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Fourier splitting and dissipation of nonlinear dispersive waves

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Presented herein is a new method for analysing the long-time behaviour of solutions of nonlinear, dispersive, dissipative wave equations. The method is applied to the generalized Korteweg–de Vries equation posed on the entire real axis, with a homogeneous dissipative mechanism included. Solutions of such equations that commence with finite energy decay to zero as time becomes unboundedly large. In circumstances to be spelled out presently, we establish the existence of a universal asymptotic structure that governs the final stages of decay of solutions. The method entails a splitting of Fourier modes into long and short wavelengths which permits the exploitation of the Hamiltonian structure of the equation obtained by ignoring dissipation. We also develop a helpful enhancement of Schwartz's inequality. This approach applies particularly well to cases where the damping increases in strength sublinearly with wavenumber. Thus the present theory complements earlier work using centre-manifold and group-renormalization ideas to tackle the situation wherein the nonlinearity is quasilinear with regard to the dissipative mechanism.

1. Introduction

This paper is concerned with the long-time behaviour of solutions of the class of one-dimensional model wave equations having the form

$$u_t + u_x + g(u)_x - Lu_x + Mu = 0, \tag{1.1}$$

where subscripts connote partial differentiation, $g : \boldsymbol{R} \to \boldsymbol{R}$ is a smooth, usually polynomial function with $g(0) = 0$, and L and M are Fourier multiplier operators given by

$$\widehat{Lv}(\xi) = \eta(\xi)\hat{v}(\xi) \quad \text{and} \quad \widehat{Mv}(\xi) = \alpha(\xi)\hat{v}(\xi). \tag{1.2}$$

A circumflex adorning a function defined on $\boldsymbol{R}$ denotes that function's Fourier transform. The dependent variable $u = u(x,t)$ is a real-valued function of the two real variables $x \in \boldsymbol{R}$ and $t \geqslant 0$. The symbols η and α of the operators L and M, respectively, are typically real-valued, even, non-negative, functions that increase at $\pm\infty$. Consequently, the terms Lu_x and Mu model dispersive and dissipative effects, respectively.

When equations in the class (1.1) arise as models of physical phenomena, u often represents a velocity or a displacement, x is typically proportional to distance in the direction of primary propagation and t is proportional to elapsed time. When $\alpha \equiv 0, \eta(\xi) = \xi^2$ and $g(y) = \frac{1}{2}y^2$, the equation recovers the classical Korteweg–de Vries equation first derived as a model for unidirectional propagation of small-amplitude, long water waves on the surface of a canal. Equations of the form depicted in (1.1) with $M = 0$ have arisen in a variety of physical contexts where the effects of nonlinearity and dispersion comprise weak, but non-negligible perturbations to a basic unidirectional propagation represented by the simple equation $u_t + u_x = 0$. If the equation without the dissipative operator M is an adequate approximation, the the evolution of disturbances is governed by a Hamiltonian system that often features solitary waves as an important aspect of the resolution of disturbances (see [2, 4, 12–15]).

In many practical situations, however, the effect of dissipation comes in at the same general level as nonlinearity and dispersion. For example, waves propagating in a channel or in near-shore zones of large bodies of water suffer significant dissipation (see [9, 19]). This has led to the development of models of the form displayed in (1.1) when quantitatively accurate predictions are needed. Interestingly, both the dispersion operator L and the dissipation operator M may have non-polynomial symbols in modelling situations that arise in practice. Hence they need not be local operators and the level of generality encompassed in the class (1.1) is thus seen not to be excessive. For example, when considering the propagation of surface water waves in a horizontal channel [16] (see also [21]), the authors derived an approximate description of damping based on ignoring dissipation at the surface layer and at the moving contact line at the channel wall while analysing the viscous boundary layers on the sides and bottom. The symbol associated with this approximation has real part $\rho|\xi|^{1/2}$ with $\rho > 0$, corresponding to dissipative effects (the imaginary part of the symbol corresponds to a real operator that makes a small, non-local contribution to dispersion).

The class (1.1) of model equations has been discussed in several recent works [1, 3, 7, 8, 11, 25], and the survey monograph of [20]. Note that the works cited above focus on the case that the nonlinearity is at most quasilinear with respect to the dissipation. By the latter terminology the following is meant. Let $L^2(\boldsymbol{R})$ denote the usual Hilbert space of real-valued, Lebesgue measurable functions which are square integrable over $\boldsymbol{R}$ and let $\mathcal{D}(M) = \{v \in L^2(\boldsymbol{R}) | Mv \in L^2(\boldsymbol{R})\}$. A nonlinearity F is quasilinear with respect to M if $F(v) \in L^2(\boldsymbol{R})$ whenever $v \in \mathcal{D}(M)$. In the present context, this is implied if, for example, g is smooth and the symbol α of M has the property

$$\liminf_{|\xi|\to\infty} \frac{\alpha(\xi)}{|\xi|^\mu} > 0 \tag{1.3}$$

for some $\mu \geqslant 1$. When the nonlinearity is quasilinear with respect to M, a natural approach to understanding solutions of (1.1) is to write it as an equivalent integral equation, namely,

$$u(x,t) = \exp[(M + L_x)t]u_0(x) + \int_0^t \exp[(M + L_x)(t-s)]g(u(x,s))_x \,\mathrm{d}s$$
$$= (Tu)(x,t), \tag{1.4}$$

