

Numerical Simulation of Singular Solutions of the Generalized Korteweg-de Vries Equation

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ABSTRACT. Presented here are some detailed results from a computer-assisted study of singularity formation in solutions of generalized Korteweg-de Vries equations $u_t + u^p u_x + u_{xxx} = 0$. This report supplements an earlier paper by the same authors. Of special interest here will be the path of the peak of the solution as it nears blow-up. Also, more detailed information is provided than heretofore about the structure of potentially singular solutions in the critical exponent case $p = 4$.

1. Introduction. This paper is concerned with the initial-value problem for the generalized Korteweg-de Vries equation (GKdV equation henceforth)

$$(1.1a) \quad u_t + u^p u_x + \epsilon u_{xxx} = 0.$$

In (1.1a), $u = u(x, t)$ is a real-valued function of the spatial variable $x \in [0, 1]$ and the temporal variable $t \geq 0$, p is a positive integer, ϵ is a positive constant and subscripts connote partial differentiation. The partial differential equation (1.1a) is considered in conjunction with initial data

$$(1.1b) \quad u(x, 0) = u_0(x) \text{ for } 0 \leq x \leq 1,$$

where u_0 is a given smooth, real-valued function of x that is periodic of period 1. Solutions u of (1.1) are sought that are likewise 1-periodic in the spatial variable.

Equation (1.1a) arises in many instances in modeling wave propagation in weakly nonlinear, weakly dispersive regimes where dissipative effects can safely be ignored. Because of their considerable role in describing a variety of physical systems, and because of their very interesting mathematical properties, these equations have come in for substantial scrutiny in the last three decades.

It is our purpose here to study the initial- and periodic-boundary-value problem (1.1) for values of $p \geq 4$. We will use high-order accurate numerical methods for approximating solutions of (1.1). These tools will cast light on some fairly delicate aspects of the evolution of relatively large initial data to be discussed presently.

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This article fits into the development by many authors of theory and intuition about (1.1). Making no claim whatever to completeness, we mention as a sample the papers of Kato [7], Bona, Strauss & Souganidis [4], Bona, Dougalis & Karakashian [1], Pego & Weinstein [11], [12], some of which will inform the further discussion. The present contribution particularly supplements two earlier studies by the same group of authors (Bona *et al.* [2], [3]).

One of the primary focuses of Bona *et al.* [2] was to better understand the instability of the special traveling-wave solutions of (1.1a) called solitary waves that was predicted theoretically for $p \geq 4$ by Bona *et al.* [1] (see also Pego & Weinstein [12]). Using a specially designed, adaptive version of our high-order numerical scheme, it was observed that small perturbations of solitary-wave initial profiles form similarity structures under the evolution (1.1a), and that these similarity profiles lead to the formation of singularities in the solution at a finite time. Indeed, the numerical simulations point to the conjecture that there is a point (x^*, t^*) such that $u(x, t) \rightarrow +\infty$ as $(x, t) \rightarrow (x^*, t^*)$. A detailed analysis of computed blow-up rates of various spatial norms of u as t approaches t^* suggest that u has the form

$$(1.2) \quad u(x, t) = \frac{1}{(t^* - t)^{2/3p}} \chi\left(\frac{x^* - x}{(t^* - t)^{1/3}}\right) + \text{bounded term},$$

where χ is a smooth, bounded function. The existence of such similarity solutions χ has recently been settled by Bona & Weissler [5], but this does not at once imply the validity of (1.2) for a broad class of initial data, nor even for appropriate perturbations of solitary waves.

In our companion paper Bona *et al.* [3], consideration was given to the effect of dissipation on the singularity formation imputed to exist from the paper [2]. In the main, we studied the initial-value problem for the GKdV-Burgers equation

$$(1.3) \quad \begin{aligned} u_t + u^p u_x - \delta u_{xx} + \epsilon u_{xxx} &= 0, & 0 \leq x \leq 1, & t \geq 0, \\ u(x, 0) &= u_0(x), & 0 \leq x \leq 1, \end{aligned}$$

where δ is a positive constant, p is an integer greater than or equal to 4, and u_0 is as in (1.1b). Among other things, the following interesting result came to light. Let u_0 be given initial data and suppose the solution u emanating from u_0 blows up in finite time in case $\delta = 0$ in (1.3) (i.e. for (1.1)). Our theory and simulations indicate that in this circumstance, there is a $\delta_c > 0$ so that if $\delta > \delta_c$, then the associated solution $u = u_\delta$ is uniformly bounded for $(x, t) \in [0, 1] \times [0, \infty)$. However, if $\delta < \delta_c$, the solutions u_δ appear to form singularities in finite time with similarity-form blow-up just as displayed in (1.2).

