$-*terms*) built in the usual recursive way from the countably infinite set Va of propositional variables $p_0, p_1, \ldots$. The *algebra of* $\mathcal{L}$-*formulas* is the algebra $\mathbf{Fm}_{\mathcal{L}} = \langle \mathrm{Fm}_{\mathcal{L}}, \langle f^{\mathbf{Fm}_{\mathcal{L}}} : f \in \mathcal{L} \rangle \rangle$, where if $f \in \mathcal{L}$ is k-ary and $\varphi_0, \ldots, \varphi_{k-1} \in \mathrm{Fm}_{\mathcal{L}}$, $f^{\mathbf{Fm}_{\mathcal{L}}}(\varphi_0, \ldots, \varphi_{k-1}) = f\varphi_0, \ldots, \varphi_{k-1}$ (using prefix notation). If $\boldsymbol{A}$ is an $\mathcal{L}$-algebra, any map from the set Va of variables to A can be extended to a homomorphism from $\mathbf{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$. An endomorphism $\sigma \colon \mathbf{Fm}_{\mathcal{L}} \rightarrow \mathbf{Fm}_{\mathcal{L}}$ is called a *substitution*.

A *consequence relation* on $\mathrm{Fm}_{\mathcal{L}}$ is a binary relation $\vdash$ between sets of formulas and formulas that satisfies the following conditions for all $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\varphi \in \mathrm{Fm}_{\mathcal{L}}$:

(i) $\varphi \in \Gamma$ implies $\Gamma \vdash \varphi$,

(ii) $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$ imply $\Delta \vdash \varphi$,

(iii) $\Gamma \vdash \varphi$ and $\Delta \vdash \gamma$ for every $\gamma \in \Gamma$ imply $\Delta \vdash \varphi$.

A consequence relation $\vdash$ is *finitary* if

$$\Gamma \vdash \varphi \text{ implies } \Gamma' \vdash \varphi \text{ for some finite } \Gamma' \subseteq \Gamma,$$

and is *structural* if

$$\Gamma \vdash \varphi \text{ implies } \sigma(\Gamma) \vdash \sigma(\varphi) \text{ for every substitution } \sigma.$$

For $\Gamma, \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ we write $\Gamma \vdash \Delta$ for the conjunction of the statements $\Gamma \vdash \delta$, $\delta \in \Delta$, and $\Gamma \dashv\vdash \Delta$ for $\Gamma \vdash \Delta$, $\Delta \vdash \Gamma$.

A *deductive system* is a pair $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash_{\mathcal{S}}$ is a finitary and structural consequence relation on $\mathrm{Fm}_{\mathcal{L}}$.

A *rule of inference* is a pair $\langle \Gamma, \varphi \rangle$, also denoted

$$\frac{\Gamma}{\varphi},$$

where Γ is a finite set of formulas (the *premises* of the rule) and φ is a formula. An *axiom* is a rule with an empty set of premises, i.e., a rule of the form $\langle \emptyset, \varphi \rangle$, usually just denoted φ. With a (possibly infinite) set of axioms and rules of inference we associate a deductive system $\mathcal{S}$ by defining a consequence relation $\vdash_{\mathcal{S}}$ as follows:

$\Gamma \vdash_{\mathcal{S}} \varphi$ if there is a sequence of formulas $\varphi_0, \ldots, \varphi_{n-1}$, $n < \omega$, such that $\varphi_{n-1} = \varphi$ and for every $i < n$ one of the following conditions holds:

(i) $\varphi_i \in \Gamma$,

(ii) there is an axiom ψ and a substitution σ such that $\varphi_i = \sigma\psi$,

(iii) there is a rule $\langle \Delta, \psi \rangle$ and a substitution σ such that $\varphi_i = \sigma\psi$ and $\sigma(\Delta) \subseteq \{\varphi_j : j < i\}$.

The sequence $\varphi_0, \ldots, \varphi_{n-1}$ is then called a *proof* of φ from Γ. Every deductive system can be defined by a set of axioms and rules of inference (see [7]); the set of axioms and rules is called an *axiomatization* of $\mathcal{S}$.

If $\mathcal{S}$ is a deductive system, a set T of formulas is called an $\mathcal{S}$-*theory* if $\varphi \in T$ whenever $T \vdash_{\mathcal{S}} \varphi$, for every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. If Γ is a set of formulas, $\langle \Gamma \rangle_{\mathcal{S}} = \{\varphi : \Gamma \vdash_{\mathcal{S}} \varphi\}$ is the smallest $\mathcal{S}$-theory containing Γ. The set of $\mathcal{S}$-theories is denoted by $\mathrm{Th}\,\mathcal{S}$. A deductive system $\mathcal{S}' = \langle \mathcal{L}, \vdash_{\mathcal{S}'} \rangle$ is an *extension* of the deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ if $\Gamma \vdash_{\mathcal{S}'} \varphi$ whenever $\Gamma \vdash_{\mathcal{S}} \varphi$, for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ or, equivalently, if $\mathrm{Th}\,\mathcal{S}' \subseteq \mathrm{Th}\,\mathcal{S}$. $\mathcal{S}'$ is an *axiomatic extension* if it can be axiomatized by adding axioms only to the axioms and rules of some axiomatization of $\mathcal{S}$.

If $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ and $\mathcal{L}' \subseteq \mathcal{L}$, the *$\mathcal{L}'$-fragment* of $\mathcal{S}$ is the deductive system $\mathcal{S}' = \langle \mathcal{L}', \vdash_{\mathcal{S}'} \rangle$ defined by

$$\Gamma \vdash_{\mathcal{S}'} \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi,$$

for all $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$.

A *matrix* is a pair $\mathcal{A} = \langle \boldsymbol{A}, F \rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and F is an arbitrary subset of A. If $\mathcal{A}$ is a matrix, $\models_{\mathcal{A}}$ is the consequence relation on $\mathrm{Fm}_{\mathcal{L}}$ defined by

$$\begin{aligned}&\Gamma \models_{\mathcal{A}} \varphi \text{ iff}\\ &\qquad h(\Gamma) \subseteq F \text{ implies } h(\varphi) \in F, \text{ for every homomorphism } h\colon \mathbf{Fm}_{\mathcal{L}} \to \boldsymbol{A}.\end{aligned}$$

If M is a class of matrices, $\models_{\mathsf{M}}$ is the consequence relation defined by

$$\Gamma \models_{\mathsf{M}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathcal{A}} \varphi, \text{ for every } \mathcal{A} \in \mathsf{M}.$$

Let $\mathcal{S}$ be a deductive system. A matrix $\mathcal{A} = \langle \boldsymbol{A}, F \rangle$ is called an *$\mathcal{S}$-matrix* if $\Gamma \vdash_{\mathcal{S}} \varphi$ implies $\Gamma \models_{\mathcal{A}} \varphi$; F is then called an *$\mathcal{S}$-filter* on $\boldsymbol{A}$. A class of matrices M is a *matrix semantics* of $\mathcal{S}$ if the consequence relations $\vdash_{\mathcal{S}}$ and $\models_{\mathsf{M}}$ coincide. It follows from structurality that the $\mathcal{S}$-filters on $\mathbf{Fm}_{\mathcal{L}}$ are exactly the $\mathcal{S}$-theories. The following result is a consequence of this fact.

THEOREM 1.1. *For any deductive system $\mathcal{S}$, the class of all $\mathcal{S}$-matrices forms a matrix semantics of $\mathcal{S}$.*

The following proposition is well known.

PROPOSITION 1.2. *Let $\mathcal{A} = \langle \boldsymbol{A}, F \rangle$ and $\mathcal{B} = \langle \boldsymbol{B}, G \rangle$ be matrices and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ a surjective homomorphism such that $F = h^{-1}G$. Then $\models_{\mathcal{A}}$ and $\models_{\mathcal{B}}$ coincide.*

Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$. The *Leibniz congruence* of F on $\boldsymbol{A}$, denoted $\Omega_{\boldsymbol{A}} F$, is the relation

$$\begin{aligned}&\{(a,b) \in A^2 :\ \varphi^{\boldsymbol{A}}(a, c_0, \ldots, c_{k-1}) \in F \text{ iff } \varphi^{\boldsymbol{A}}(b, c_0, \ldots, c_{k-1}) \in F,\\ &\qquad \text{for all } \varphi(p, q_0, \ldots, q_{k-1}) \in \mathrm{Fm}_{\mathcal{L}},\ k < \omega, \text{ and all } c_0, \ldots, c_{k-1} \in A\}.\end{aligned}$$

It is the largest congruence of $\boldsymbol{A}$ *compatible* with F, i.e., the largest congruence Θ of $\boldsymbol{A}$ such that for all $a \in A$ we have either $a/\Theta \subseteq F$ or $(a/\Theta) \cap F = \emptyset$.

A matrix $\langle \boldsymbol{A}, F\rangle$ is said to be *reduced* if $\Omega_{\boldsymbol{A}} F = \Delta_{\boldsymbol{A}}$, the identity relation on A. For a matrix $\mathcal{A} = \langle \boldsymbol{A}, F\rangle$, let $\mathcal{A}^*$ denote the matrix $\langle \boldsymbol{A}/\Omega_{\boldsymbol{A}} F, F/\Omega_{\boldsymbol{A}} F\rangle$.▌ $\mathcal{A}^*$ is a reduced matrix and it follows from Proposition 1.2 that $\models_{\mathcal{A}}$ and $\models_{\mathcal{A}^*}$ coincide.

COROLLARY 1.3. *Let $\mathcal{S}$ be a deductive system. The class of all reduced $\mathcal{S}$-matrices is a matrix semantics of $\mathcal{S}$.*

1.2. Equational consequence

Let $\mathcal{L}$ be a propositional language and $Eq_{\mathcal{L}} = \{\varphi \approx \psi : \varphi, \psi \in \mathrm{Fm}_{\mathcal{L}}\}$, the set of $\mathcal{L}$-equations. Let K be a class of algebras of type $\mathcal{L}$. The *equational consequence relation* determined by K is the relation between sets of equations and equations defined by

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi$$

if for every $\boldsymbol{A} \in \mathsf{K}$ and every homomorphism $h\colon \mathbf{Fm}_{\mathcal{L}} \to \boldsymbol{A}$

$$h(\varphi_i) = h(\psi_i) \text{ for every } i \in I \text{ implies } h(\varphi) = h(\psi).$$

When I is finite, say, $I = \{0, \dots, n-1\}$, we have $\{\varphi_i \approx \psi_i : i < n\} \models_{\mathsf{K}} \varphi \approx \psi$ if and only if K satisfies the quasi-identity

$$\forall \overline{p}((\varphi_0 \approx \psi_0 \wedge \cdots \wedge \varphi_{n-1} \approx \psi_{n-1}) \to \varphi \approx \psi).$$

Here and elsewhere $\overline{p}$ stands for a list of the variables occurring in $\{\varphi_i \approx \psi_i : i < n\} \cup \{\varphi \approx \psi\}$.

If Σ, Σ' are sets of equations, then we write often $\Sigma \models_{\mathsf{K}} \Sigma'$ for the conjunction of the statements $\Sigma \models_{\mathsf{K}} \varphi \approx \psi$, $\varphi \approx \psi \in \Sigma'$, and $\Sigma \mathrel{=\!\mid\!\models_{\mathsf{K}}} \Sigma'$ for $\Sigma \models_{\mathsf{K}} \Sigma'$, $\Sigma' \models_{\mathsf{K}} \Sigma$.