where u_0 is the initial value, and attempt to apply a contraction-mapping argument in a suitable function space to the operator T whose action is defined by the right-hand side of (1.4). When successful, one conclusion of such an analysis is existence of global solutions of the initial-value problem for (1.1), but more subtle information about the long-time asymptotics can be won by this approach. This idea is at the heart of the centre-manifold analysis carried out by Wayne [26] in determining the long-time behaviour of nonlinear heat equations in higher space dimensions. The renormalization-group methods introduced by Bricmont *et al.* [10] and applied in [7, 8] to equations like (1.1) have as an essential ingredient the analysis of the nonlinear portion of the operator T in (1.4). When the symbol α grows at least linearly at infinity as in (1.3) with $\mu \geqslant 1$, the theory for the large-time asymptotics of solutions of (1.1) can proceed by essentially ignoring dispersive effects as embodied in the operator L. Indeed, if $\eta(\xi) \equiv 0$, $\alpha(\xi) = |\xi|^{2\nu}$ with $\nu \geqslant \frac{1}{2}$ and $g(z) = z^{p+1}/p+1$ are homogeneous, then the conviction that damping dominates the long-time behaviour indicates considering the problem in the frame of reference moving to the right with speed one and then making the change of variables

$$u(x,t) = t^{-1/2\nu}v(y,\tau), \tag{1.5}$$

where $y = x/t^{1/2\nu}$ and $\tau = \log(t)$. The evolution equation for v is

$$v_\tau + Mv - \frac{1}{2\nu}(yv)_y + \mathrm{e}^{-\gamma\tau}v^p v_y = 0, \tag{1.6}$$

where $\gamma = p + 1/2\nu - 1$ is taken to be positive. The latter restriction appears already in Dix's [11] pioneering work; he refers to this situation as the case of asymptotically weak nonlinearity. In appropriate function classes (cf. [7]), the operator $-M + (1/2\nu)\partial_y y$ has a simple eigenvalue at 0 whose eigenfunction, denoted by f^*, is given by $\widehat{f^*}(k) = \exp[-|k|^{2\nu}]$. The remainder of the spectrum is sufficiently buried in the left-half complex plane that there follows the exponential convergence of v to an invariant manifold that is tangent to the subspace spanned by f^*. In the original variables, this implies algebraic temporal convergence of u to the self-similar form $c/t^{1/2\nu}f^*(\cdot/t^{1/2\nu})$ as $t \to \infty$, for a constant c determined by the mass $\hat{u}_0(0)$ of the initial data.

In the case $\nu \geqslant \frac{1}{2}$, where the analysis just sketched is effective, the conclusion can even be strengthened to include higher-order asymptotics (see [8]). However, the crucial estimates on which this theory depends are currently unavailable in case $\nu < \frac{1}{2}$, even when account is taken of dispersive effects. Since the case $\nu = \frac{1}{4}$ arises in an important application of models in the class (1.1), it is natural to attempt to extend the range of dissipative symbols α for which the general conclusions about long-time asymptotics are valid. We introduce here a new technical aspect which avoids the restriction that the nonlinearity be quasilinear with respect to the dissipation. The analysis to be presented is similar in its overall aspect to that just outlined. The details are decidedly different. Employing a family of orthogonal projections, the solution u of (1.1) is decomposed into a long-wavelength part y and a short-wavelength component z. To compensate for the relatively weak dissipation, the dispersion and the conservative form of the nonlinearity are exploited. The evolution of y, which can be thought of as the centre portion of the solution, is shown to be governed by the linear, parabolic equation $y_t + My = 0$, up to an asymptotically negligible contribution from the nonlinearity and dispersion. On the other hand, the L^2-norm of the short-wave component $|z|_{L^2}$ can be written as a sum of a term of order quadratic in $|y|_{L^2}$ plus a term that decays exponentially with time (see equation (3.10)). Thus z is eventually negligible, and the results follow.

It is worth emphasizing that the theory developed in the manner just indicated does not rely upon assumptions about the size of the initial data. If the initial state is large, however, then the proof proceeds by deriving suboptimal decay which results in a time τ, say, when the solution $u(\cdot, \tau)$ is small in a suitable sense. For $t \geqslant \tau$, we are faced with an initial value problem with small data, and in this regime the analysis may be applied in its full extent to obtain the ultimate convergence to a universal asymptotic structure and associated optimal decay rates. It is possible that this waiting time τ is not just an artifact of the proof in the case of large initial data. Indeed, in the absence of dissipation, large initial data may decompose into a train of solitary waves followed by a dispersive disturbance. In the presence of small, but non-zero dissipation, the solitary waves may still emerge (cf. the numerical simulation in [5]), but will then decay, though because of their strong stability, perhaps not so rapidly as genuinely small-amplitude disturbances. Thus the waiting period that arises in our proof may reflect a real aspect of the solution in which the ghosts of solitary waves must move sufficiently close to the centre-manifold before the long-term asymptotic structure becomes visible.

The paper is organized as follows. In § 2 the generalized KdV equations which are the focus of discussion are set forth, notation is introduced, and global well-posedness of the initial-value problem for these evolutionary systems is demonstrated. These results are contained in theorem 2.2. Section 3 contains the main technical work required to establish the sharp temporal decay estimates, including the enhancement of Schwartz's inequality. Finally, in § 4 the form f^* introduced above is shown to provide a universal, self-similar asymptotic for the long-time flow.

2. Generalized KdV

In the sequel, attention will be focused on the class of initial value problems

$$\left.\begin{aligned} u_t + u_{xxx} + u^p u_x + Mu &= 0, \qquad x \in R, \quad t \geqslant 0, \\ u(x,0) &= u_0(x), \qquad x \in R, \end{aligned}\right\} \tag{2.1}$$

with p is positive integer and M the dissipative operator defined by

$$\widehat{Mu}(\xi) = |\xi|^{2\nu}\hat{u}(\xi), \tag{2.2}$$

where, in the present discussion, $0 < \nu \leqslant 1$. Here and below, the Fourier transform of a function f will be denoted variously as

$$\mathcal{F}(f)(k) = \hat{f}(k) = \int_{-\infty}^{\infty} \exp[\mathrm{i}kx] f(x)\,\mathrm{d}x.$$

Denote by D^k the operators of the form (2.2) with $k = 2\nu$.