In the present paper, two issues are addressed that were not elaborated previously, but which are related to the results just described. The first is concerned with the putative blow-up due to instability, of perturbed solitary waves. Very careful observations indicate that the peak of a blowing-up solution, which is the spatial point $x = X(t)$ at which the solution at time t makes its maximum positive excursion, propagates as it becomes infinite. In Section 3, we report computational evidence that $X(t)$ moves according to the law $X(t) \sim x^* - c(t^* - t)^{1/3}$ as t approaches t^* , where c is a positive constant. Another point taken up here which has been largely neglected in the earlier work is the structure of solutions in the critical case $p = 4$. In this case it was noted before that the computations relating to the instability of solitary waves yielded much less definite results than for supercritical values $p > 4$. A very careful set of numerical experiments is reported in Section

4 that suggests strongly there is singularity formation as a result of the instability, but that the structure of the blow-up may not follow so closely the paradigm illustrated in (1.2).

The description of the numerical simulations in Sections 3 and 4 are preceded in Section 2 by a summary of the algorithms used and a report of the analytical facts pertaining to these algorithms. The paper concludes with a short section in which open questions suggested by and related to the contents of Sections 3 and 4 are brought out.

2. The Computational Procedures. The computer-generated data presented later in this report was obtained by a fully discrete Galerkin-finite-element scheme that utilizes splines for the spatial discretization and high-order, conservative, implicit Runge-Kutta methods of the Gauss-Legendre class for the time stepping. The stability limitations are kept minimal by using an implicit scheme, but there results as a consequence a nonlinear system of algebraic equations to solve at each time step. These systems are solved approximately by a few iterations of Newton-type, starting with an appropriate extrapolation of the solution based on previous time levels.

The methods just outlined are developed in detail in Bona *et al.* [2]. They comprise what we will refer to as the *base scheme*, and they feature uniform spatial and temporal grids. The base scheme as realized by a Fortran code has been extensively checked for accuracy and convergence as reported in Section 3 of [2]. It is worth mentioning the recent, very satisfactory theoretical results pertaining to the base scheme detailed in Karakashian & McKinney [6]. Let U^n connote the fully discrete approximation to the exact solution u at the time $t_n = nk$, where k is the constant time step, generated by applying the base scheme with splines of order r and a constant spatial grid of size h and a q -stage Gauss-Legendre time-stepping. (Referring to the detailed aspects of the determination of U^n set forth in [2], this involves one "outer" and two "inner" Newton iterations.) The theorem that applies here is that there is a constant $c = c(u, t)$ depending on a low-order Sobolev-norm of the solution $u(\cdot, t)$ for $0 \leq t \leq T$ such that

$$\max_{0 \leq n \leq T/k} \|U^n - u(\cdot, t_n)\|_{L_2} \leq c(h^r + k^{2q}),$$

uniformly for small h and k . The exponents r and $2q$ appearing above are the optimal rates achievable by these specific approximation techniques.

Since most of the solutions we aim to approximate feature large variations and very steep gradients, it will be necessary to equip the base scheme with adaptive capabilities in both space and time. The decision to refine spatially is based on the inverse inequality

$$\|v\|_{L_\infty} \leq \frac{c_*}{h^{1/2}} \|v\|_{L_2}$$

which is valid for members of the finite-dimensional subspace S_h^r consisting of splines of order r on a mesh of size h . The temporal adaptivity employed is based upon limiting the variation of a discrete version of an integral invariant of (1.1a) (a Hamiltonian functional in fact). We refer to [2] for more details.

3. Trajectory of the Center of the Peak Near Blow Up. The solitary-wave solution of the GKdV-equation has the form $u(x, t) = \varphi(x - ct)$ where

$$(3.1) \quad \varphi(x) = A \operatorname{sech}^{2/p}(K(x - x_0))$$

with $A > 0$. Here K , A and c are related by the formulas

$$K = p \left[\frac{A^p}{\epsilon(p+1)(p+2)} \right]^{1/2} \quad \text{and} \quad c = \frac{2kA^p}{(p+1)(p+2)}.$$

It was found in our earlier study (Bona *et al.* [2]) that if initial data u_0 in (1.1) is taken as

$$(3.2) \quad u_0(x) = \lambda\varphi(x),$$

say, where $\lambda > 1$, and $p > 4$, then the associated solution u apparently evolves into a similarity structure of the form depicted in (1.2). Thus the solution forms a single peak that becomes infinite in finite time. Detailed aspects of the formation of the singularity are presented in [2] in terms of the rate of blow-up of various norms of the solution. In addition to evolving into a similarity structure and forming a single peak, the peak itself propagates in the process of blowing up. It is this propagation we wish to investigate in the present section. By means of numerical experiments, the details of the propagation of the peak for times t near the blow-up time t^* will be investigated both for initial-value problems for the GKdV-equation (1.1) and for the GKdV-Burgers equation (1.3). A couple of different types of initial profiles will be considered.