The relation $\models_{\mathsf{K}}$ is *finitary* if $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi$ implies $\{\varphi_i \approx \psi_i : i \in J\} \models_{\mathsf{K}} \varphi \approx \psi$ for some finite $J \subseteq I$. In this paper we will be concerned only with relations $\models_{\mathsf{K}}$ which are finitary. It is well-known (see for instance [3, p.13]) that $\models_{\mathsf{K}}$ is finitary if and only if it coincides with $\models_{\mathsf{Q}(\mathsf{K})}$, where $\mathsf{Q}(\mathsf{K})$ is the quasivariety generated by K.

If K is a quasivariety axiomatized by a set Φ of identities and quasi-identities, then $\models_{\mathsf{K}}$ can also be viewed as a consequence relation over the set of $\mathcal{L}$-equations defined by axioms and inference rules as follows. For the axioms use

$$p \approx p, \tag{i}$$

together with

$$\varphi \approx \psi, \text{ for every identity } \forall \overline{p}(\varphi \approx \psi) \text{ in } \Phi, \tag{ii}$$

and for the rules take

$$\frac{p \approx q}{q \approx p}, \qquad \frac{p \approx q, \quad q \approx r}{p \approx r}, \tag{iii}$$

$$\frac{\{p_i \approx q_i : i < m\}}{f(p_0, \ldots, p_{m-1}) \approx f(q_0, \ldots, q_{m-1})}, \text{ for every } f \in \mathcal{L}, \text{ with } m \text{ the arity of } f, \tag{iv}$$

and finally

$$\frac{\{\varphi_i \approx \psi_i : i < n\}}{\varphi \approx \psi}, \text{ for every quasi-identity} \tag{v}$$
$$\forall \overline{p}((\varphi_0 \approx \psi_0 \wedge \cdots \wedge \varphi_{n-1} \approx \psi_{n-1}) \to \varphi \approx \psi) \text{ in } \Phi.$$

It is easy to verify that now

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi$$

if and only if there is a finite sequence of equations $\xi_0 \approx \eta_0, \ldots, \xi_{n-1} \approx \eta_{n-1}$ such that $\xi_{n-1} \approx \eta_{n-1}$ is $\varphi \approx \psi$, and for every $i < n$ one of the following conditions holds:

(i) $\xi_i \approx \eta_i$ is $\varphi_j \approx \psi_j$ for some $j \in I$,

(ii) there is an axiom $\gamma \approx \delta$ and a substitution σ such that $\xi_i \approx \eta_i$ is $\sigma\gamma \approx \sigma\delta$,

(iii) there is a rule $\dfrac{\{\gamma_l \approx \delta_l : l < r\}}{\gamma \approx \delta}$ and a substitution σ such that $\xi_i \approx \eta_i$ is $\sigma\gamma \approx \sigma\delta$ and $\{\sigma\gamma_l \approx \sigma\delta_l : l < r\} \subseteq \{\xi_j \approx \eta_j : j < i\}$.

The consequence relation $\models_{\mathsf{K}}$ over the set of $\mathcal{L}$-equations associated with a quasivariety K is an example of what was called a 2-*deductive system* in [6]; all notions applicable to deductive systems tranfer naturally to 2-deductive systems, and in particular to $\models_{\mathsf{K}}$. Thus a set of equations Σ is called a $\models_{\mathsf{K}}$-*theory*, or K-*theory* for short, if $\Sigma \models_{\mathsf{K}} \varphi \approx \psi$ implies $\varphi \approx \psi \in \Sigma$. If Σ is a set of equations, $\langle \Sigma \rangle_{\mathsf{K}} = \{\varphi \approx \psi : \Sigma \models_{\mathsf{K}} \varphi \approx \psi\}$ is the smallest K-theory containing Σ. The set of K-theories is denoted by $\mathrm{Th}\,\mathsf{K}$. If K is a quasivariety then a set of equations Σ is a K-theory if and only if $\Theta =$

$\{(\varphi, \psi) : \varphi \approx \psi \in \Sigma\}$ is a K-*congruence* of $\mathrm{Fm}_{\mathcal{L}}$, i.e., Θ is a congruence of $\mathbf{Fm}_{\mathcal{L}}$ and $\mathbf{Fm}_{\mathcal{L}}/\Theta \in \mathsf{K}$.

Conversely, every 2-deductive system that is axiomatized in the above way—i.e., that contains the axiom (i) and rules (iii) and (iv), possibly with additional axioms and rules such as (ii) and (v), is of the form $\models_{\mathsf{K}}$ for some quasivariety K. The 2-deductive systems of the form $\models_{\mathsf{K}}$ are called *algebraic* deductive systems. Observe that if K, K' are quasivarieties, and $\mathsf{K}' \not\subseteq \mathsf{K}$, then there is a quasi-identity

$$\forall \overline{p}((\varphi_0 \approx \psi_0 \wedge \cdots \wedge \varphi_{n-1} \approx \psi_{n-1}) \rightarrow \varphi \approx \psi)$$

which holds in K but fails in K'. It follows that $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{K}} \varphi \approx \psi$, but

$$\{\varphi_i \approx \psi_i : i \in I\} \not\models_{\mathsf{K}'} \varphi \approx \psi,$$

and hence $\models_{\mathsf{K}} \not\subseteq \models_{\mathsf{K}'}$.

COROLLARY 1.4. *The assignment* $\mathsf{K} \mapsto \models_{\mathsf{K}}$ *establishes an order reversing bijection between quasivarieties and algebraic 2-deductive systems.*

We refer to [5] for a more thorough discussion.

2. Algebraic semantics

The notion of algebraic semantics we will use was introduced in its general form in [3].

DEFINITION 2.1. *Let* $\mathcal{S}$ *be a deductive system and* K *a class of algebras.* K *is called an* **algebraic semantics** *of* $\mathcal{S}$ *if there exists a finite set* $\{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ *of equations in a single variable such that for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \textit{ iff } \{\delta_i(\gamma) \approx \varepsilon_i(\gamma) : \gamma \in \Gamma, i < n\} \models_{\mathsf{K}} \delta_j(\varphi) \approx \varepsilon_j(\varphi), j < n. \qquad \text{(vi)}$$

The equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$, *are called the* **defining equations** *for the algebraic semantics.*

A finite set τ of single variable equations will be called a *translation*. If $\tau = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ is a translation and $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\tau(\Gamma)$ will denote the set $\{\delta_i(\gamma) \approx \varepsilon_i(\gamma) : \gamma \in \Gamma, i < n\}$. Condition (vi) may then be abbreviated as

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{if and only if} \quad \tau(\Gamma) \models_{\mathsf{K}} \tau(\varphi).$$

PROPOSITION 2.2. [3, Cor 2.3] If K is an algebraic semantics of a deductive system $\mathcal{S}$, then so is the quasivariety $\mathsf{Q}(\mathsf{K})$, with the same defining equations.

If $\tau = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ is a translation and $\boldsymbol{A}$ is an algebra, $F^{\tau}_{\boldsymbol{A}}$ will denote the set $\{a \in A : \delta_i^{\boldsymbol{A}}(a) = \varepsilon_i^{\boldsymbol{A}}(a),\, i < n\}$.

THEOREM 2.3. [3, Thm 2.4] *Let $\mathcal{S}$ be a deductive system, K a class of $\mathcal{L}$-algebras and τ a translation. The following are equivalent:*

(i) K *is an algebraic semantics of $\mathcal{S}$ with defining equations τ.*

(ii) *The class $\mathsf{M} = \{\langle \boldsymbol{A}, F^{\tau}_{\boldsymbol{A}}\rangle : \boldsymbol{A} \in \mathsf{K}\}$ is a matrix semantics of $\mathcal{S}$.*

Thus the deductive systems that have an algebraic semantics are precisely those deductive systems for which there is a matrix semantics M and a finite set τ of single variable equations such that for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ we have $F = F^{\tau}_{\boldsymbol{A}}$.

Let $\mathcal{L} = \{\to, \wedge, \vee, \neg, \bot, \top\}$, let $\mathbf{CPC} = \langle \mathcal{L}, \vdash_{\mathbf{CPC}}\rangle$ denote the classical propositional calculus and let BA denote the variety of Boolean algebras. As $\{\langle \boldsymbol{A}, \{\top^{\boldsymbol{A}}\}\rangle : \boldsymbol{A} \in \mathsf{BA}\}$ is a matrix semantics of $\mathbf{CPC}$, we have that BA is an algebraic semantics of $\mathbf{CPC}$ with defining equation $p \approx \top$. Similarly, the variety HA of Heyting algebras is an algebraic semantics for the intuitionistic propositional calculus $\mathbf{IPC} = \langle \mathcal{L}, \vdash_{\mathbf{IPC}}\rangle$, with the same defining equation.

As pointed out in [3, Cor 2.5], if a deductive system $\mathcal{S}$ over a language $\mathcal{L}$ has an algebraic semantics K, then for any language $\mathcal{L}' \subseteq \mathcal{L}$ the $\mathcal{L}'$-fragment of $\mathcal{S}$ has an algebraic semantics as well, provided $\mathcal{L}'$ contain all the primitive connectives occurring in the defining equations. The class of all $\mathcal{L}'$-reducts of members of K is then an algebraic semantics for the fragment. In particular, any fragment of $\mathbf{CPC}$ or $\mathbf{IPC}$ which contains the symbol $\top$ (or in which $\top$ is definable) has an algebraic semantics.

BA and HA are in fact *equivalent* algebraic semantics of $\mathbf{CPC}$ and $\mathbf{IPC}$ respectively. The notions of an equivalent algebraic semantics and an algebraizable deductive system were also introduced and studied in [3].

DEFINITION 2.4. *Let $\mathcal{S}$ be a deductive system and K an algebraic semantics of $\mathcal{S}$ with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. K is an* **equivalent algebraic semantics** *of $\mathcal{S}$ if there exists a finite set $\{\Delta_j(p,q) : j < m\}$ of formulas in two variables such that for every $\varphi \approx \psi \in Eq_{\mathcal{L}}$,*

$$\varphi \approx \psi \dashv\models_{\mathsf{K}} \{\delta_i(\Delta_j(\varphi,\psi)) \approx \varepsilon_i(\Delta_j(\varphi,\psi)) : i < n, j < m\}. \qquad \text{(vii)}$$

The set $\{\Delta_j(p,q) : j < m\}$ is called an **equivalence system**. *A deductive system is* **algebraizable** *if it has an equivalent algebraic semantics.*

Both **CPC** and **IPC** are algebraizable, with equivalent algebraic semantics the classes of Boolean algebras and Heyting algebras, respectively, both with defining equation $\{p \approx \top\}$ and equivalence system $\{p \to q, q \to p\}$.

It turns out that if K is an equivalent algebraic semantics of $\mathcal{S}$ then the dual conditions of (vi) and (vii) hold too: for all $\Sigma \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$,

$$\Sigma \models_{\mathsf{K}} \varphi \approx \psi \text{ iff } \{\Delta_j(\xi, \eta) : \xi \approx \eta \in \Sigma, j < m\} \vdash_{\mathcal{S}} \Delta_j(\varphi, \psi),\ j < m, \tag{viii}$$

and, for every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$,

$$\varphi \dashv\vdash_{\mathcal{S}} \{\Delta_j(\delta_i(\varphi), \varepsilon_i(\varphi)) : j < m, i < n\}. \tag{ix}$$

A result similar to that of Proposition 2.2 holds for equivalent algebraic semantics: if K is an equivalent algebraic semantics of a deductive system $\mathcal{S}$, then so is $\mathsf{Q}(\mathsf{K})$; $\mathsf{Q}(\mathsf{K})$ is then called an *equivalent quasivariety semantics* of $\mathcal{S}$.