The norm of a function f in the standard class $L^p = L^p(R)$ is denoted $|f|_{L^p}$, for $1 \leqslant p \leqslant \infty$. Weighted spaces with weight function $q > 0$ are denoted by $L^p(q)$ with norm $|f|_{L^p(q)}$. The classes H^k, $k = 1, 2, \dots$, comprise the functions f, which, along with their first k derivatives, lie in L^2. The Hilbert space H^k is given its usual norm

$$|f|_{H^k} = \left(\int_R |f|^2 + \left| \frac{\partial^k f}{\partial x^k} \right|^2 \mathrm{d}x \right)^{1/2}.$$

The space comprising functions u of a spatial variable $x \in \boldsymbol{R}$ and temporal variable $t \in \boldsymbol{R}^+$ whose H^k-norm in x lies temporally in L^p will be denoted $L^p(\boldsymbol{R}^+; H^k(\boldsymbol{R}))$.

A note about the use of constants is merited. An unadorned constant c will denote any continuous function of the parameters ν and p and of the initial data u_0 in various norms. In particular, if a spatial norm of the solution $\|u(\cdot, t)\|$ is known to be uniformly bounded in terms of the initial data, then it may be absorbed into such a constant c without comment.

2.1. Linearized equations

Our objective is to describe the decay of solutions of (2.1) as t grows unboundedly. It is instructive to examine the behaviour of the solutions of the linearized initial-value problem

$$\left. \begin{aligned} u_t + u_{xxx} + Mu &= 0, \qquad x \in R, \quad t \geqslant 0, \\ u(x, 0) &= u_0(x), \qquad x \in R. \end{aligned} \right\} \tag{2.3}$$

The large-time asymptotic behaviour of solutions of (2.3) is easily obtained, and is summarized in the result below.

THEOREM 2.1. *If $u_0 \in L^2 \cap L^1$, then the solution u of (2.3) satisfies*

$$|D^k u|_{L^2}^2 \leqslant \frac{1}{\nu} \Gamma\left(\frac{2k+1}{2\nu} \right) |u_0|_{L^1}^2 (2t)^{-(2k+1)/2\nu} \tag{2.4}$$

for any $k \geqslant 0$, where Γ denotes the gamma function.

Proof. Taking the Fourier transform of (2.3) and solving the resulting ordinary differential equation yields the explicit formula,

$$\hat{u}(\xi, t) = \exp[-(|\xi|^{2\nu} + i\xi^3)t] \widehat{u_0}(\xi).$$

Thus, by Plancherel's theorem, it follows that

$$|D^k u|_{L^2}^2 = \int_{-\infty}^{\infty} |\xi|^{2k} \exp[-2|\xi|^{2\nu} t] |\widehat{u_0}(\xi)|^2 \, \mathrm{d}\xi.$$

Changing the variable of integration to $z = 2t\xi^{2\nu}$ and bounding $\hat{u}_0$ by its sup norm results in the inequality

$$|D^k u|_{L^2}^2 \leqslant |\widehat{u_0}|_{L^\infty} \frac{1}{\nu} (2t)^{-(2k+1)/2\nu} \int_0^\infty z^{(2k+1-2\nu)/2\nu} \mathrm{e}^{-z} \, \mathrm{d}z. \tag{2.5}$$

The inequality $|\widehat{u_0}|_{L^\infty} \leqslant |u_0|_{L^1}$ and the observation that the integral on the right-hand side of (2.5) is the gamma function $\Gamma((2k+1)/2\nu)$ yield theorem 2.1. $\square$

2.2. *A priori* estimates

Local existence theory for large classes of conservation laws with general dispersive and dissipative effects has been established, for example in theorem 2 of ch. 4 of [20]. Equations of the form depicted in (2.1) fall into the category covered by these results. As a first step towards resolving the long-time behaviour of solutions of (2.1), we now obtain uniform temporal bounds on the spatial norms of the solutions. These results are collected in the following theorem.

THEOREM 2.2. *Let $0 < \nu \leqslant 1$ and $p = 1, 2$, or 3 be given. If the initial data $u_0 \in H^k(\boldsymbol{R})$ for some $k \geqslant 2$, then the solution u of (2.1) satisfies $u \in L^\infty(R^+; H^k)$ and $D^\nu u \in L^2(R^+; H^k)$.*

Proof. To see that u lies in $L^\infty(R^+; L^2)$, take the L^2-scalar product of (2.1) with u and integrate by parts to reach the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_{L^2}^2 + |D^\nu u|_{L^2}^2 = 0. \tag{2.6}$$

Integrating (2.6) over the interval $(0, t)$ yields

$$|u(t)|_{L^2}^2 + 2\int_0^t |D^\nu u(s)|_{L^2}^2 \,\mathrm{d}s \leqslant |u_0|_{L^2}^2,$$

and one readily deduces that $u \in L^\infty(R^+; L^2)$ and $D^\nu u \in L^2(R^+; L^2)$. For a similar H^1-estimate, take the L^2-inner product of equation (2.1) with the quantity $u - u_{xx} - u^{p+1}/(p+1)$ and integrate by parts to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2}|u(\cdot,t)|_{H^1}^2 - \frac{1}{(p+1)(p+2)}\int_R u^{p+2}(x,t)\,\mathrm{d}x\right\} + |D^\nu u|_{H^1}^2 \leqslant \left|\left(D^{2\nu}u, \frac{u^{p+1}}{p+1}\right)\right|. \tag{2.7}$$

To proceed, a Liebnitz rule for fractional derivatives is helpful.