In the first groups of numerical experiments, the GKdV-equation (1.1a) was integrated numerically with initial data u_0 as indicated in (3.1)–(3.2), a perturbed solitary wave. We took $\lambda = 1.01$, $A = 2$ and $\epsilon = 5 \times 10^{-4}$ and computed the solution for values of $p = 5, 6$ and 7 . The relevant parameters appearing in our numerical scheme were taken to be the same as those appearing in Tables 11, 15 and 16 for $p = 5, 6$, and 7 , respectively, in [2].

For solutions u that develop a dominant peak like those considered here, let $X(t)$ connote the location of the peak, which is to say

$$u(X(t), t) = \max_{0 \leq x \leq 1} u(x, t).$$

Let (x^*, t^*) denote the point of blow-up. As an Ansatz, suppose that as t approaches t^* , $X(t)$ has the form

$$(3.3) \quad X(t) = x^* + C(t^* - t)^\gamma,$$

where C and γ are constants to be determined. As a consequence of the presumption (3.3), for t near t^* the speed of the peak of the solution would be

$$\dot{X}(t) = -\gamma C(t^* - t)^{\gamma-1}.$$

The evaluation of the parameters C and γ and the estimation of how closely the power law (3.3) describes what actually transpires is effected by making use of technical aspects of the adaptive computer code to do with cutting the temporal step, refining locally the spatial mesh and translating the peak into the region of finest spatial mesh. These procedures are explained in Bona *et al.* [2]. The refinement of the spatial structure takes place statically, with new grid points being inserted in smaller and smaller, nested neighborhoods of $x = 1/2$. This refined mesh is effective in following a blowing-up solution because the code systematically takes advantage of the translation-invariance of the differential equations to place the peak of the solution at or near $x = 1/2$. These spatial shifts occur at times τ_i , $1 \leq i \leq f$, where τ_i is the time when the i^{th} spatial refinement takes place. The τ_i

get very close to t^* as i increases to f and $X(\tau_i)$ can be estimated by the quantities $x_i + 0.5$, where x_i is the accumulated amount of translation of the peak at the time of the i^{th} refinement. Approximating x^* and t^* by $x_f + 0.5$ and τ_f , respectively, it is reasonable to expect that γ should be approximated well, for i large, but not very close to $i = f$, by the quantities

$$(3.4) \quad \gamma_i = \frac{\log((x_i - x_f)/(x_{i+1} - x_f))}{\log((\tau_i - \tau_f)/(\tau_{i+1} - \tau_f))}.$$

On the right-hand side of (3.4), there appear differences of terms that get extremely close to each other as i increases. Care must be exercised therefore in their computation in order to avoid loss of accuracy due to cancelation. For example, $\tau_f - \tau_i$ is actually computed as the sum of the differences $\Delta\tau_j = \tau_{j+1} - \tau_j$ formed by accumulating small, but positive quantities during the refinement process.

In Table 1 the values of γ_i computed with (3.4) are recorded for the GKdV equation with $p = 5$ and 6 , for $i = 5, 10, \dots, 30$. It was found in Bona *et al.* [2], that the $p = 5$ solitary wave with the aforementioned parameters evolved into a similarity profile whose evolution was followed numerically up to its apparent blow-up at $x^* \cong x_f = 0.61333$, $t^* \cong \tau_f = 0.022543$. The adaptive code performed $f = 42$ refinements achieving a maximum amplitude $U_{\max} = 224,766$. For $p = 6$ the analogous parameters were $x^* \cong x_f = .54732$, $t^* \cong \tau_f = 0.51541 \times 10^{-2}$, $f = 41$, $U_{\max} = 26,099$.

i	p = 5		p = 6	
	γ_i	C_i	γ_i	C_i
5	.3388	-.4128	.3482	-.3219
10	.3306	-.4082	.3152	-.3111
15	.3393	-.4168	.3052	-.3054
20	.3374	-.4148	.3362	-.3175
25	.3336	-.4181	.3409	-.3153
30	.3348	-.4086	.3101	-.3104

TABLE 1. Values of γ_i and C_i at the time τ_i of the i^{th} spatial refinement for GKdV. Solitary-wave initial profile (3.2).

The data in Table 1 is consistent with the hypothesis that γ is a constant independent of p , quite probably equal to $1/3$; the same value appears from the analogous data in the case $p = 7$ not shown here.

The parameter C postulated as a constant in (3.3) may be approximated by the numbers $(x_i - x_f)/(\tau_f - \tau_i)^{\gamma_i}$. This computation was found to be very sensitive to small variations in the computed values of the γ_i , since $\tau_f - \tau_i$ can become as small as 10^{-40} in our computations. Instead, taking into account the robust value $\gamma = 1/3$ that emerged by use of (3.4), the computation of the approximations C_i of C at τ_i was effected by the formula

$$C_i = (x_i - x_f)/(\tau_f - \tau_i)^{1/3}.$$

The entries in Table 1 for C_i stabilize and suggest that C is indeed a constant for each p . Its values are about $-.41$ for $p = 5$, $-.31$ for $p = 6$, and $-.28$ for the $p = 7$

data not shown here. Hence, for the family of solutions evolving from perturbed solitary-wave initial profiles of the form (3.1) with $A = 2$ and $\epsilon = 5 \times 10^{-4}$, C seems to be well approximated by $-2/p$.