2.1. Largest algebraic semantics

It is shown in [3, Thm 2.15] that an equivalent quasivariety semantics of a deductive system, if it exists, is unique. In general however there may be many quasivarieties that serve as an algebraic semantics for the same deductive system, using the same defining equations. This was shown in [3, Section 2.1]; we give here a particularly natural example of this phenomenon, the so-called Glivenko interpretation of the classical propositional calculus in the intuitionistic propositional calculus.

LEMMA 2.5. *Let $\mathcal{L}$ be the language of* **CPC** *and* **IPC** *as above, and let $\{\varphi_i \approx \psi_i : i \in I\} \cup \{\varphi \approx \psi\} \subseteq Eq_{\mathcal{L}}$. Then*

$$\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{BA}} \varphi \approx \psi \textit{ iff } \{\neg\neg\varphi_i \approx \neg\neg\psi_i : i \in I\} \models_{\mathsf{HA}} \neg\neg\varphi \approx \neg\neg\psi.$$

PROOF. The implication from right to left follows from the facts that $\mathsf{BA} \subseteq \mathsf{HA}$ and $p \approx q \mathrel{=\mkern-3mu|\mkern-3mu\models}_{\mathsf{BA}} \neg\neg p \approx \neg\neg q$.

For the converse, assume $\{\varphi_i \approx \psi_i : i \in I\} \models_{\mathsf{BA}} \varphi \approx \psi$. Let $\boldsymbol{A} \in \mathsf{HA}$ and $h\colon \mathbf{Fm}_{\mathcal{L}} \to \boldsymbol{A}$ be a homomorphism such that $h(\neg\neg\varphi_i) = h(\neg\neg\psi_i)$, for every $i \in I$. It is well-known that the set $\{\neg a : a \in A\}$ is the universe of a Boolean algebra $\boldsymbol{B}$ such that the map $f : \boldsymbol{A} \to \boldsymbol{B}$ given by $f(a) = \neg\neg a$ is a homomorphism (see for instance [1].) Then $f \circ h$ is a homomorphism from $\mathbf{Fm}_{\mathcal{L}}$ to $\boldsymbol{B}$ and $fh(\varphi_i) = fh(\psi_i)$ for every $i \in I$. Therefore $fh(\varphi) = fh(\psi)$, i.e., $h(\neg\neg\varphi) = h(\neg\neg\psi)$. ■

PROPOSITION 2.6. *The varieties* BA *and* HA *are both algebraic semantics of* **CPC** *with defining equation* $\neg\neg p \approx \top$.

PROOF. Since $p \approx \top \mathrel{=\!\mid\!\models_{\mathsf{BA}}} \neg\neg p \approx \top$, the variety BA is an algebraic semantics for **CPC** with defining equation $\neg\neg p \approx \top$. The same holds for the variety HA, since by the above lemma we have

$$\begin{aligned} \Gamma \vdash_{\mathbf{CPC}} \varphi \quad &\text{iff} \quad \{\gamma \approx \top : \gamma \in \Gamma\} \models_{\mathsf{BA}} \varphi \approx \top \\ &\text{iff} \quad \{\neg\neg\gamma \approx \neg\neg\top : \gamma \in \Gamma\} \models_{\mathsf{HA}} \neg\neg\varphi \approx \neg\neg\top \\ &\text{iff} \quad \{\neg\neg\gamma \approx \top : \gamma \in \Gamma\} \models_{\mathsf{HA}} \neg\neg\varphi \approx \top. \end{aligned}$$

■

In the example above we saw that in addition to the equivalent algebraic semantics BA for **CPC** with defining equation $\neg\neg p \approx \top$, there is the larger algebraic semantics HA. Many more quasivarieties serve as an algebraic semantics for **CPC**, but as we will see now, they are all contained in a largest algebraic semantics; furthermore, this is a general phenomenon.

DEFINITION 2.7. *Let* $\mathcal{S}$ *be a deductive system,* $\tau = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ *a translation. An algebra* $\boldsymbol{A}$ *is a* **τ-model** *of* $\mathcal{S}$ *if for all* $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \text{ implies } \tau(\Gamma) \models_{\boldsymbol{A}} \tau(\varphi).$$

The class of all τ*-models of* $\mathcal{S}$ *is denoted* $\mathsf{K}(\mathcal{S}, \tau)$.

The next proposition is immediate:

PROPOSITION 2.8 ([3, p.16]). *If* $\mathcal{S}$ *has an algebraic semantics with defining equations* τ*, then* $\mathsf{K}(\mathcal{S}, \tau)$ *is the largest such semantics.*

It is easy to see that $\mathsf{K}(\mathcal{S}, \tau)$ is a quasivariety. We now provide an axiomatization of the algebraic 2-deductive system $\models_{\mathsf{K}(\mathcal{S},\tau)}$ and the quasivariety $\mathsf{K}(\mathcal{S}, \tau)$ in terms of an axiomatization of $\mathcal{S}$ and the translation τ.

PROPOSITION 2.9. *Let* $\mathcal{S}$ *be a deductive system and* $\tau = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ *a translation. Suppose* $\mathcal{S}$ *is axiomatized by a set Ax of axioms and a set IR of inference rules. The algebraic 2-deductive system* $\models_{\mathsf{K}(\mathcal{S},\tau)}$ *is axiomatized by the axiom (i) and the rules (iii) and (iv), together with the axioms* $\tau(\varphi)$*, or, more explicitly,*

$$\delta_i(\varphi) \approx \varepsilon_i(\varphi),\ i < n, \tag{x}$$

for every axiom $\varphi \in Ax$*, and the rules*

$$\frac{\bigcup\{\tau(\varphi_i) : i < m\}}{\tau(\psi)},$$

or, more explicitly, the rules

$$\frac{\{\delta_i(\varphi_j) \approx \varepsilon_i(\varphi_j) \ : \ i < n, \ j < m\}}{\delta_i(\psi) \approx \varepsilon_i(\psi)}, \quad i < n, \tag{xi}$$

for every rule $\langle\{\varphi_j : j < m\}, \psi\rangle$ *in IR.*

Proof. Let $\models$ be the 2-deductive system axiomatized by the axioms and rules as stated. Since $\models$ is algebraic (see the end of section 1.2), the relation $\models$ coincides with $\models_{\mathsf{Q}}$ for some quasivariety Q of algebras. We will show $\mathsf{K}(\mathcal{S}, \tau) = \mathsf{Q}$.

Clearly $\models_{\mathsf{Q}} \subseteq \models_{\mathsf{K}(\mathcal{S},\tau)}$, so $\mathsf{K}(\mathcal{S}, \tau) \subseteq \mathsf{Q}$. To prove the converse, suppose $\boldsymbol{A} \in \mathsf{Q}$, and let $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathcal{S}} \varphi$. We need to verify that $\tau(\Gamma) \models_{\boldsymbol{A}} \tau(\varphi)$. This is done by induction on the length of a proof of φ from Γ in the deductive system $\mathcal{S}$. ■

Corollary 2.10. *Let* $\mathcal{S}$ *be a deductive system and* $\tau = \{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ *a translation. Suppose* $\mathcal{S}$ *is axiomatized by a set Ax of axioms and a set IR of inference rules. The quasivariety* $\mathsf{K}(\mathcal{S}, \tau)$ *is axiomatized by the axioms*

$$\delta_i(\varphi) \approx \varepsilon_i(\varphi), \ i < n,$$

for every axiom $\varphi \in Ax$*, and the quasiidentities*

$$\Big(\bigwedge_{i<n} \bigwedge_{j<m} \delta_i(\varphi_j) \approx \varepsilon_i(\varphi_j)\Big) \to \delta_i(\psi) \approx \varepsilon_i(\psi), \ i < n,$$

for every inference rule $\langle\{\varphi_j : j < m\}, \psi\rangle \in IR$.

2.2. Extensions of deductive systems with an algebraic semantics

Our next aim is to show that if $\mathcal{S}$ is a deductive system with an algebraic semantics, then any extension $\mathcal{S}'$ of $\mathcal{S}$ has an algebraic semantics as well, with the same defining equations. We will do this in Theorem 2.15, by showing that a suitable subclass of $\mathsf{K}(\mathcal{S}, \tau)$ is an algebraic semantics of $\mathcal{S}'$. We need some preliminary results.

DEFINITION 2.11. *Given a deductive system $\mathcal{S}$, a class of algebras K and a translation τ, let*

$$\tau_{\mathcal{S},\mathsf{K}} : \mathrm{Th}\,\mathcal{S} \to \mathrm{Th}\,\mathsf{K}$$

be the map defined by

$$\tau_{\mathcal{S},\mathsf{K}}(T) = \langle \tau(T) \rangle_{\mathsf{K}},$$

for $T \in \mathrm{Th}\,\mathcal{S}$.

LEMMA 2.12. *Let $\mathcal{S}$ be a deductive system, τ a translation, and $\mathsf{K} \subseteq \mathsf{K}(\mathcal{S},\tau)$ a class of τ-models of $\mathcal{S}$. Then for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$,*

$$\tau_{\mathcal{S},\mathsf{K}}(\langle \Gamma \rangle_{\mathcal{S}}) = \langle \tau(\Gamma) \rangle_{\mathsf{K}}.$$

PROOF. Clearly $\tau(\Gamma) \subseteq \tau(\langle \Gamma \rangle_{\mathcal{S}})$ and so $\langle \tau(\Gamma) \rangle_{\mathsf{K}} \subseteq \langle \tau(\langle \Gamma \rangle_{\mathcal{S}}) \rangle_{\mathsf{K}} = \tau_{\mathcal{S},\mathsf{K}}(\langle \Gamma \rangle_{\mathcal{S}})$. Conversely, if $\varphi \approx \psi \in \tau_{\mathcal{S},\mathsf{K}}(\langle \Gamma \rangle_{\mathcal{S}})$ then $\tau(\langle \Gamma \rangle_{\mathcal{S}}) \models_{\mathsf{K}} \varphi \approx \psi$, i.e., $\{\tau(\xi) : \Gamma \vdash_{\mathcal{S}} \xi\} \models_{\mathsf{K}} \varphi \approx \psi$. Since K consists of τ-models of $\mathcal{S}$, for every formula ξ such that $\Gamma \vdash_{\mathcal{S}} \xi$ we have $\tau(\Gamma) \models_{\mathsf{K}} \tau(\xi)$. Therefore $\tau(\Gamma) \models_{\mathsf{K}} \varphi \approx \psi$ and thus $\varphi \approx \psi \in \langle \tau(\Gamma) \rangle_{\mathsf{K}}$. ■

THEOREM 2.13. *Let $\mathcal{S}$ be a deductive system, τ a translation, and $\mathsf{K} \subseteq \mathsf{K}(\mathcal{S},\tau)$ a class of τ-models of $\mathcal{S}$. The following are equivalent:*

- (i) K *is an algebraic semantics of $\mathcal{S}$ with defining equations τ.*
- (ii) $\tau_{\mathcal{S},\mathsf{K}}$ *is injective.*

PROOF. (i)⇒(ii): Let T and T' be $\mathcal{S}$-theories such that $\tau_{\mathcal{S},\mathsf{K}}(T) = \tau_{\mathcal{S},\mathsf{K}}(T')$. Let $\varphi \in T$. Then $\tau(\varphi) \subseteq \tau_{\mathcal{S},\mathsf{K}}(T) = \tau_{\mathcal{S},\mathsf{K}}(T')$ and so $\tau(T') \models_{\mathsf{K}} \tau(\varphi)$. By (i) we get $T' \vdash_{\mathcal{S}} \varphi$ and so $\varphi \in T'$. Thus $T \subseteq T'$ and by symmetry $T' \subseteq T$.