LEMMA 2.3. *Let $0 \leqslant \nu \leqslant 1$ and conjugate exponents p_i, q_i, $i = 1, 2$ satisfying*

$$\frac{1}{p_i} + \frac{1}{q_i} = \frac{3}{2}$$

be given. Then there exists a constant $c > 0$ such that for all functions f and g satisfying $\widehat{D^\nu f} \in L^{p_1}$, $\hat{f} \in L^{p_2}$, $\hat{g} \in L^{q_1}$, and $\widehat{D^\nu g} \in L^{q_2}$, the estimate

$$|D^\nu(fg)|_{L^2} \leqslant c|\widehat{D^\nu f}|_{L^{p_1}}|\hat{g}|_{L^{q_1}} + |\hat{f}|_{L^{p_2}}|\widehat{D^\nu g}|_{L^{q_2}}, \tag{2.8}$$

is valid. In particular, for p a positive integer, there exists a constant c depending only upon p such that

$$|D^\nu u^{p+1}|_{L^2} \leqslant c|\widehat{D^\nu u}|_{L^1}|\hat{u}|_{L^1}^{p-1}|u|_{L^2}. \tag{2.9}$$

Proof. With the aim of applying Plancherel's theorem, consider the quantity $D^\nu(fg)$ in Fourier transformed variables, namely,

$$\mathcal{F}(D^\nu(fg))(k) = |k|^\nu \int_R \hat{f}(k - k_1)\hat{g}(k_1)\,\mathrm{d}k_1. \tag{2.10}$$

Using the sublinearity of $|k|^\nu$ for $0 \leqslant \nu \leqslant 1$, there appears the inequality

$$|\mathcal{F}(D^\nu(fg))(k)| \leqslant \int_R (|k-k_1|^\nu + |k_1|^\nu)|\hat{f}(k-k_1)||\hat{g}(k_1)|\,dk_1$$
$$= |\widehat{D^\nu f}| * |\hat{g}|(k) + |\hat{f}| * |\widehat{D^\nu g}|(k). \tag{2.11}$$

To finish the demonstration of (2.8), take the L^2-norm of both sides of the above inequality, use Plancheral on the left-hand side and on the right-hand side apply the convolution inequality

$$|f_1 * f_2|_{L^r} \leqslant c|f_1|_{L^{r_1}}|f_2|_{L^{r_2}},$$

where

$$\frac{1}{r_1} + \frac{1}{r_2} = 1 + \frac{1}{r},$$

with the choice $r_1 = p_1$, $r_2 = q_1$, and $r = 2$ for the first term of the right-hand side above and with $r_1 = p_2$, $r_2 = q_2$, and $r = 2$ for the second term. The inequality (2.9) follows from applying (2.8) p times. □

To complete the estimation begun in (2.7), the following inequalities play a key role.

LEMMA 2.4. *Let $0 \leqslant \nu \leqslant 1$ be given. For each ϵ_1 and ϵ_2 satisfying $0 < \epsilon_1 < 1/\nu$ and $0 < \epsilon_2 < 1$, there exist positive constants c_1, c_2, and c_3 such that (2.12a)–(2.12c) below are valid:*

$$|\hat{u}|_{L^1} \leqslant c_1|u|_{L^2}^{(2\nu+1-\epsilon_1)/2(\nu+1)}|D^\nu u|_{H^1}^{(1+\epsilon_1)/2(\nu+1)}, \tag{2.12 a}$$
$$|\widehat{D^\nu u}|_{L^1} \leqslant c_2|u|_{L^2}^{(1-\epsilon_2)/2(\nu+1)}|D^\nu u|_{H^1}^{(2\nu+1+\epsilon_2)/2(\nu+1)}, \tag{2.12 b}$$
$$|D^\nu u|_{L^2} \leqslant c_3|u|_{L^2}^{1/(1+\nu)}|D^\nu u|_{H^1}^{\nu/(1+\nu)}. \tag{2.12 c}$$

Proof. We prove (a); the proof of (b) is similar to that of (a) and (c) is a standard interpolation inequality. First it appears that

$$|\hat{u}|_{L^1} = \int_R |\hat{u}(k)|\,dk$$
$$= \int_R (|\hat{u}(k)|^{1-r})(|\hat{u}(k)||k|^\nu(1+|k|^2)^{1/2})^r(|k|^\nu(1+|k|^2)^{1/2})^{-r}\,dk,$$

for $0 < r < 1$. Hölder's inequality with exponents $2/(1-r)$, $2/r$, 2 applied to the functions within the three parentheses comprising the integrand in the last integral and Plancherel's relation yields

$$|\hat{u}|_{L^1} \leqslant |u|_{L^2}^{1-r}|D^\nu u|_{H^1}^r\left(\int_R |k|^{-2r\nu}(1+|k|^2)^{-r}\,dk\right)^{1/2}.$$

The last integral above is the desired constant c_1 and is finite if $1/(2(\nu+1)) < r < 1/2\nu$. Taking $r = (1+\epsilon_1)/2(\nu+1)$ gives (2.12 a). □

Armed with these two lemmas, attention is returned to (2.7). Observe that $D^{2\nu}$ is a positive operator with self-adjoint square root D^ν. This fact, together with Schwartz's inequality leads to the bound

$$|(D^{2\nu}u, u^{p+1})| = |(D^\nu u, D^\nu u^{p+1})| \leqslant |D^\nu u|_{L^2}|D^\nu u^{p+1}|_{L^2}.$$

For arbitrary θ satisfying $0 \leqslant \theta \leqslant 1$, split the term $|D^\nu u|_{L^2}$ into $|D^\nu u|^\theta_{L^2}|D^\nu u|^{1-\theta}_{L^2}$ and apply the Liebnitz rule of lemma 2.3 to the last term in the inequality above to obtain

$$|(D^{2\nu}u, u^{p+1})| \leqslant c|D^\nu u|^\theta_{L^2}|D^\nu u|^{1-\theta}_{L^2}|\widehat{D^\nu u}|_{L^1}|\hat{u}|^{p-1}_{L^1}|u|_{L^2}.$$