When consideration is given to the Korteweg-de Vries-Burgers model equation (1.3) for small positive values of δ so that solutions emanating from, say, initial profiles of the type (3.1) still blow up in finite time as the computations in Bona *et al.* (1996) suggest, practically the same values of γ and C emerge. This is consistent with the findings of the last-quoted paper in that it is again verified that values of δ in the range of blow-up of GKdV-B do not affect the leading-order asymptotics of the blow-up structure.

Analogous computations were performed using as initial data the Gaussian profile

$$(3.5) \quad u_0(x) = \exp\left(-100\left(x - \frac{1}{2}\right)^2\right)$$

not specifically tied to a traveling-wave solution of this class of equations. For example, using (3.5) as initial data for the dissipative equation in (1.3) with $p = 5$, $\epsilon = 2 \times 10^{-4}$ and $\delta = 10^{-4}$, it was found that the solution apparently blew up in finite time at the point $(x^*, t^*) = (.72886, .37376)$ where the numerical solution achieved a value $U_{\max} = 17,148$. The mechanism of blow-up was identical to the example discussed in Bona *et al.* [2] (cf. Figure 9 and Table 18 of that reference): the initial Gaussian profile quickly resolved into a solitary-wave type pulse traveling to the right, followed by a hump. Subsequently, the solitary-wave portion became unstable, evolving into a similarity solution apparently of the form displayed in (1.2). The computed values of γ_i and C_i associated with the trajectory of this leading pulse as it blew up are given in Table 2.

i	5	10	15	20	25	30
γ_i	.3384	.3320	.3315	.3301	.3380	.3288
C_i	-.3049	-.3053	-.3019	-.3021	-.3067	-.3028

TABLE 2. Values of γ_i and C_i at the time τ_i of the i^{th} spatial refinement for GKdV with $p = 5$. Gaussian initial profile as in (3.5).

The emerging value of γ is again approximately $1/3$, a number that seems therefore to be stable under perturbations of ϵ , p , δ and u_0 , provided of course that there is a well-defined, dominating peak that blows up. The value of the negative constant C seems to depend in general on the particular initial profile u_0 and on ϵ and p as well.

4. On the Critical Exponent Case $p = 4$.

It was remarked in Bona *et al.* [2] that the outcome of the computations to determine whether the instability of solitary waves for (1.1a) in the critical exponent case $p = 4$ leads to blow-up in finite time was not as convincing as for $p \geq 5$. In fact, perturbations of solitary-wave initial profiles of the form (3.2) with perturbation factors $\lambda > 1$ did not apparently lead to blow-up. In the above-mentioned reference (cf. in particular formula (5.8)) we constructed a special initial condition u_0 obtained from the $p = 4$ solitary wave (corresponding to $A = 2$,