(ii)⇒(i): We need to show that, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, $\tau(\Gamma) \models_{\mathsf{K}} \tau(\varphi)$ implies $\Gamma \vdash_{\mathcal{S}} \varphi$. If $\tau(\Gamma) \models_{\mathsf{K}} \tau(\varphi)$ then $\langle \tau(\Gamma) \rangle_{\mathsf{K}} = \langle \tau(\Gamma) \cup \tau(\varphi) \rangle_{\mathsf{K}}$. By Lemma 2.12 we have $\tau_{\mathcal{S},\mathsf{K}}(\langle \Gamma \rangle_{\mathcal{S}}) = \tau_{\mathcal{S},\mathsf{K}}(\langle \Gamma \cup \{\varphi\} \rangle_{\mathcal{S}})$. Therefore $\langle \Gamma \rangle_{\mathcal{S}} = \langle \Gamma \cup \{\varphi\} \rangle_{\mathcal{S}}$ and so $\Gamma \vdash_{\mathcal{S}} \varphi$. ■

THEOREM 2.14. *Let $\mathcal{S}$ be a deductive system, τ a translation, $\mathsf{K} = \mathsf{K}(\mathcal{S},\tau)$, and suppose K is an algebraic semantics for $\mathcal{S}$ with defining equations τ. Let $\mathcal{S}'$ be an extension of $\mathcal{S}$, and $\mathsf{K}' = \mathsf{K}(\mathcal{S}',\tau)$. Then $\tau_{\mathcal{S}',\mathsf{K}'}$ equals $\tau_{\mathcal{S},\mathsf{K}}$ restricted to $\mathrm{Th}\,\mathcal{S}'$.*

PROOF. Let $T \in \mathrm{Th}\,\mathcal{S}'$; we will show that

$$\langle \tau(T) \rangle_{\mathsf{K}} = \langle \tau(T) \rangle_{\mathsf{K}'}.$$

As $\mathcal{S}'$ is an extension of $\mathcal{S}$, $\mathsf{K}' \subseteq \mathsf{K}$, so $\models_{\mathsf{K}'}$ is an extension of $\models_{\mathsf{K}}$; hence $\langle\tau(T)\rangle_{\mathsf{K}} \subseteq \langle\tau(T)\rangle_{\mathsf{K}'}$. For the converse inclusion, first observe that by Proposition 2.9 $\models_{\mathsf{K}'}$ can be axiomatized by the axiom (i) and the rules (iii) and (iv), together with the axioms $\tau(\varphi)$, for every theorem φ of $\mathcal{S}'$, and the rules

$$\frac{\bigcup\{\tau(\varphi_i) : i < n\}}{\tau(\psi)}$$

for every inference $\{\varphi_i : i < n\} \vdash_{\mathcal{S}'} \psi$.

We claim that $\langle\tau(T)\rangle_{\mathsf{K}}$ contains all substitution instances of the axioms of $\models_{\mathsf{K}'}$ and is closed under the rules of $\models_{\mathsf{K}'}$. This clearly applies to the axiom (i) and the rules (iii) and (iv). Next assume φ is a theorem of $\mathcal{S}'$ and σ a substitution. Then since $\emptyset \vdash_{\mathcal{S}'} \sigma\varphi$ and T is an $\mathcal{S}'$-theory we have $\sigma\varphi \in T$ and so $\sigma\tau(\varphi) = \tau(\sigma\varphi) \subseteq \tau(T) \subseteq \langle\tau(T)\rangle_{\mathsf{K}}$. Thus $\langle\tau(T)\rangle_{\mathsf{K}}$ contains all substitution instances of the axioms of $\models_{\mathsf{K}'}$. Now let $\{\varphi_i : i < n\} \vdash_{\mathcal{S}'} \psi$ be an inference of $\mathcal{S}'$, and σ a substitution such that $\{\sigma\tau(\varphi_i) : i < n\} \subseteq \langle\tau(T)\rangle_{\mathsf{K}}$. Then $\tau(T) \models_{\mathsf{K}} \sigma\tau(\varphi_i)$, $i < n$, and since $\sigma\tau(\varphi_i) = \tau(\sigma\varphi_i)$, we conclude $\tau(T) \models_{\mathsf{K}} \tau(\sigma\varphi_i)$, $i < n$. Since K is an algebraic semantics of $\mathcal{S}$, it follows that $T \vdash_{\mathcal{S}} \sigma\varphi_i$, $i < n$, and hence $\{\sigma\varphi_i : i < n\} \subseteq T$. Since $\{\sigma\varphi_i : i < n\} \vdash_{\mathcal{S}'} \sigma\psi$ and T is an $\mathcal{S}'$-theory we have $\sigma\psi \in T$. Then $\sigma\tau(\psi) = \tau(\sigma\psi) \subseteq \tau(T) \subseteq \langle\tau(T)\rangle_{\mathsf{K}}$.

Thus $\langle\tau(T)\rangle_{\mathsf{K}}$ does indeed contain all substitution instances of the axioms of $\models_{\mathsf{K}'}$ and is closed under the rules of $\models_{\mathsf{K}'}$. This immediately gives that $\langle\tau(T)\rangle_{\mathsf{K}}$ is a K'-theory and obviously $\langle\tau(T)\rangle_{\mathsf{K}}$ contains $\tau(T)$. Thus $\langle\tau(T)\rangle_{\mathsf{K}'} \subseteq \langle\tau(T)\rangle_{\mathsf{K}}$. ■

We are now ready to show that the property of possessing an algebraic semantics is preserved on passing from a deductive system to any extension. This corrects a claim to the contrary, made in [3, p. 17].

THEOREM 2.15. *If a deductive system $\mathcal{S}$ has an algebraic semantics, then so does any extension of $\mathcal{S}$, with the same defining equations.*

PROOF. Assume the deductive system $\mathcal{S}$ has an algebraic semantics, with defining equations τ, and let $\mathcal{S}'$ be an extension of $\mathcal{S}$, with $\mathsf{K}' = \mathsf{K}(\mathcal{S}', \tau)$. By Proposition 2.8 the class $\mathsf{K} = \mathsf{K}(\mathcal{S}, \tau)$ is an algebraic semantics of $\mathcal{S}$ with defining equations τ. By Theorem 2.13 the map $\tau_{\mathcal{S},\mathsf{K}}$ is one-to-one. By Theorem 2.14 so is the map $\tau_{\mathcal{S}',\mathsf{K}'}$. Applying Theorem 2.13 again we see that $\mathcal{S}'$ has an algebraic semantics. ■

A special case of this theorem appeared in [8], where it was shown that any *axiomatic* extension of a deductive system possessing an algebraic semantics has itself an algebraic semantics.

2.3. Deductive systems without an algebraic semantics

We now give some examples of deductive systems that do not possess an algebraic semantics. First we establish a property shared by all deductive systems that have an algebraic semantics.

In [3, Thm 2.7] it is shown that any deductive system $\mathcal{S}$ that has an algebraic semantics with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$, must satisfy

$$\{p, \delta_i(p)\} \vdash_{\mathcal{S}} \varepsilon_i(p) \text{ and } \{p, \varepsilon_i(p)\} \vdash_{\mathcal{S}} \delta_i(p) \text{ for } i < n. \tag{xii}$$

We need a stronger version of this result.

Theorem 2.16. *Let $\mathcal{S}$ be a deductive system which has an algebraic semantics with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. Then*

$$\{p, \gamma(\delta_i(p), \psi_0, \ldots, \psi_{k-1})\} \vdash_{\mathcal{S}} \gamma(\varepsilon_i(p), \psi_0, \ldots, \psi_{k-1}),\ i < n, \tag{xiii}$$

and

$$\{p, \gamma(\varepsilon_i(p), \psi_0, \ldots, \psi_{k-1})\} \vdash_{\mathcal{S}} \gamma(\delta_i(p), \psi_0, \ldots, \psi_{k-1}),\ i < n, \tag{xiv}$$

for every formula $\gamma(p, q_0, \ldots, q_{k-1})$, $k < \omega$, and formulas $\psi_0, \ldots, \psi_{k-1}$.

Proof. Let K be an algebraic semantics of $\mathcal{S}$ with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. Then (xiii) follows immediately from the fact that, for $i < n$,

$$\{\delta_j(p) \approx \varepsilon_j(p) : j < n\} \cup \{\delta_j(\gamma(\delta_i(p), \overline{\psi})) \approx \varepsilon_j(\gamma(\delta_i(p), \overline{\psi})) : j < n\} \models_{\mathsf{K}} \delta_j(\gamma(\varepsilon_i(p), \overline{\psi})) \approx \varepsilon_j(\gamma(\varepsilon_i(p), \overline{\psi})),\ j < n,$$

where $\overline{\psi}$ stands for $\psi_0, \ldots, \psi_{k-1}$. Likewise we obtain (xiv). ■

We will see at the end of Section 2.4 that conditions (xiii) and (xiv) do not guarantee the existence of an algebraic semantics. The following proposition sheds some light on the significance of the conditions in terms of the Leibniz relation.

Proposition 2.17. *A deductive system $\mathcal{S}$ satisfies the conditions (xiii) and (xiv) above with respect to a set of equations $\{\delta_i(p) \approx \varepsilon_i(p) : i < n\}$ iff $(\delta_i(p), \varepsilon_i(p)) \in \Omega_{\mathbf{Fm}_{\mathcal{L}}} T$, $i < n$, for every $\mathcal{S}$-theory T such that $p \in T$.*

PROOF. $\Rightarrow$: Let T be an $\mathcal{S}$-theory such that $p \in T$. Let $\gamma(p, q_0, \dots, q_{k-1}) \in \mathrm{Fm}_{\mathcal{L}}$ and $\psi_0, \dots, \psi_{k-1} \in \mathrm{Fm}_{\mathcal{L}}$. By (xiii) and (xiv) we get $\gamma(\delta_i(p), \overline{\psi}) \in T$ iff $\gamma(\varepsilon_i(p), \overline{\psi}) \in T$, for every $i < n$. Thus $(\delta_i(p), \varepsilon_i(p)) \in \Omega_{\mathbf{Fm}_{\mathcal{L}}} T$, $i < n$, by the definition of $\Omega_{\mathbf{Fm}_{\mathcal{L}}}$.
$\Leftarrow$: Let $\gamma(p, q_0, \dots, q_{k-1}), \psi_0, \dots, \psi_{k-1} \in \mathrm{Fm}_{\mathcal{L}}$. For $i < n$, let T_i be the $\mathcal{S}$-theory generated by p and $\gamma(\delta_i(p), \overline{\psi})$. Since $(\delta_i(p), \varepsilon_i(p)) \in \Omega_{\mathbf{Fm}_{\mathcal{L}}} T_i$ and $\gamma(\delta_i(p), \overline{\psi}) \in T_i$ we have $\gamma(\varepsilon_i(p), \overline{\psi}) \in T_i$. Thus $\{p, \gamma(\delta_i(p), \overline{\psi})\} \vdash_{\mathcal{S}} \gamma(\varepsilon_i(p), \overline{\psi})$ for every $i < n$. This shows (xiii) and likewise we show (xiv). ■

Theorem 2.16 is a useful tool in producing examples of deductive systems that fail to have an algebraic semantics. Such an example was given in [2], using [3, Thm 2.7]. An even simpler example is the following.