Applying (2.12 *a*)–(2.12 *c*), respectively, to the fourth, third, and second terms on the right-hand side above yields

$$|(D^{2\nu}u, u^{p+1})| \leqslant c|D^\nu u|^\theta_{L^2}|D^\nu u|^\beta_{H^1}|u|^\omega_{L^2}, \tag{2.13}$$

where

$$\beta = \frac{2\nu(1-\theta) + 2\nu + 1 + \epsilon_2 + (1+\epsilon_1)(p-1)}{2(1+\nu)}$$

and

$$\omega = 1 + \frac{2(1-\theta) + 1 - \epsilon_2 + (p-1)(2\nu + 1 - \epsilon_1)}{2(\nu+1)}.$$

If for some choice of θ, ϵ_1, and ϵ_2 satisfying $0 < \theta \leqslant 1$, $0 < \epsilon_1 < 1/\nu$, and $0 < \epsilon_2 \leqslant 1$, it transpires that $\beta + \theta = 2$, then Young's inequality may be applied to (2.13) to conclude

$$|(D^{2\nu}u, u^{p+1})| \leqslant \tfrac{1}{2}|D^\nu u|^2_{H^1} + c|D^\nu u|^2_{L^2}|u|^{2\omega/\theta}_{L^2}. \tag{2.14}$$

But, the constraint $\beta + \theta = 2$ is equivalent to

$$2\theta + \epsilon_2 + \epsilon_1(p-1) = 4 - p, \tag{2.15}$$

and for $p = 1$, this obtains by choosing $\theta = \epsilon_2 = 1$, while for $p = 2, 3$ it is clear that the right-hand side of (2.15) is contained in the interior of the range of the left-hand side (namely $(0, 3 + (p-1)/\nu)$). In conjunction with (2.7), inequality (2.14) implies, for ω and θ as above, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{|u(\cdot,t)|^2_{H^1} - \frac{2}{(p+1)(p+2)}\int_R u^{p+2}(x,t)\,\mathrm{d}x\right\} + |D^\nu u|^2_{H^1} \leqslant c|u|^{2\omega/\theta}_{L^2}|D^\nu u|^2_{L^2}.$$

Since $u \in L^\infty(R^+, L^2)$ and $D^\nu u \in L^2(R^+, L^2)$, the right-hand side of this differential inequality is bounded in $L^1(R^+)$ in terms of the L^2-norm of the initial data u_0. Integration over $(0, t)$ then shows that

$$|u(\cdot,t)|^2_{H^1} + \int_0^t |D^\nu u(\cdot,s)|^2_{H^1}\,\mathrm{d}s \leqslant \int_R u^{p+2}(x,s)\,\mathrm{d}x\bigg|_{s=0}^{s=t} + c, \tag{2.16}$$

for some constant c depending on the H^1-norm of the initial data u_0. It remains to bound the integral term on the right-hand side of (2.16). We commence with the inequality

$$\left|\int_R u^{p+2}(x,s)\,\mathrm{d}x\right| \leqslant |u(\cdot,s)|_{L^\infty}^p |u(\cdot,s)|_{L^2}^2$$
$$\leqslant |u_x(\cdot,s)|_{L^2}^{p/2} |u(\cdot,s)|_{L^2}^{2+p/2}$$
$$\leqslant |u(\cdot,s)|_{L^2}^{2+p/2} |u(\cdot,s)|_{H^1}^{p/2}.$$

From the uniform temporal bounds on $|u(\cdot,t)|_{L^2}$, it is thus adduced that

$$\left|\int_R u^{p+2}(x,s)\,\mathrm{d}x\Big|_{s=0}^{s=t}\right| \leqslant c(1+|u(\cdot,t)|_{H^1}^{p/2}),$$

and since $p/2 < 2$ for the values of p under consideration, Young's inequality leads to

$$\left|\int_R u^{p+2}(x,s)\,\mathrm{d}x\Big|_{s=0}^{s=t}\right| \leqslant c + \tfrac{1}{2}|u(\cdot,t)|_{H^1}^2.$$

Substituting the estimate above into (2.16) yields

$$\tfrac{1}{2}|u(\cdot,t)|_{H^1}^2 + \int_0^t |D^\nu u(\cdot,s)|_{H^1}^2\,\mathrm{d}s \leqslant c, \tag{2.17}$$

where the constant c is a polynomial function of $|u_0|_{H^1}$. Since $t > 0$ is arbitrary, it is thereby concluded that $u \in L^\infty(R^+; H^1)$ and $D^\nu u \in L^2(R^+; H^1)$.

Attention is now turned to the case $k = 2$. To establish *a priori* bounds in H^2, we generalize a relation of Kato [17] and Schechter [22]. For notational simplicity, the convention is adopted that $u^n \equiv 0$ if $n < 0$. Take the inner product of (2.1) with the quantity $\{2u + 2u_{xxxx} - \frac{5}{3}(pu^{p-1}u_x^2 - 2(u^p u_x)_x)\}$ and integrate by parts to arrive at the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|_{H^2}^2 - \frac{5}{3}\int_R u^p u_x^2\,\mathrm{d}x\right) + 2|D^\nu u|_{H^2}^2$$
$$= -\tfrac{5}{3}(D^{2\nu}u, pu^{p-1}u_x^2 + 2u^p u_{xx}) + \frac{5p}{3}(u^{2p-1}, u_x^3) + \frac{p(p-1)(p-2)}{12}(u_x^5, u^{p-3}). \tag{2.18}$$

Using the self-adjointness of D^ν and the Schwartz inequality, the first term on the right-hand side of (2.18) may be bounded from above as follows:

$$|(D^{2\nu}u, pu^{p-1}u_x^2 + 2u^p u_{xx})| \leqslant c|D^\nu u|_{L^2}(|D^\nu(u^{p-1}u_x^2)|_{L^2} + |D^\nu(u^p u_{xx})|_{L^2}). \tag{2.19}$$