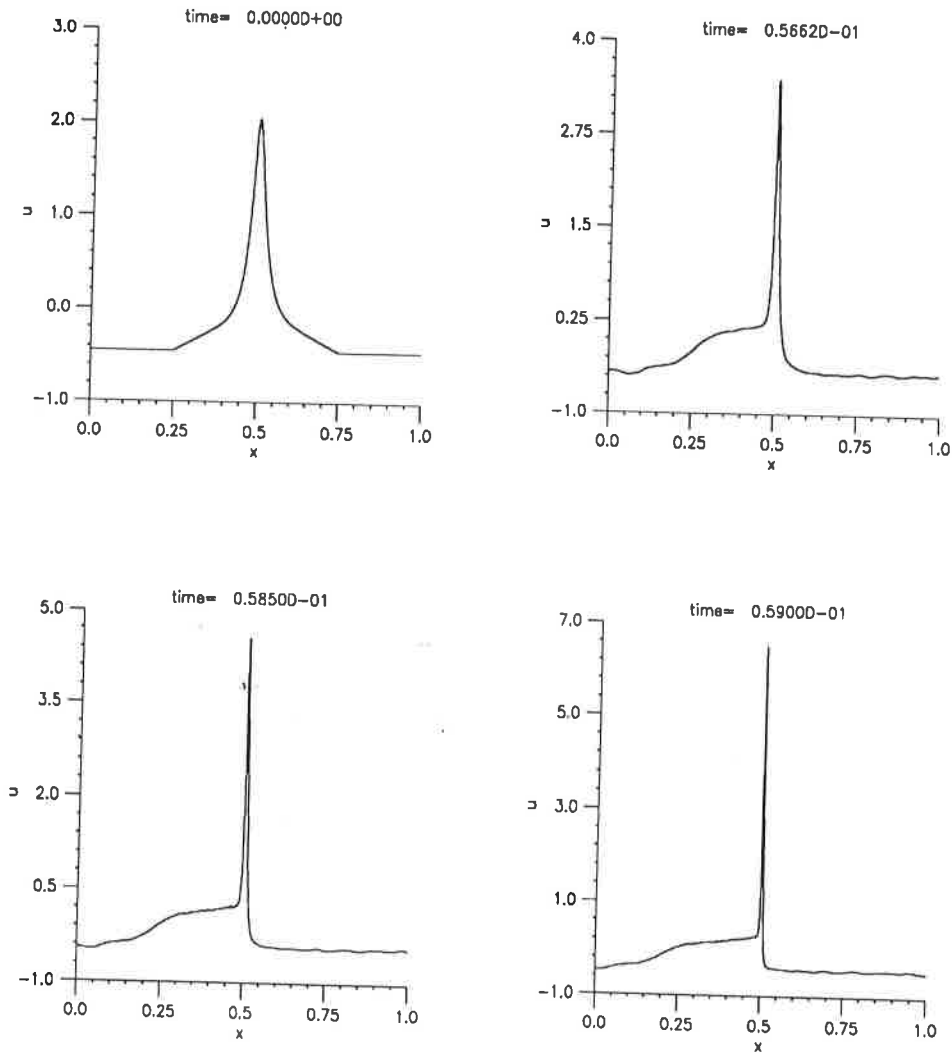


FIGURE 1. Presumed blow-up of a perturbed solitary-wave for $p = 4$. (Parameters as in Table 3.) (a): $u_0(x)$, (b)–(d): computed approximations at the times of the first three spatial grid refinements, respectively.

$\epsilon = 5 \times 10^{-1}$) by first adding a constant which results in non-zero asymptotic values of about -0.446 at $x = 0$ and $x = 1$. The resulting profile was then perturbed in a special direction (motivated by the theory in Bona *et al.* [4]) and then truncated for periodicity. The result is the initial datum shown in (a) of Figure 1. This was integrated numerically with the adaptive code starting with $h_0 = 0.1 \times 10^{-2}$, $k_0 = 0.25 \times 10^{-3}$ and using tolerance levels (cf. [2]) $TOL1 = 0.1 \times 10^{-5}$, $TOL2 = 0.1$. These parameters allowed the code to perform $f = 13$ spatial refinements while