Let $\mathcal{L} = \{\top\}$ and let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be the deductive system determined by the one axiom $\top$, and the empty set of inference rules. Then we have $\Gamma \vdash_{\mathcal{S}} \varphi$ iff $\varphi \in \Gamma \cup \{\top\}$, for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. Note that the $\mathcal{L}$-formulas are the variables and the constant $\top$. It is straightforward to see that the class of all $\mathcal{L}$-algebras is an algebraic semantics of $\mathcal{S}$ with defining equation $p \approx \top$.

We modify $\mathcal{S}$ by adding to the language a unary symbol. Thus let $\mathcal{L}' = \{f, \top\}$, where f is a unary connective, and let $\mathcal{S}' = \langle \mathcal{L}', \vdash_{\mathcal{S}'} \rangle$ be the deductive system determined again by the one axiom $\top$ and the empty set of inference rules. We have again $\Gamma \vdash_{\mathcal{S}'} \varphi$ iff $\varphi \in \Gamma \cup \{\top\}$, for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}'}$. Although the deductive system $\mathcal{S}'$ does satisfy (xii) with respect to the equation $p \approx \top$, i.e., we have $\{p\} \vdash_{\mathcal{S}'} \top$ and $\{p, \top\} \vdash_{\mathcal{S}'} p$, it does not possess an algebraic semantics. Indeed, suppose it did, say with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. Then by Theorem 2.16 we have $\{p, f\delta_i(p)\} \vdash_{\mathcal{S}'} f\varepsilon_i(p)$, $i < n$. Since the formula $f\varepsilon_i(p)$ is neither p nor $\top$ it must equal $f\delta_i(p)$. This implies that $\delta_i(p)$ is equal to $\varepsilon_i(p)$ for $i < n$, and then $\Gamma \vdash_{\mathcal{S}'} \varphi$ for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$; contradicting, for example, the fact that $\nvdash_{\mathcal{S}'} p$, for p a variable.

We now present an example of a more interesting deductive system that does not possess an algebraic semantics. A deductive system $\mathcal{S}$ is *protoalgebraic* if it possesses a system of *implication formulas*, i.e., if there are $\mathcal{L}$-formulas $\Delta_i(p, q)$, $i < n$, for some $n < \omega$, such that

$$\vdash_{\mathcal{S}} \Delta_i(p, p),$$

for all $i < n$, and that collectively have the detachment property:

$$\{p\} \cup \{\Delta_i(p, q) \, : \, i < n\} \vdash_{\mathcal{S}} q.$$

It is well-known that protoalgebraic deductive systems need not be algebraizable, but the class of reduced matrices of a protoalgebraic system is well-behaved, and shares many of the characteristic features of an equivalent algebraic semantics (see [3] and [4]). Our next example shows that protoalgebraicity does not however guarantee the existence of an algebraic semantics.

Let $\mathcal{L}$ consist of just one binary connective $\rightarrow$, and let $\mathcal{S}$ be the deductive system over $\mathcal{L}$ with the single axiom

$$p \rightarrow p,$$

and the single rule

$$\frac{p,\ p \rightarrow q}{q}.$$

The formula $p \rightarrow q$ constitutes then by itself a system of implication formulas, and hence $\mathcal{S}$ is protoalgebraic. We will now show it does not have an algebraic semantics.

LEMMA 2.18. *Let* p, q *be different variables and* φ, ψ *formulas. Then* $\{p, q \rightarrow \varphi\} \vdash_{\mathcal{S}} \psi$ *if and only if* $\psi \in \{p, q \rightarrow \varphi\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$.

PROOF. The implication from right to left is trivial. Now assume $\{p, q \rightarrow \varphi\} \vdash_{\mathcal{S}} \psi$. We apply induction on the length of a proof to show $\psi \in \{p, q \rightarrow \varphi\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$.

If ψ is either one of the hypotheses or a substitution instance of the axiom then $\psi \in \{p, q \rightarrow \varphi\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$.

Next assume ψ is obtained by the application of the rule, say from ξ and $\xi \rightarrow \psi$. By induction hypothesis we have $\xi, \xi \rightarrow \psi \in \{p, q \rightarrow \varphi\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$. Obviously $\xi \rightarrow \psi \neq p$ and if $\xi \rightarrow \psi = q \rightarrow \varphi$ then $\xi = q$, contradicting the induction hypothesis. Thus $\xi = \psi$ and so $\psi \in \{p, q \rightarrow \varphi\} \cup \{\gamma \rightarrow \gamma : \gamma \in \mathrm{Fm}_{\mathcal{L}}\}$. ■

THEOREM 2.19. $\mathcal{S}$ *does not have an algebraic semantics.*

PROOF. We argue by contradiction. Let K be an algebraic semantics of $\mathcal{S}$ with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. Let $q \in \mathrm{Va}$, $q \neq p$. By Theorem 2.16 we have $\{p, q \rightarrow \delta_i(p)\} \vdash_{\mathcal{S}} q \rightarrow \varepsilon_i(p)$, $i < n$. Since $q \rightarrow \varepsilon_i(p) \neq p$ and $q \neq \varepsilon_i(p)$ (p is the only variable allowed in the defining equations), by Lemma 2.18 we get $q \rightarrow \delta_i(p) = q \rightarrow \varepsilon_i(p)$. Hence $\delta_i(p)$ is equal to $\varepsilon_i(p)$, $i < n$. It follows that $\Gamma \vdash_{\mathcal{S}} \varphi$ for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, a contradiction. ■

2.4. Algebraic semantics for mono-unary deductive systems

No satisfying characterization of the deductive systems that possess an algebraic semantics exists at present. The example given in [3] of a deductive system that fails to have an algebraic semantics is one over a language consisting of just one unary connective. We will now characterize the deductive systems over this mono-unary language that have an algebraic semantics.

THEOREM 2.20. *Let $\mathcal{L} = \{f\}$, f a unary operation, and let $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}} \rangle$ be a deductive system. Then $\mathcal{S}$ has an algebraic semantics if and only if $p \vdash_{\mathcal{S}} fp$. Furthermore, if $\mathcal{S}$ has an algebraic semantics then it has an algebraic semantics with defining equation $p \approx fp$.*

PROOF. $\Rightarrow$: Let K be an algebraic semantics of $\mathcal{S}$ with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. Since any unary term $\delta(p)$ is of the form $f^n(p)$ for some $n < \omega$, the defining equations $\delta_i(p) \approx \varepsilon_i(p)$ take the form $f^{m_i}p \approx f^{n_i}p$, for $m_i, n_i < \omega$. It also follows that for any unary term δ we have $\delta(fp) = f\delta(p)$; hence the equations $\delta_i(fp) \approx \varepsilon_i(fp)$ coincide with $f\delta_i(p) \approx f\varepsilon_i(p)$. Since K is an algebraic semantics for $\mathcal{S}$ with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$, we know $p \vdash_{\mathcal{S}} fp$ if and only if $\{\delta_i(p) \approx \varepsilon_i(p) : i < n\} \models_{\mathsf{K}} \delta_j(fp) \approx \varepsilon_j(fp)$, $j < n$. But it is now obvious that the right hand side of this equivalence holds, and hence that $p \vdash_{\mathcal{S}} fp$.

$\Leftarrow$: We know by Theorem 2.15 that any extension of a deductive system that has an algebraic semantics possesses an algebraic semantics as well. Therefore, in order to prove that the condition $p \vdash_{\mathcal{S}} fp$ guarantees the existence of an algebraic semantics with defining equation $\tau = \{p \approx fp\}$, it suffices to show that the deductive system $\mathcal{S}$ whose only rule is $\langle \{p\}, fp \rangle$ has an algebraic semantics with defining equation $\tau = \{p \approx fp\}$. Note that if $\boldsymbol{A}$ is an arbitrary $\mathcal{L}$-algebra, then $p \approx fp \models_{\boldsymbol{A}} fp \approx ffp$, that is, $\tau(p) \models_{\boldsymbol{A}} \tau(fp)$, so $\boldsymbol{A}$ is a τ-model of $\mathcal{S}$. Therefore in this case $\mathsf{K}(\mathcal{S}, \tau)$ is the class K of all $\mathcal{L}$-algebras.

We claim that K is an algebraic semantics of $\mathcal{S}$ with defining equation $p \approx fp$. To see this, it suffices to show that if $\Gamma \nvdash_{\mathcal{S}} \varphi$, then $\tau(\Gamma) \not\models_{\boldsymbol{A}} \tau(\varphi)$, for some mono-unary algebra $\boldsymbol{A}$. So suppose $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that $\Gamma \nvdash_{\mathcal{S}} \varphi$. Now $\varphi = f^n p$, for some $n < \omega$, and some variable p. Suppose $f^i p \in \Gamma$ for some i, $i \leq n$. Then we see $\Gamma \vdash_{\mathcal{S}} f^n p$, by applying the rule $\langle \{p\}, fp \rangle$ repeatedly; a contradiction—so $f^i p \notin \Gamma$, for all i, $i \leq n$. Now let $\boldsymbol{A}_{n+1}$ be the mono-unary algebra $\langle \{0, 1, \ldots, n+1\}, f^{\boldsymbol{A}} \rangle$, where

$$f^{\boldsymbol{A}}(i) = \begin{cases} i+1 & \text{if } i \leq n \\ n+1 & \text{if } i = n+1. \end{cases}$$

Let $h : \mathbf{Fm}_{\mathcal{L}} \to \boldsymbol{A}$ be determined by $h(p) = 0$, and $h(q) = n + 1$, for all $q \in \mathrm{Va}$, $q \neq p$. Then $h(\varphi) = n \neq n + 1 = h(f\varphi)$, but $h(\psi) = h(f\psi)$, for all $\psi \in \Gamma$. Thus $\tau(\Gamma) \not\models_{\boldsymbol{A}} \tau(\varphi)$, as desired. ■

Since $p \vdash_{\mathcal{S}} \neg p$ is not a rule of **CPC** or **IPC**, we infer from the theorem that the $\{\neg\}$-fragment of **CPC** and the $\{\neg\}$-fragment of **IPC** do not have an algebraic semantics. However, both deductive systems satisfy conditions (xiii) and (xiv) of Theorem 2.16 with respect to the equations $p \approx \neg\neg p$ and $\neg p \approx \neg\neg\neg p$ respectively. Hence the conditions (xiii) and (xiv) of Theorem 2.16 do not provide a sufficient condition guaranteeing the existence of an algebraic semantics.

