MATHEMATICAL BACKGROUND FOR ESSAY 1

The following provides some background for the mathematics underlying the Essay 1 questions. Do not use these argumets in your essay—a more geometric argument, involving the stacking or packing of blocks is required.

You may assume basic facts about geometric sums and series for Essay 1. Let r be any real number and let n be a non-negative integer. The sum

$$1 + r + r^2 + \cdots + r^n \tag{1}$$

is a *geometric sum* and the infinite series

$$1 + r + r^2 + \cdots + r^n + \cdots \tag{2}$$

is a *geometric series*.

Suppose further that $r \neq 1$. Then the geometric sum (1) can be computed by the formula

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

This fact, which you may assume, is easily proved by mathematical induction.

Now suppose that $|r| < 1$. Then $\lim_{n \longrightarrow \infty} r^n = 0$ which means the geometric series (2) converges to $\frac{1}{1-r}$ by the preceding equation. We write

$$1 + r + r^2 + \cdots + r^n + \cdots = \frac{1}{1 - r} \tag{3}$$

to indicate that the series converges and to designate the limit of the sequence of partial sums.

Your essay will involve the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots . \tag{4}$$

Since $|\frac{1}{2}| < 1$, it follows by (3) that (4) converges and

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots = 2.$$

The Delux blocks are cubes with side lengths 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$, ... Your essay involves analyzing the sum of their side lengths

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{16}+\cdots.$$

The preceding series is called the *harmonic series*. Think of the terms of the geometric series (4) as markers for grouping terms of the harmonic series as follows:

$$\mathbf{1}+\mathbf{\frac{1}{2}}+(\frac{1}{3}+\mathbf{\frac{1}{4}})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\mathbf{\frac{1}{8}})+(\frac{1}{9}+\cdots+\mathbf{\frac{1}{16}})+\cdots. \tag{5}$$

We will find an overestimate and an underestimate for the sum of the terms in each of the parenthesized groups. You will see a pattern emerging in our calculations:

$$1=\frac{1}{2}+\frac{1}{2}>\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2},$$
$$1=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}>\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2},$$
$$1=\frac{1}{8}+\cdots+\frac{1}{8}=8(\frac{1}{8})>\frac{1}{9}+\cdots+\frac{1}{16}>\frac{1}{16}+\cdots+\frac{1}{16}=8(\frac{1}{16})=\frac{1}{2},$$
$$\vdots$$

Using (5) and our underestimates, we see that

$$\begin{aligned}
&1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{8}+\frac{1}{9}+\cdots+\frac{1}{16}+\cdots\\
&= \quad 1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+(\frac{1}{9}+\cdots+\frac{1}{16})+\cdots\\
&> \quad 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.
\end{aligned}$$

Thus the partial sums of the harmonic series grow without bound which is expressed by

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\infty.$$

Below is a formal proof of the the fact that the sums of terms in parenthesized groupings lie between $\frac{1}{2}$ and 1. You should *not* include the proof in your essay; the mathematics of your essay is to be treated informally. Observe that the terms of a parenthesized group are given by $\frac{1}{2^n+1}, \ldots, \frac{1}{2^{n+1}}$ for some $n \geq 1$.

Lemma 1. *Let n be a positive integer. Then* $\frac{1}{2} < \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} < 1$.

Proof: Since $2^{n+1} = 2^n + 2^n$ the sum in the statement of the lemma has 2^n terms. Each term has the form $\frac{1}{2^n+\ell}$ for some $1 \leq \ell \leq 2^n$ and thus satisfies

$$\frac{1}{2^{n+1}} \leq \frac{1}{2^n+\ell} \leq \frac{1}{2^n}.$$

At least one of the terms is larger than $\frac{1}{2}$ and one at least one is smaller then 1. Therefore

$$\frac{1}{2} = 2^n(\frac{1}{2^{n+1}}) < \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} < 2^n(\frac{1}{2^n}) = 1.$$

□