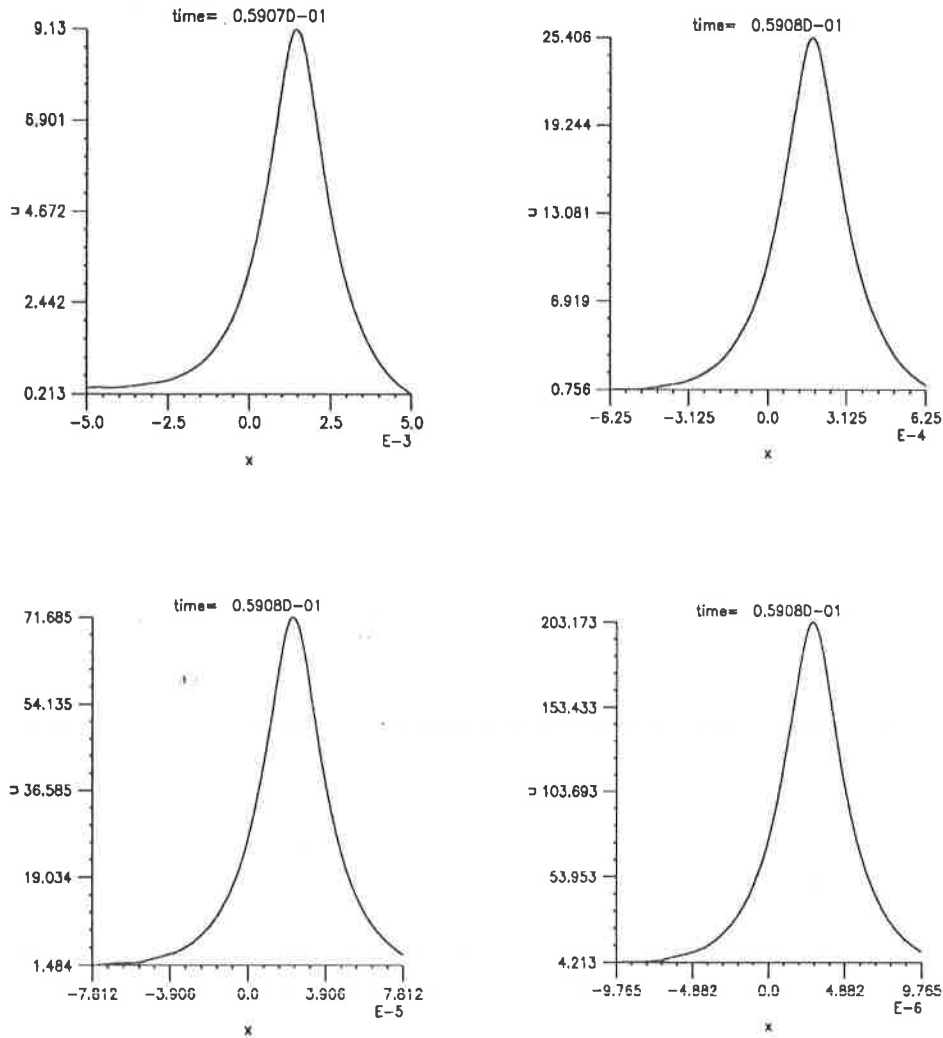


FIGURE 2. Presumed blow-up of a perturbed solitary-wave for $p = 4$. (Continuation of Figure 1). (a), (b), (c), (d): solution at the 4th, 7th, 10th, 13th spatial grid refinements, respectively. The x -axis comprises the finest and the next finest grid region.

sustaining good accuracy. The resulting solution traveled to the right and formed a thin spike which is shown (translated back to the region of the finest grid which is always a small interval symmetric about $x_0 = 1/2$ and called Ω^* in [2], at the instances of the first three spatial refinements in frames (b), (c) and (d) of Figure 1. The spike grew in height achieving a maximum amplitude $U_{\max} = 203$ at $t^* \cong \tau_{13} = 0.059086$ and $x^* = 0.73244$, as may be seen in Figures 2 and 3, where the x - and the u -axes have been rescaled. In particular, in Figure 2 the profile of the

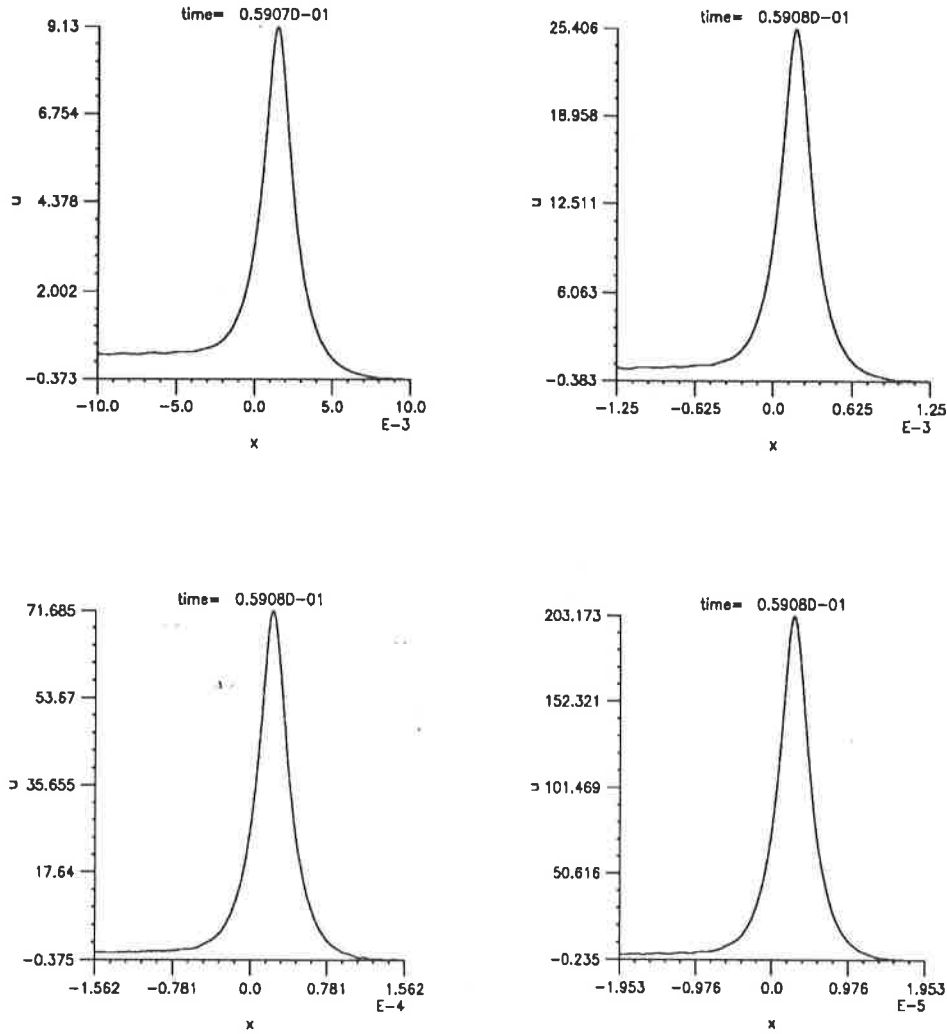


FIGURE 3. Data of Figure 2 with the x -axis scaled to contain the finest grid and the next two grid regions.

peak is shown on an interval about $1/2$ (translated to a symmetric interval about 0 in the pictures) that contains the current interval Ω^* of finest grid size together with one more adjacent region of the next-to-finest mesh intervals at the instances of the 4th, 7th, 10th and 13th refinement ((a), (b), (c), and (d), respectively). In Figure 3 the x -interval has been enlarged to contain the interval Ω^* and the next two regions of successively coarser mesh enclosing Ω^* , with the purpose of exhibiting a detailed view of the peak profile and of the trailing, almost horizontal "shelf".