Applying Liebnitz's rule (2.9) shows that

$$|D^\nu(u^{p-1}u_x^2)|_{L^2} \leqslant c|D^\nu u|_{L^2}|\hat{u}|_{L^1}^{p-2}|\widehat{u_x}|_{L^1}^2 + c|\hat{u}|_{L^1}^{p-1}|\widehat{u_x}|_{L^1}|D^\nu u_x|_{L^2}. \tag{2.20}$$

In light of the embeddings

$$|\hat{u}|_{L^1} + |D^\nu u|_{L^2} \leqslant c|u|_{H^1}$$

and

$$|D^\nu u_x|_{L^2} \leqslant |D^\nu u|_{L^2}^{1/2}|D^\nu u|_{H^2}^{1/2},$$

and the estimate arising from (2.12 *a*) applied to u_x with $\epsilon_1 = \nu < 1/\nu$, namely,

$$|\widehat{u_x}|_{L^1} \leqslant c|u_x|_{L^2}^{1/2}|D^\nu u_x|_{H^1}^{1/2} \leqslant c|u|_{H^1}^{1/2}|D^\nu u|_{H^2}^{1/2},$$

the inequality (2.20) may be reduced to

$$|D^\nu(u^{p-1}u_x^2)|_{L^2} \leqslant c|u|_{H^1}^p|D^\nu u|_{H^2}. \tag{2.21}$$

Similarly, the Liebnitz rule followed by (2.12 *b*) with $\epsilon_2 = 1$ and the simple embedding $|u_{xx}|_{L^2} \leqslant |D^\nu u|_{H^2}$ leads to the inequalities

$$\begin{aligned}|D^\nu(u^p u_{xx})|_{L^2} &\leqslant c|\hat{u}|_{L^1}^{p-1}|\widehat{D^\nu u}|_{L^1}|u_{xx}|_{L^2} + c|\hat{u}|_{L^1}^p|D^\nu u_{xx}|_{L^2}\\ &\leqslant c(|u|_{H^1}^{p-1}|D^\nu u|_{H^1} + |u|_{H^1}^p)|D^\nu u|_{H^2}.\end{aligned} \tag{2.22}$$

Combining (2.21) and (2.22) with (2.19), judiciously bounding $|D^\nu u|_{L^2}$ by either $|u|_{H^1}$ or $|D^\nu u|_{H^1}$, using the previously established result $|u(\cdot,t)|_{H^1} \in L^\infty(R^+)$, and applying Young's inequality leads to the desired majorization of the first term on the right-hand side of (2.18), namely

$$|(D^{2\nu}u, pu^{p-1}u_x^2 + 2u^p u_{xx})| \leqslant c|D^\nu u|_{H^1}^2 + \tfrac{1}{2}|D^\nu u|_{H^2}^2, \tag{2.23}$$

where the constant c in (2.23) depends upon the H^1-norm of the initial data u_0. A simpler series of inequalities leads to the bound

$$|(u^{2p-1}, u_x^3)| \leqslant c|D^\nu u|_{H^1}^2 + \tfrac{1}{4}|D^\nu u|_{H^2}^2, \tag{2.24}$$

on the second term of the right-hand side of (2.18), where again c depends on the H^1-norm of u_0.

Finally, we bound the last term on the right-hand side of (2.18). This term is zero except in the case $p = 3$, when it reduces to $c\int_R u_x^5\,\mathrm{d}x$. The inequalities

$$|u_x|_{L^5}^5 \leqslant c|u_x|_{L^2}^2|u_x|_{L^\infty}^3 \leqslant c|u_x|_{L^2}^{7/2}|u_{xx}|_{L^2}^{3/2},$$

an application of Young's inequality, the uniform temporal bounds on $|u_x|_{L^2}$, and the inequality $|u_{xx}|_{L^2} \leqslant c|D^\nu u|_{H^2}$, yield

$$|u_x|_{L^5}^5 \leqslant c|D^\nu u|_{H^1}^2 + \tfrac{1}{4}|D^\nu u|_{H^2}^2.$$

The latter inequality together with (2.23), (2.24), and (2.18) leads to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|u|_{H^2}^2 - \frac{5}{3}\int_R u^p u_x^2\,\mathrm{d}x\right) + |D^\nu u|_{H^2}^2 \leqslant c|D^\nu u|_{H^1}^2. \tag{2.25}$$

As $D^\nu u \in L^\infty(R^+; H^1)$, integrating (2.25) over the interval $(0,t)$ yields

$$\begin{aligned}|u(\cdot,t)|_{H^2}^2 + \int_0^t |D^\nu u(\cdot,s)|_{H^2}\,\mathrm{d}s &\leqslant c\left(\left|\int_R (u^p u_x^2)(x,s)\,\mathrm{d}x\Big|_{s=0}^{s=t}\right| + 1\right)\\ &\leqslant c(|u(\cdot,s)|_{H^1}^{p+2}|_{s=0}^{s=t} + 1) \leqslant c,\end{aligned} \tag{2.26}$$

where the constants c above depend only upon the H^2-norm of u_0. It is concluded that $u \in L^\infty(R^+; H^2)$ and $D^\nu u \in L^2(R^+, H^2)$, and the case $k = 2$ is complete.