3. Sufficient conditions

In this section we obtain conditions that ensure that a deductive system have an algebraic semantics. The first main result, Theorem 3.1, is very general, and applicable to a wide class of deductive systems. It shows in particular that if the underlying algebras of the matrices of some matrix semantics of a deductive system $\mathcal{S}$ satisfy an identity of the form $f(p, \dots, p) \approx p$, for some connective f in the language of $\mathcal{S}$, then the deductive system has an algebraic semantics.

THEOREM 3.1. *Let $\mathcal{S}$ be a deductive system over a language $\mathcal{L}$, let M be a matrix semantics of $\mathcal{S}$, let $f \in \mathcal{L}$ be an n-ary connective, with $1 \leq n < \omega$, and let $\varepsilon(p)$ be an at most unary $\mathcal{L}$-term whose main symbol (if any) is different from f. If for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and every $a \in F$*

$$f^{\boldsymbol{A}}(a, \dots, a) = \varepsilon^{\boldsymbol{A}}(a)$$

then $\mathcal{S}$ has an algebraic semantics with defining equation $f(p, \dots, p) \approx \varepsilon(p)$.

We give the proof of the theorem in two stages. In the first stage we consider the case in which the term $\varepsilon(p)$ is the variable p. This case is subdivided in two subcases. We begin by treating the simpler but important case in which the deductive system has a matrix semantics with the property that there is an equation of the form $f(p \dots, p) \approx p$ satisfied by the underlying algebras of the matrices in M (Theorem 3.3). Next we deal with the case in which the deductive system has a matrix semantics with the weaker property that there is an equation of the form $f(p \dots, p) \approx p$ satisfied by the the elements of the filters of the matrices in M (Theorem 3.6). Having dealt with the case $\varepsilon(p) = p$ we turn to the next stage, where ε is a term that contains at least one fundamental operation symbol. In all cases we appeal to the following lemma.

LEMMA 3.2. *Let* M *be a matrix semantics of a deductive system* $\mathcal{S}$*, and let* $\delta_i(p) \approx \varepsilon_i(p)$*,* $i < n$*, be a finite set of unary equations of type* $\mathcal{L}$*. Suppose for every* $\mathcal{A} = \langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ *there is a matrix* $\mathcal{A}' = \langle \boldsymbol{A}', F'\rangle$*, and a surjective homomorphism* $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ *such that* $h^{-1}F = F'$ *and* $F' = \{a \in A' : \delta_i^{\boldsymbol{A}'}(a) = \varepsilon_i^{\boldsymbol{A}'}(a),\ i < n\}$*; let* M' *denote the class* $\{\mathcal{A}' : \mathcal{A} \in \mathsf{M}\}$ *of all such matrices. Then the class* K' *of algebra reducts of matrices in* M'*, i.e., the class* $\mathsf{K}' = \{\boldsymbol{A}' : \langle \boldsymbol{A}', F'\rangle \in \mathsf{M}'\}$*, is an algebraic semantics of* $\mathcal{S}$ *with defining equations* $\delta_i(p) \approx \varepsilon_i(p)$*,* $i < n$*.*

PROOF. For $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ let $\mathcal{A}'$ and $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ be as stipulated. Since $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ is onto and $F' = h^{-1}F$, by Proposition 1.2 the consequence relations $\models_{\mathcal{A}}$ and $\models_{\mathcal{A}'}$ coincide. It follows that $\models_{\mathsf{M}}$ and $\models_{\mathsf{M}'}$ coincide, and hence M' is a matrix semantics of $\mathcal{S}$. Since $F' = \{a \in A' : \delta_i^{\boldsymbol{A}'}(a) = \varepsilon_i^{\boldsymbol{A}'}(a),\ i < n\}$, by Theorem 2.3 K' is an algebraic semantics of $\mathcal{S}$ with defining equations $\delta_i(p) \approx \varepsilon_i(p)$, $i < n$. ■

THEOREM 3.3. *Let* $\mathcal{S}$ *be a deductive system over* $\mathcal{L}$ *and* M *a matrix semantics of* $\mathcal{S}$*. If there is an* n*-ary connective* f*,* $n \geq 1$*, such that for every* $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ *and every* $a \in A$ $f^{\boldsymbol{A}}(a, \dots, a) = a$*, then* $\mathcal{S}$ *has an algebraic semantics with defining equation* $f(p, \dots, p) \approx p$*. In addition, if* $\mathcal{L}' \subseteq \mathcal{L}$ *such that* $f \in \mathcal{L}'$*, then the* $\mathcal{L}'$*-fragment* $\mathcal{S}'$ *of* $\mathcal{S}$ *also has an algebraic semantics with defining equation* $f(p, \dots, p) \approx p$*.*

PROOF. Let $f \in \mathcal{L}$, f n-ary, $1 \leq n < \omega$, such that for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ we have $f^{\boldsymbol{A}}(a, \dots, a) = a$, for all $a \in A$. Suppose $\mathcal{A} = \langle \boldsymbol{A}, F\rangle \in \mathsf{M}$. Let A' be the set A together with an extra copy of $A \setminus F$; formally $A' = A \cup \{a' : a \in A \setminus F\}$, where for $a, b \in A \setminus F$ we choose $a' \notin A$ and $a' \neq b'$ if $a \neq b$. Let $h\colon A' \to A$ be defined by $h(a) = a$ if $a \in A$ and $h(a') = a$ if $a \in A \setminus F$.

We define a new algebra $\boldsymbol{A}' = \langle A', \langle g^{\boldsymbol{A}'} : g \in \mathcal{L}\rangle\rangle$ as follows. If $g \in \mathcal{L}$, say g is k-ary, then

$$g^{\boldsymbol{A}'}(a_0, \dots, a_{k-1}) = g^{\boldsymbol{A}}(ha_0, \dots, ha_{k-1}),$$

for all $a_0, \dots, a_{k-1} \in A'$, except if $g = f$ and $a_0 = a_1 = \ldots = a_{k-1} = a$ for some $a \in A$, $a \notin F$, in which case we define

$$f^{\boldsymbol{A}'}(a, \dots, a) = a'.$$

Note in particular that since f is n-ary for some $1 \leq n < \omega$ we have $c^{\boldsymbol{A}'} = c^{\boldsymbol{A}}$ for any constant symbol $c \in \mathcal{L}$.

Let $\mathcal{A}'$ be the matrix $\langle \boldsymbol{A}', F\rangle$ associated in this way with $\mathcal{A}$, and M' the class of all matrices $\mathcal{A}'$, $\mathcal{A} \in \mathsf{M}$. Clearly $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ is a surjective homomorphism, and $h^{-1}(F) = F$. We verify that

$$F = \{a \in A' : f^{\boldsymbol{A}'}(a, \dots, a) = a\}. \tag{xv}$$

$\subseteq$: If $a \in F$ then $f^{\boldsymbol{A}'}(a, \dots, a) = f^{\boldsymbol{A}}(ha, \dots, ha) = ha = a$.

$\supseteq$: Let $a \notin F$. If $a \notin A$ then $f^{\boldsymbol{A}'}(a, \dots, a) \in A$ and so $f^{\boldsymbol{A}'}(a, \dots, a) \neq a$. If $a \in A$, then $a \in A \backslash F$, so $f^{\boldsymbol{A}'}(a, \dots, a) = a' \notin A$ and so $f^{\boldsymbol{A}'}(a, \dots, a) \neq a$.

The conditions of the previous lemma are now satisfied, and the class $\mathsf{K}' = \{\boldsymbol{A}' : \mathcal{A}' \in \mathsf{M}'\}$ is thus an algebraic semantics of $\mathcal{S}$ with defining equation $f(p, \dots, p) \approx p$.

To prove the last statement of the theorem it suffices to check that every $\mathcal{L}'$-fragment of $\mathcal{S}$ with $f \in \mathcal{L}'$ has a matrix semantics such that the equation $f(p, \dots, p) \approx p$ holds in the algebras underlying the matrices of the semantics. Now observe that if M is a matrix semantics of a deductive system $\mathcal{S} = \langle \mathcal{L}, \vdash_{\mathcal{S}}\rangle$ and $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathsf{M}_{\mathcal{L}'} = \{\langle \boldsymbol{A}_{\mathcal{L}'}, F\rangle : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$ is a matrix semantics of the $\mathcal{L}'$-fragment of $\mathcal{S}$; here $\boldsymbol{A}_{\mathcal{L}'}$ denotes the $\mathcal{L}'$-reduct of $\boldsymbol{A}$. Clearly the underlying algebras of the matrices in $\mathsf{M}_{\mathcal{L}'}$ satisfy the equation $f(p, \dots, p) \approx p$ if those of the matrices in M do. Now apply the first part of the theorem. ■

Corollary 3.4. *Every fragment of* **CPC** *or* **IPC** *which contains conjunction (resp. disjunction) has an algebraic semantics with defining equation* $p \wedge p \approx p$ *(resp.* $p \vee p \approx p$*).*

We consider an example of the construction for the $\{\wedge, \vee\}$-fragment $\mathbf{CPC}_{\wedge,\vee}$ of **CPC**. Let $\mathbf{B_2} = \langle\{0,1\}, \to, \wedge, \vee, \neg, 0, 1\rangle$ be the two-element Boolean algebra and $\mathbf{D_2} = \langle\{0,1\}, \wedge, \vee\rangle$ its $\{\wedge, \vee\}$-reduct. Since $\{\langle \mathbf{B_2}, \{1\}\rangle\}$ is a matrix semantics of **CPC**, we have that $\{\langle \mathbf{D_2}, \{1\}\rangle\}$ is a matrix semantics of $\mathbf{CPC}_{\wedge,\vee}$. Since the equation $p \wedge p \approx p$ holds in $\mathbf{D_2}$ Theorem 3.3 is applicable. In this case the algebra $\mathbf{D_2}'$ is $\langle\{0, 0', 1\}, \wedge, \vee\rangle$, where $\wedge$ and $\vee$ are given by the following tables.

$\wedge$	0	$0'$	1
0	$0'$	0	0
$0'$	0	0	0
1	0	0	1

$\vee$	0	$0'$	1
0	0	0	1
$0'$	0	0	1
1	1	1	1

Thus $\mathbf{D_2}'$ (or the quasivariety generated by $\mathbf{D_2}'$) is an algebraic semantics of $\mathbf{CPC}_{\wedge,\vee}$ with defining equation $p \wedge p \approx p$.

Applying similarly the construction of the proof of Theorem 3.3 to **CPC** and the 2-element boolean algebra $\mathbf{B_2}$ with respect to the equation $p \wedge p \approx p$, we get an algebraic semantics, say $\{\mathbf{B_2}'\}$, for **CPC** with defining equation $p \wedge p \approx p$. Note that the quasivariety generated by $\{\mathbf{B_2}'\}$ differs from the variety of Boolean algebras; by the uniqueness of equivalent algebraic semantics (see [3, Theorem 2.15]) and from the well-known fact the variety of Boolean algebras is an equivalent algebraic semantics for **CPC**, we conclude that $\{\mathbf{B_2}'\}$ is not an equivalent algebraic semantics for **CPC**, and that hence the translation $p \approx p \wedge p$ is not 'invertible'.