The maximum amplitude $U_{\max} = 203$ reached in this computation was an improvement by almost an order of magnitude on the analogous value that was

achieved in the computations reported in Bona, *et al.* [2]. In addition, the monitoring of other quantities of interest in the refinement process (like the number of spatial refinements, the size of the time step etc.) adds confidence to the conjecture that the solution indeed blows up. Because of the better accuracy evinced in the present calculations, one expects more robust rates of blow-up of the various spatial norms of the solution and its derivative as $t \rightarrow t^*$. By a rate or exponent of blow-up for some spatial norm $\|u(\cdot, t)\|$ of the solution u , we mean a value $\rho > 0$ for which $\|u(\cdot, t)\| \sim c(t^* - t)^{-\rho}$ as $t \rightarrow t^*$. This is indeed what is observed; in Table 3 we record the computed blow-up rates of the norms used with the same notation in the analogous tables of [2] at the instances of spatial grid refinement τ_i , for $1 \leq i \leq 7$. For $i \geq 8$ the computed rates were not as stable, losing accuracy as i approached 13, the total number of refinements. (In Tables 3 and 4, the columns labeled $L_{2,D}$ and $L_{\infty,D}$ denote the computed rates of blow-up of the L_2 -norm and L_∞ -norm of u_x , respectively.)

i	L_4	L_5	L_6	L_∞	$L_{2,D}$	$L_{\infty,D}$
1	0.9124 (-1)	0.1111	0.1236	0.1854	0.3693	0.5568
2	0.9193 (-1)	0.1113	0.1238	0.1858	0.3712	0.5571
3	0.9228 (-1)	0.1112	0.1236	0.1854	0.3706	0.5570
4	0.9244 (-1)	0.1112	0.1235	0.1851	0.3700	0.5553
5	0.9391 (-1)	0.1128	0.1254	0.1882	0.3761	0.5671
6	0.9160 (-1)	0.1099	0.1221	0.1831	0.3654	0.5500
7	0.9289 (-1)	0.1115	0.1238	0.1853	0.3714	0.5541

TABLE 3. Blow-up rates. Perturbed Solitary wave,
 $p = 4$, $\epsilon = 5 \times 10^{-4}$, $\tau_f = 0.059086$, $f = 13$,
 $x^* = 0.73244$, $U_{\max} = 203.17$, $k_{\min} = 0.36 \times 10^{-14}$,
 $\Delta\tau_f = 0.40 \times 10^{-11}$, $\text{TOL1} = 0.1 \times 10^{-5}$, $\text{TOL2} = 0.1$.

In Bona *et al.* [2], assuming that the singular solution that develops from an unstable solitary wave of the GKdV equation is of the form

$$u = \frac{1}{(t^* - t)^\alpha} \chi \left(\frac{x^* - x}{(t^* - t)^\beta} \right) + \text{bounded term},$$

we computed the rates of blow-up of the L_q -norms of $u(\cdot, t)$ and $u_x(\cdot, t)$ as $t \rightarrow t^*$ as functions of α , β and q . Then using the fact that the third invariant of (1.1), namely the quantity

$$I_3 = \int_0^1 (u^{p+2} - \frac{1}{2}(p+1)(p+2)\epsilon u_x^2) dx,$$

is constant in time (implying that $\|u_x\|_{L_2}^2$ and $\|u\|_{L_{p+2}}^{p+2}$ have the same rates of blow-up) one may deduce that blow-up can occur only for $p \geq 4$ and that α and β must be related by the formula

$$\alpha = 2\beta/p.$$

As a consequence of the foregoing considerations, for $q > p/2$ the rates of blow-up ρ of the L_q -norms of u and u_x were predicted to be

$$(4.1) \quad \rho = \rho(\|u\|_{L_q}) = \alpha(1 - p/2q)$$

and

$$(4.2) \quad \rho = \rho(\|u_x\|_{L_q}) = \alpha(1 + p(q - 1)/2q).$$

The evidence accumulated in Bona *et al.* [2] strongly suggested that for $p \geq 5$, $\rho(\|u\|_{L_\infty}) = 2/3p$. This determines a conjectured value of $\alpha = 2/3p$ (implying $\beta = 1/3$). Moreover, this conjecture for the value of α was strongly supported for $p \geq 5$ by the good agreement of the other blow-up rates determined from the numerical experiments with the predicted values displayed in (4.1) and (4.2).