The remainder of the proof of theorem 2.2 is accomplished by an inductive argument. Assume that $k \geqslant 2$ and that $u \in L^\infty(R^+; H^k)$ and $D^\nu u \in L^2(R^+; H^k)$. It will then be shown that $u \in L^\infty(R^+; H^{k+1})$ and $D^\nu u \in L^2(R^+; H^{k+1})$ provided that $u_0 \in H^{k+1}$. Take the L^2-inner product of (2.1) with $u + (-1)^{k+1}\partial_x^{2k+2}u$ and integrate by parts to find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{H^{k+1}} + |D^\nu u|^2_{H^{k+1}} \leqslant c|(\partial_x^{k+2}u^{p+1}, \partial_x^{k+1}u)|$$
$$\leqslant c|D^{k+1}u|^2_{L^2}|u_x|_{L^\infty} + \text{lower-order terms}.$$

From the bound,

$$|D^{k+1}u|_{L^2} \leqslant c|D^k u|^{\nu/(\nu+1)}_{L^2}|D^\nu u|^{1/(\nu+1)}_{H^{k+1}}$$

and Young's inequality, there obtains,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{H^{k+1}} + |D^\nu u|^2_{H^{k+1}} \leqslant c|D^k u|^{2\nu/\nu+1}_{L^2}|D^\nu u|^{2/\nu+1}_{H^{k+1}}|u_x|_{L^\infty} + \text{lower-order terms}$$
$$\leqslant \tfrac{1}{2}|D^\nu u|^2_{H^{k+1}} + c|D^k u|^2_{L^2}|u_x|^{(1+\nu)/\nu}_{L^\infty} + \text{lower-order terms}.$$

But the inductive assumption implies that $|D^k u(\cdot,t)|_{L^2}$ and $|u_x(\cdot,t)|_{L^\infty}$, as well as the lower-order terms, are in $L^\infty(R^+) \cap L^2(R^+)$. Integration over $(0,t)$ for arbitrary $t > 0$ implies $u \in L^\infty(R^+, H^{k+1})$ and $D^\nu u \in L^2(R^+; H^{k+1})$ provided $|u_0|_{H^{k+1}}$ is finite. Theorem 2.2 is established. □

3. Temporal decay rates

The primary subject of this section is the derivation of temporal decay rates for various norms of solutions u of (2.1). The theory subsists on assuming that the initial value u_0 satisfies the hypothesis of theorem 2.2, thereby assuring that the solution u lies in $L^\infty(\boldsymbol{R}^+; H^k)$ for some $k \geqslant 2$. Because $|u(\cdot,t)|_{H^2}$ is uniformly bounded, the arguments needed to settle the case where the nonlinearity $u^p u_x$ has $p > 1$ are quite similar to those that come to the fore for the case $p = 1$. In consequence, the detailed proofs are presented for this latter case, though the theorems will be stated for the range of p for which they apply.

The central tool used here to extract the decay estimates is a pair of complementary orthogonal projections P_δ and Q_δ defined for each $\delta > 0$ by the relation

$$\left.\begin{aligned} \widehat{P_\delta u}(k) &= \chi_{[-\delta,\delta]}(k)\hat{u}(k), \\ \widehat{Q_\delta u}(k) &= (1 - \chi_{[-\delta,\delta]}(k))\hat{u}(k), \end{aligned}\right\} \tag{3.1}$$

where $\chi_{[-\delta,\delta]}$ denotes the characteristic function of the interval $[-\delta,\delta]$. For notational convenience, we write $y = P_\delta u$ and $z = Q_\delta u$. Thus $u = y + z$ where y contains the large wavelength components which are weakly dissipated and z contains the small wavelengths which are more strongly dissipated. The components y and z satisfy the equations

$$\left.\begin{aligned} y_t + My + y_{xxx} + P_\delta(u^p u_x) &= 0, \\ y(x,0) &= P_\delta u_0(x), \end{aligned}\right\} \tag{3.2 a}$$

$$\left.\begin{aligned} z_t + Mz + z_{xxx} + Q_\delta(u^p u_x) &= 0, \\ z(x,0) &= Q_\delta u_0(x). \end{aligned}\right\} \tag{3.2 b}$$

In the sequel y_0 and z_0 will denote $P_\delta u_0$ and $Q_\delta u_0$.

The approach taken to studying the coupled system (3.2) is to obtain pointwise bounds on $|\widehat{y}(k,t)|$ which depend upon $u(\cdot,t)$ only through the temporal decay of $|u(\cdot,t)|_{L^2}$ and to estimate the Sobolev norms $|D^j z(\cdot,t)|_{L^2}$ in terms of δ and t. For fixed values of δ, it is found that there is a waiting time $\tau_j(\delta,u_0)$ such that $|D^j z(\cdot,t)|_{L^2}$ decays slowly for $0<t<\tau_j$, but that for $t>\tau_j$, the dissipative terms dominate nonlinearity in an appropriate sense and optimal decay estimates are obtained. The time τ_j can be interpreted as that required for the nonlinear self-interaction of the short-wavelength terms to be small with respect to dissipation. For times $t>\tau_j$, the evolution of $|D^j z(\cdot,t)|_{L^2}$ is dominated by the linearization of (3.2 b) about y. Exploiting an enhancement of Schwartz's inequality, optimal decay rates on z are obtained. Piecing together the different estimates on y and z yields temporal decay of the Sobolev-norms of u in terms of t and δ; choosing an optimal curve in the t–δ-plane leads to sharp decay rates for u. These results are summarized in theorem 3.7.

The analysis of the high-frequency term z depends sensitively on the value of the dissipation parameter ν. Only the case $0<\nu<\frac{3}{4}$ is treated here, the range $\frac{1}{2}<\nu\leqslant 1$ having been considered elsewhere [8, 11].

The following lemma collects together elementary properties of the orthogonal projections P_δ and Q_δ which will be used frequently in the sequel.