We next consider some modal logics. We take for our language $\mathcal{L} = \{\wedge, \vee, \to, \neg, \Box, \bot, \top\}$, where $\Box$ is the unary "necessity" operator. The basic deductive system **K** of modal logic is given by the axiom

$$\Box(p \to q) \to (\Box p \to \Box q)$$

together with all classical tautologies, and the rules

$$\begin{array}{ll} \langle \{p, p \to q\}, q \rangle & \text{(modus ponens)} \\ \langle \{p\}, \Box p \rangle & \text{(necessitation).} \end{array}$$

This deductive system is algebraizable (see [3]). Its equivalent algebraic semantics is the variety MA of modal algebras, defined by the equations

$$\Box(p \wedge q) \approx \Box p \wedge \Box q$$

$$\Box(\top) \approx \top$$

together with the equations defining Boolean algebras.

Let $\mathbf{K}'$ be the deductive system axiomatized by the set of theorems of **K** together with the single rule of inference of modus ponens. The system $\mathbf{K}'$ is an example of a *non-normal* modal logic, and it was studied in some depth using matrices in [2], as no natural algebraic semantics seemed available. It was shown in [3, Cor 5.6] that $\mathbf{K}'$ indeed does not possess an *equivalent* algebraic semantics, and thus fails to be algebraizable. As another application of Theorem 3.3, we show now that $\mathbf{K}'$ in fact does have an algebraic semantics.

COROLLARY 3.5. $\mathbf{K}'$ *and all of its extensions have an algebraic semantics with defining equation* $p \wedge p \approx p$.

PROOF. By Theorem 2.15, we only need to verify that $\mathbf{K}'$ has an algebraic semantics with defining equation $p \wedge p \approx p$. This follows immediately from Theorem 3.3, since the class $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{MA}, F \text{ a Boolean filter}\}$ is a matrix semantics of $\mathbf{K}'$ (see for instance [2] for a proof.) ■

Next we weaken the assumption of Theorem 3.3 that an equation of the form $f(p, \dots, p) \approx p$ hold in the underlying algebra of every matrix of a matrix semantics of a deductive system, and require it to hold only for all elements belonging to the *filter* of every matrix of a matrix semantics of a deductive system, thus allowing it to fail outside of the filter. We use again Lemma 3.2, but the construction of the algebraic semantics is a little more involved.

THEOREM 3.6. *Let $\mathcal{S}$ be a deductive system over a language $\mathcal{L}$, and M a matrix semantics of $\mathcal{S}$. If there is an n-ary connective $f \in \mathcal{L}$, $1 \leq n < \omega$, such that for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ and every $a \in F$ $f^{\boldsymbol{A}}(a, \dots, a) = a$, then $\mathcal{S}$ has an algebraic semantics with defining equation $f(p, \dots, p) \approx p$.*

PROOF. Let f be as stipulated. In order to apply Lemma 3.2, we need to exhibit for every matrix $\mathcal{A} = \langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ a matrix $\mathcal{A}' = \langle \boldsymbol{A}', F'\rangle$ and a surjective homomorphism $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ satisfying $F' = h^{-1}(F)$ and $F' = \{a \in A' : f^{\boldsymbol{A}'}(a, \dots, a) = a\}$.

For $\mathcal{A} = \langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ write $B = \{f^{\boldsymbol{A}}(a, \dots, a) : a \in A\}$, and let $B^* = \{b^* : b \in B\}$ be a copy of B disjoint from A, with for $c, d \in B$, if $c \neq d$ then $c^* \neq d^*$. Note that $B \supseteq F$ by assumption. Now let $A' = A \cup B^*$, and let $h\colon A' \to A$ be defined by $h(a) = a$ if $a \in A$, and $h(b^*) = b$ for $b^* \in B^*$. Finally let $F' = h^{-1}F$.

We define a new algebra $\boldsymbol{A}' = \langle A', \langle g^{\boldsymbol{A}'} : g \in \mathcal{L}\rangle\rangle$ as follows. If $g \in \mathcal{L}$, say g is k-ary, then

$$g^{\boldsymbol{A}'}(a_0, \dots, a_{k-1}) = g^{\boldsymbol{A}}(ha_0, \dots, ha_{k-1}),$$

for all $a_0, \dots, a_{k-1} \in A'$, except if $g = f$, and $a_0 = a_1 = \ldots = a_{k-1} = a$ for some $a \in A'$. In this case we define

$$f^{\boldsymbol{A}'}(a, \dots, a) = \begin{cases} f^{\boldsymbol{A}}(ha, \dots, ha) & \text{if } a \in F' \text{ and } a \in A \ (\textit{i.e.}, a \in F), \\ f^{\boldsymbol{A}}(ha, \dots, ha)^* & \text{if } a \in F' \text{ and } a \notin A, \\ f^{\boldsymbol{A}}(ha, \dots, ha)^* & \text{if } a \notin F' \text{ and } a \in A, \\ f^{\boldsymbol{A}}(ha, \dots, ha) & \text{if } a \notin F' \text{ and } a \notin A. \end{cases}$$

Note in particular that since f is n-ary for some n, $1 \leq n < \omega$, we have $c^{\boldsymbol{A}'} = c^{\boldsymbol{A}}$ for any constant symbol $c \in \mathcal{L}$.

Clearly $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ is a surjective homomorphism. We verify now that

$$F' = \{a \in A' : f^{\boldsymbol{A}'}(a, \dots, a) = a\}.$$

$\subseteq$: Let $a \in F'$. Then $ha \in F$ and so by assumption $f^{\boldsymbol{A}}(ha, \dots, ha) = ha$. If $a \in A$ then $f^{\boldsymbol{A}'}(a, \dots, a) = f^{\boldsymbol{A}}(ha, \dots, ha) = ha = a$. If $a \notin A$ then $a = (ha)^*$ and $f^{\boldsymbol{A}'}(a, \dots, a) = f^{\boldsymbol{A}}(ha, \dots, ha)^* = (ha)^* = a$.

$\supseteq$: Let $a \notin F'$. If $a \in A$ then $f^{\boldsymbol{A}'}(a, \dots, a) = f^{\boldsymbol{A}}(ha, \dots, ha)^* \notin A$ and so $f^{\boldsymbol{A}'}(a, \dots, a) \neq a$. If $a \notin A$ then $f^{\boldsymbol{A}'}(a, \dots, a) = f^{\boldsymbol{A}}(ha, \dots, ha) \in A$ and so $f^{\boldsymbol{A}'}(a, \dots, a) \neq a$ as well.

If we set $\mathcal{A}' = \langle \boldsymbol{A}', F' \rangle$, and $\mathsf{M}' = \{\mathcal{A}' : \mathcal{A} \in \mathsf{M}\}$, then the assumptions of Lemma 3.2 are satisfied, and $\mathsf{K}' = \{\boldsymbol{A}' : \mathcal{A}' \in \mathsf{M}'\}$ is an algebraic semantics for $\mathcal{S}$ with defining equation $f(p, \dots, p) \approx p$.

■

We are now ready to prove Theorem 3.1, formulated at the beginning of this section.

PROOF OF THEOREM 3.1: Let $\mathcal{S}$ be a deductive system over a language $\mathcal{L}$, let M be a matrix semantics of $\mathcal{S}$, let $f \in \mathcal{L}$ be an n-ary connective, with $1 \leq n < \omega$, and let $\varepsilon(p)$ be an at most unary $\mathcal{L}$-term whose main symbol is different from f. Assume that for every $\langle \boldsymbol{A}, F \rangle \in \mathsf{M}$ and every $a \in F$

$$f^{\boldsymbol{A}}(a, \dots, a) = \varepsilon^{\boldsymbol{A}}(a).$$

We will show that $\mathcal{S}$ has an algebraic semantics with defining equation $f(p, \dots, p) \approx \varepsilon(p)$.

If $\varepsilon(p)$ is just p, we appeal to Theorem 3.6. So now assume $\varepsilon(p) \neq p$, i.e., the term $\varepsilon(p)$ contains at least one occurrence of a fundamental operation, and the main symbol of $\varepsilon(p)$ is different from f.

For $\mathcal{A} = \langle \boldsymbol{A}, F \rangle \in \mathsf{M}$ write $B = \{f^{\boldsymbol{A}}(a, \dots, a) : a \in A \setminus F\}$, and let $B^* = \{b^* : b \in B\}$ be a copy of B disjoint from A, where, for $c, d \in B$, if $c \neq d$ then $c^* \neq d^*$. Now let $A' = A \cup B^*$, and let $h\colon A' \to A$ be defined by $h(a) = a$ if $a \in A$, and $h(b^*) = b$ for $b^* \in B^*$. Finally let $F' = h^{-1}F$.

We define a new algebra $\boldsymbol{A}' = \langle A', \langle g^{\boldsymbol{A}'} : g \in \mathcal{L} \rangle \rangle$ as follows. If $g \in \mathcal{L}$, say g is k-ary, then

$$g^{\boldsymbol{A}'}(a_0, \dots, a_{k-1}) = g^{\boldsymbol{A}}(ha_0, \dots, ha_{k-1}),$$

for all $a_0, \ldots, a_{k-1} \in A'$, except if $g = f$, and $a_0 = a_1 = \ldots = a_{k-1} = a$ for some $a \in A'$. In this case we define

$$f^{\boldsymbol{A}'}(a, \ldots, a) = \begin{cases} f^{\boldsymbol{A}}(ha, \ldots, ha) & \text{if } a \in F' \\ f^{\boldsymbol{A}}(ha, \ldots, ha)^* & \text{if } a \notin F' \end{cases}$$

Note in particular that since f is n-ary for some n, $1 \leq n < \omega$, we have $c^{\boldsymbol{A}'} = c^{\boldsymbol{A}}$ for all constant symbols $c \in \mathcal{L}$.

Clearly $h \colon \boldsymbol{A}' \to \boldsymbol{A}$ is a surjective homomorphism. We have:

(i) $\varepsilon^{\boldsymbol{A}'}(a) = \varepsilon^{\boldsymbol{A}}(ha)$ for every $a \in A'$:

If ε is a constant symbol then $\varepsilon^{\boldsymbol{A}'} = \varepsilon^{\boldsymbol{A}}$ by the above remark. Otherwise ε is $g(\varepsilon_0, \ldots, \varepsilon_{k-1})$, for some $g \in \mathcal{L}$, with g k-ary, for $1 \leq k < \omega$, and $g \neq f$. Then

$$\begin{aligned} \varepsilon^{\boldsymbol{A}'}(a) = g^{\boldsymbol{A}'}(\varepsilon_0^{\boldsymbol{A}'}(a), \ldots, \varepsilon_{k-1}^{\boldsymbol{A}'}(a)) &= g^{\boldsymbol{A}}(h\varepsilon_0^{\boldsymbol{A}'}(a), \ldots, h\varepsilon_{k-1}^{\boldsymbol{A}'}(a)) \\ &= g^{\boldsymbol{A}}(\varepsilon_0^{\boldsymbol{A}}(ha), \ldots, \varepsilon_{k-1}^{\boldsymbol{A}}(ha)) = \varepsilon^{\boldsymbol{A}}(ha). \end{aligned}$$

(ii) $F' = \{a \in A' : f^{\boldsymbol{A}'}(a, \ldots, a) = \varepsilon^{\boldsymbol{A}'}(a)\}$:

If $a \in F'$ then $ha \in F$ and by assumption $f^{\boldsymbol{A}}(ha, \ldots, ha) = \varepsilon^{\boldsymbol{A}}(ha)$. So then $f^{\boldsymbol{A}'}(a, \ldots, a) = f^{\boldsymbol{A}}(ha, \ldots, ha) = \varepsilon^{\boldsymbol{A}}(ha) = \varepsilon^{\boldsymbol{A}'}(a)$. Conversely, if $a \notin F'$ then $f^{\boldsymbol{A}'}(a, \ldots, a) \notin A$ and so $f^{\boldsymbol{A}'}(a, \ldots, a) \neq \varepsilon^{\boldsymbol{A}}(ha) = \varepsilon^{\boldsymbol{A}'}(a)$.