In the critical exponent case $p = 4$ at hand, the predicted value of $\alpha = 1/6$ does not agree with the outcome of the numerical experiment. For example, examination of the column with the blow-up rate entries for the L_∞ -norm in Table 3 shows that $\rho(\|u\|_{L_\infty})$ might well be .185, which is some 11% larger than the value $1/6$ predicted by descent from higher values of p . Similarly, the blow-up rates suggested by the data of Table 3 for the other norms were also about 11% higher than the values suggested by (4.1) and (4.2) for $p = 4$ and $\alpha = 1/6$.

The anomaly for the rates of blow-up in the critical case $p = 4$ was further investigated by attempting several other expressions for the blow-up, including

$$(4.3) \quad \|u(\cdot, t)\| \sim c \left(\frac{t^* - t}{|\log(t^* - t)|} \right)^{-\rho} \quad \text{as } t \rightarrow t^*.$$

The idea of having a logarithmic term appear in the critical exponent for blow-up in nonlinear dispersive evolution equations is well-known in the context of the nonlinear Schrödinger equation (cf. McLaughlin *et al.* [10], Landman *et al.* [9] and Kosmatov *et al.* [8]). The computed values of ρ using (4.3) as the form of the blow-up are provided in Table 4 and are seen to be closer to the theoretical rates than the values of ρ in Table 3.

i	L_4	L_5	L_6	L_∞	$L_{2,D}$	$L_{\infty,D}$
1	0.7939 (-1)	0.9665 (-1)	0.1076	0.1613	0.3213	0.4844
2	0.8212 (-1)	0.9941 (-1)	0.1106	0.1660	0.3316	0.4976
3	0.8408 (-1)	0.1013	0.1126	0.1689	0.3377	0.5075
4	0.8538 (-1)	0.1027	0.1141	0.1710	0.3418	0.5128
5	0.8762 (-1)	0.1053	0.1170	0.1756	0.3510	0.5291
6	0.8613 (-1)	0.1034	0.1148	0.1722	0.3436	0.5172
7	0.8789 (-1)	0.1055	0.1172	0.1753	0.3515	0.5243

TABLE 4. Blow-up rates. Perturbed Solitary wave, same data as in Table 3, rates computed with the log formula in (4.3).

Since the adaptive refinement procedure used in the numerical schemes is based (among other things) upon controlling the time step by checking that properly weighted values of I_3 remain close to each other from time step to time step, it is reasonable to expect that the computed values of $\|u\|_{L_6}^6$ become infinite like $\|u_x\|_{L_2}^2$, something that is confirmed experimentally by examining the columns labelled L_6 and $L_{2,D}$ in Table 3. On the other hand, assuming self-similarity of the blow-up

profile and using the constancy of I_3 (for $p = 4$), one is led to the conjecture

$$\rho = \rho(\|u\|_{L_q}) = \alpha(1 - 2/q), \quad \rho = \rho(\|u_x\|_{L_q}) = \alpha(3 - 2/q),$$

which seems to be consistent with the data of Table 3 if $\alpha \cong .185$ and Table 4 if $\alpha \cong .172$ and $\beta = 2\alpha$. Of course the blow-up is not necessarily self-similar, though the graphs in Figure 2 make a good case for such a hypothesis.

Another point of interest is that the exponent γ in the assumed form of the trajectory of the center of the peak, (see 2.3), in this experiment was found to be about .26 and not $1/3$ as in the cases where $p \geq 5$.

5. Conclusions. The present study has added some detailed information connected with the apparent singularity formation in solutions of the GKdV equation and the GKdV-Burgers equation. The results in Section 3 indicating the peak of a blowing-up solution propagates at speed $(t^* - t)^{1/3}$ have been incorporated into the theoretical study of similarity solutions of the GKdV equation in Bona & Weessler [5].

The computations connected with the critical case $p = 4$ are improved in various ways over those reported previously. Especially the better agreement with predicted values of blow-up rate one obtains by incorporating a logarithm into the Ansatz about the singularity is intriguing.

It must be candidly remarked, however, that the computations connected with $p = 4$ seem to stretch the capabilities of our computer code in its present form. One possible reason for this may be seen already in Figures 2 and 3. The peak in the profile of the solution has wider "support" than the analogous peaks for the cases where $p \geq 5$. Moreover, the support does not shrink as rapidly as occurs for $p \geq 5$. It may be that the refinement process of our code (or indeed as embodied in the choices of parameters) is so restrictive that it causes the grid in the vicinity of the peak to become unacceptably structured after 10 to 15 refinements. As it took on the order of 40 refinements to produce a peak of height about 10^5 for $p \geq 5$, we are seriously handicapped if we make no more than a dozen refinements. Together with associated theoretical work, improving our simulations of solutions when $p = 4$ is presently under study.

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