If we set $\mathcal{A}' = \langle \boldsymbol{A}', F' \rangle$, and $\mathsf{M}' = \{\mathcal{A}' : \mathcal{A} \in \mathsf{M}\}$, then the assumptions of Lemma 3.2 are satisfied, and hence $\mathsf{K}' = \{\boldsymbol{A}' : \mathcal{A}' \in \mathsf{M}'\}$ is an algebraic semantics for $\mathcal{S}$ with defining equation $f(p, \ldots, p) \approx \varepsilon(p)$. ∎

We saw earlier in this section that the (non-normal) modal logic $\mathbf{K}'$, even though it is not algebraizable, still has an algebraic semantics, with defining equation $p \wedge p \approx p$. Theorem 3.1 allows us to show it has an algebraic semantics with a very different defining equation.

COROLLARY 3.7. *The modal logic $\mathbf{K}'$ and all of its extensions have an algebraic semantics with defining equation $p \to p \approx \top$.*

PROOF. We already observed in the proof of Corollary 3.5 that the class $\{\langle \boldsymbol{A}, F \rangle : \boldsymbol{A} \in \mathsf{MA}, F \text{ a Boolean filter}\}$ is a matrix semantics of $\mathbf{K}'$. Every modal algebra $\boldsymbol{A}$ satisfies the identity $p \to p \approx \top$, and the hypotheses of Theorem 3.1 are therefore satisfied. It follows that $\mathbf{K}'$ has an algebraic semantics with defining equation $p \to p \approx \top$.

To see that every extension of $\mathbf{K}'$ has an algebraic semantics with defining equation $p \to p \approx \top$ as well, we invoke Theorem 2.15. ∎

One of the assumptions in Theorem 3.1 is that in the defining equation $f(p,\dots,p) \approx \varepsilon(p)$ the main symbol of the term $\varepsilon(p)$ be different from the fundamental operation f. We conclude the paper with a result that shows that under certain circumstances this requirement can be omitted.

THEOREM 3.8. *Let $\mathcal{S}$ be a deductive system and let M be a matrix semantics of $\mathcal{S}$. If there is a binary connective $\to$ such that every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$ satisfies the following two conditions:*

(i) $a \to^{\boldsymbol{A}} a \in F$ *for every* $a \in A$,

(ii) $a \to^{\boldsymbol{A}} a = (a \to^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} a)$ *for every* $a \in F$,

then $\mathcal{S}$ has an algebraic semantics with defining equation $p \to p \approx (p \to p) \to (p \to p)$.

PROOF. Let $\mathcal{A} = \langle \boldsymbol{A}, F\rangle \in \mathsf{M}$. Let $B = \{a \to^{\boldsymbol{A}} a : a \in A \setminus F\}$, and let $B^* = \{b^* : b \in B\}$ be a copy of B that is disjoint from A, where $c^* \neq d^*$ whenever $c, d \in B$, $c \neq d$. Let $A' = A \cup B^*$, and let $h\colon A' \to A$ be defined by $h(a) = a$ if $a \in A$ and $h(b^*) = b$ for $b \in B$. Let $F' = h^{-1}F$. We define $\boldsymbol{A}' = \langle A', \{g^{\boldsymbol{A}'} : g \in \mathcal{L}\}\rangle$ as follows.

For any fundamental operation g, with g k-ary, say, and $a_0, \dots, a_{k-1} \in A'$, we set

$$g^{\boldsymbol{A}'}(a_0,\dots,a_{k-1}) = g^{\boldsymbol{A}}(ha_0,\dots,ha_{k-1}),$$

except if $g = \to$, and $a_0 = a_1 = a$, in which case we define

$$a \to^{\boldsymbol{A}'} a = \begin{cases} ha \to^{\boldsymbol{A}} ha & \text{if } a \in F' \\ (ha \to^{\boldsymbol{A}} ha)^* & \text{if } a \notin F'. \end{cases}$$

In particular, if c is a constant symbol then $c^{\boldsymbol{A}'} = c^{\boldsymbol{A}}$.

Clearly $h\colon \boldsymbol{A}' \to \boldsymbol{A}$ is a surjective homomorphism. We have:

(a) For every $a \in A'$,

$$(a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}'} (a \to^{\boldsymbol{A}'} a) = (ha \to^{\boldsymbol{A}} ha) \to^{\boldsymbol{A}} (ha \to^{\boldsymbol{A}} ha).$$

Indeed, $h(a \to^{\boldsymbol{A}'} a) = ha \to^{\boldsymbol{A}} ha \in F$ by (i). Thus $a \to^{\boldsymbol{A}'} a \in F'$ and so

$$\begin{aligned}(a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}'} (a \to^{\boldsymbol{A}'} a) &= h(a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}} h(a \to^{\boldsymbol{A}'} a) \\ &= (ha \to^{\boldsymbol{A}} ha) \to^{\boldsymbol{A}} (ha \to^{\boldsymbol{A}} ha).\end{aligned}$$

(b) $F' = \{a \in A' : a \to^{\boldsymbol{A}'} a = (a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}'} (a \to^{\boldsymbol{A}'} a)\}$.

To see that $F' \subseteq \{a \in A' : a \to^{\boldsymbol{A}'} a = (a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}'} (a \to^{\boldsymbol{A}'} a)\}$, let $a \in F'$ be arbitrary. Then

$$\begin{aligned} a \to^{\boldsymbol{A}'} a &= ha \to^{\boldsymbol{A}} ha \\ &= (ha \to^{\boldsymbol{A}} ha) \to^{\boldsymbol{A}} (ha \to^{\boldsymbol{A}} ha) && \text{(by (ii))} \\ &= (a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}'} (a \to^{\boldsymbol{A}'} a). && \text{(by (a))} \end{aligned}$$

For the converse inclusion, if $a \notin F'$, then $a \to^{\boldsymbol{A}'} a \notin A$ and so

$$\begin{aligned} a \to^{\boldsymbol{A}'} a &\neq (ha \to^{\boldsymbol{A}} ha) \to^{\boldsymbol{A}} (ha \to^{\boldsymbol{A}} ha) \\ &= (a \to^{\boldsymbol{A}'} a) \to^{\boldsymbol{A}'} (a \to^{\boldsymbol{A}'} a). && \text{(by (a))} \end{aligned}$$

By Lemma 3.2, $\{\boldsymbol{A}' : \langle \boldsymbol{A}, F\rangle \in \mathsf{M}\}$ is an algebraic semantics of $\mathcal{S}$ with defining equation $p \to p \approx (p \to p) \to (p \to p)$. ■

The last theorem can be used to see that fragments of intuitionistic linear logic and their extensions—which need not be algebraizable—possess an algebraic semantics with defining equation $p \to p \approx (p \to p) \to (p \to p)$.

A *partially ordered commutative residuated monoid* (*pocrm*, for short) is a structure $\boldsymbol{A} = \langle A, \cdot, \to, e, \leq\rangle$ where $\langle A, \cdot, e\rangle$ is a commutative monoid with identity e, $\leq$ is a partial order on A, and for all $a, b, c \in A$

$$a \cdot c \leq b \quad \text{iff} \quad c \leq a \to b. \tag{xvi}$$

The monoid operation in a pocrm can be shown to preserve the partial order.

We verify that any pocrm $\boldsymbol{A}$ satisfies the identity

$$p \to p \approx (p \to p) \to (p \to p).$$

Let $a \in A$. We first observe that the elements of the form $(a \to a)$ satisfy $(a \to a) \cdot (a \to a) \leq (a \to a)$. To see this, we only need to verify, in virtue of (xvi), that $a \cdot ((a \to a) \cdot (a \to a)) \leq a$. This follows using associativity of $\cdot$, as $a \cdot (a \to a) \leq a$. Using (xvi) again, we infer that

$$(a \to a) \leq (a \to a) \to (a \to a).$$

The reverse inequality also holds, since $e \leq (a \to a)$, and for any element $b \in A$ with the property $e \leq b$ we have $(b \to b) = e \cdot (b \to b) \leq b \cdot (b \to b) \leq b$; so in particular

$$(a \to a) \to (a \to a) \leq (a \to a).$$

If $\boldsymbol{A}$ is a pocrm, then a set $F \subseteq A$ is called a *filter* of $\boldsymbol{A}$ if

(i) $a \to a \in F$, for all $a \in A$, and

(ii) if $a, a \to b \in F$, then $b \in F$.

The reduced matrices for *intuitionistic linear logic* (**ILL**, for short) are structures $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is a pocrm with additional operations, and $F \subseteq A$ a filter, with some additional closure properties; such matrices satisfy thus conditions (i) and (ii) of Theorem 3.8. All fragments of **ILL** possess a matrix semantics consisting of such matrices, provided their language contain the connective $\to$. Theorem 3.8 thus yields

COROLLARY 3.9. *All fragments of intuitionistic linear logic* **ILL** *and their extensions possess an algebraic semantics with defining equation*

$$p \to p \approx (p \to p) \to (p \to p),$$

provided their language contains the connective $\to$.

Some of these deductive systems are not algebraizable. For example, it follows from results in in [3, Section 5] that the $\to$-fragment of the relevance logic **R**—an extension of the $\to$-fragment of **ILL**—fails to be algebraizable.

References

[1] BALBES, B., and Ph. DWINGER, *Distributive lattices*, University of Missouri Press, Columbia (Missouri), 1974.

[2] BLOK, W. J., and P. KÖHLER, 'Algebraic semantics for quasi-classical modal logics', *J. Symbolic Logic* 48:941–964, 1983.

[3] BLOK, W. J., and D. PIGOZZI, *Algebraizable logics*, Memoirs of the American Mathematical Society, **396**, Amer. Math. Soc., Providence, 1989.

[4] BLOK, W. J., and D. PIGOZZI, 'Protoalgebraic logics', *Studia Logica* 45:337–369, 1986.

[5] BLOK, W. J., and D. PIGOZZI, 'Abstract algebraic logic and the deduction theorem', manuscript.

[6] BLOK, W. J., and D. PIGOZZI, 'Algebraic semantics for universal Horn logic without equality', in: *Universal Algebra and Quasigroup Theory*, A. Romanovska, J. D. Smith (eds.), Heldermann Verlag, Berlin, 1992.

[7] ŁOŚ, J. and R. SUSZKO, 'Remarks on sentential logics', *Indagationes Mathematicae* 20:177–183, 1958.

[8] REBAGLIATO, J. and V. VERDÚ, 'Algebraizable Gentzen systems and the deduction theorem for Gentzen systems', Mathematics Preprint Series, Universitat de Barcelona, 175, 1995.

[9] Troelstra, A. S., *Lectures on Linear Logic*, CSLI Lecture Notes, No 29, 1992.

W. J. Blok
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
851 S. Morgan Street
Chicago, Illinois 60607
wjb@uic.edu

J. Rebagliato
Sant Cugat
08190 Sant Cugat del Vallés
Barcelona, Spain
jrebagli@pie.xtec.es