LEMMA 3.1. *For any $s,\delta>0$, we have $P_\delta Q_\delta = Q_\delta P_\delta = 0, P_\delta D^s = D^s P_\delta, Q_\delta D^s = D^s Q_\delta$, and*

$$\left.\begin{aligned} |D^s P_\delta f|_{L^2} &\leqslant \delta^s |P_\delta f|_{L^2}, \\ |Q_\delta f|_{L^2} &\leqslant \delta^{-s}|D^s Q_\delta f|_{L^2}, \\ |P_\delta f|_{L^\infty} &\leqslant \delta^{1/2}|P_\delta f|_{L^2}. \end{aligned}\right\} \tag{3.3}$$

Proof. The first two inequalities are simple applications of Plancherel's theorem, while the third follows from the first and the relation

$$|f|^2_{L^\infty} \leqslant |f|_{L^2}|Df|_{L^2}.$$

This completes the proof. □

We now derive pointwise estimates on the Fourier transform of $y = P_\delta u$. In Fourier transformed variables, (3.2) reads

$$\widehat{y}_t(k,t) + |k|^{2\nu}\widehat{y}(k,t) + \mathrm{i}k^3\widehat{y}(k,t) + \tfrac{1}{2}\mathcal{F}(P_\delta(u^2)_x)(k,t) = 0.$$

Denote complex conjugation by an overbar, multiply the equation above by $\overline{\widehat{y}}$, take the real part, write the nonlinear term as a convolution and bound it by its complex magnitude, thereby obtaining

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}|\widehat{y}(k,t)|^2 + |k|^{2\nu}|\widehat{y}(k,t)|^2 &\leqslant \tfrac{1}{2}|k||(\hat{u}*\hat{u})(k,t)\widehat{y}(k,t)| \\ &\leqslant \tfrac{1}{2}|k||(\hat{u}*\hat{u})(\cdot,t)|_{L^\infty}|\widehat{y}(k,t)| \\ &\leqslant c|k||u(\cdot,t)|^2_{L^2}|\widehat{y}(k,t)|. \end{aligned} \tag{3.4}$$

In the last inequality we made use of the convolution estimate $|f*g|_{L^\infty} \leqslant c|f|_{L^2}|g|_{L^2}$ and Plancherel's theorem. Noting that if $\widehat{y}(k,t)=0$ at some time t, then it is zero for all subsequent times, we may divide (3.4) by $|\widehat{y}(k,t)|$ and integrate the resulting inequality over the interval $(0,t)$, to obtain

$$|\widehat{y}(k,t)| \leqslant \exp[-|k|^{2\nu}t]|\widehat{y}_0(k)| + c|k| \int_0^t \exp[-|k|^{2\nu}(t-s)]|u(\cdot,s)|_{L^2}^2\,\mathrm{d}s. \tag{3.5}$$

We now endeavour to show temporal decay of $|z(\cdot,t)|_{L^2}$, which in conjunction with (3.5) will yield sharp explicit bounds on $|\widehat{y}(k,t)|$. These results set the stage for the more technical estimates on the higher derivatives of z. Take the L^2-inner product of (2.1) with z, use the orthogonality of y and z, and integrate by parts to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{L^2}^2 + |D^\nu z|_{L^2}^2 \leqslant |(uu_x, z)| = |(yy_x + (yz)_x + zz_x, z)|. \tag{3.6}$$

The contribution (zz_x, z), which describes the self-interaction of the high-frequency modes within the nonlinearity, is in fact zero via integration by parts. The derivatives in the mixed term may all be moved onto the variable y, namely,

$$((yz)_x, z) = -\tfrac{1}{2}(y_x, z^2),$$

and with this simplification in hand the integrals on the right-hand side of (3.6) are bounded in obvious ways to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{L^2}^2 + |D^\nu z|_{L^2}^2 \leqslant c(|y_x|_{L^2}|y|_{L^\infty}|z|_{L^2} + |y_x|_{L^\infty}|z|_{L^2}^2). \tag{3.7}$$

In view of the estimates in lemma 3.1, the right-hand side of (3.7) may be bounded as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{L^2}^2 + |D^\nu z|_{L^2}^2 \leqslant c\delta^{3/2}|y|_{L^2}^2|z|_{L^2} + c\delta^{3/2-2\nu}|y|_{L^2}|D^\nu z|_{L^2}^2. \tag{3.8}$$

Since $|y|_{L^2} \leqslant |u|_{L^2} \leqslant |u_0|_{L^2}$, there exists a number $\delta_0 = \delta_0(|u_0|_{H^1}) > 0$ such that for all δ satisfying $0 < \delta < \delta_0$, the last term on the right-hand side of (3.8) may be bounded by $\frac{1}{2}|D^\nu z|_{L^2}^2$, and hence absorbed into the second term on the left-hand side. Employing lemma 3.1 to bound $|D^\nu z|_{L^2}^2$ from below by $\delta^{2\nu}|z|_{L^2}^2$ and dividing by $|z|_{L^2}$ (supposing this to be positive), one arrives at the informative inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|_{L^2} + \tfrac{1}{2}\delta^{2\nu}|z|_{L^2} \leqslant c\delta^{3/2}|y|_{L^2}^2. \tag{3.9}$$

If the continuous function $|z(t)|_{L^2}$ vanishes on an interval, then manifestly (3.9) holds for t in that interval. Thus (3.9) holds for almost every t in the interval of existence of the solution. Integration of (3.9) on the interval $(0,t)$ yields the following estimate on the L^2-norm of z in terms of y and δ, showing exponential convergence to within $\delta^{3/2-2\nu}$ of zero, namely

$$|z(\cdot,t)|_{L^2} \leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}t] + c\delta^{3/2}\int_0^t \exp[-\tfrac{1}{2}\delta^{2\nu}(t-s)]|y(\cdot,s)|_{L^2}^2\,\mathrm{d}s. \tag{3.10}$$

The coupled bounds (3.5) and (3.10) are self-improving in that, if for some constant $\sigma \geqslant 0$ the quantity

$$\sup_{t\geqslant 0}((1+t)^\sigma|u(\cdot,t)|_{L^2}^2) \tag{3.11}$$

is finite, then (3.5) and (3.10) imply the same result with σ replaced by $\tilde{\sigma}$, where

$$\tilde{\sigma} = \min\left\{\frac{1}{2\nu}, \frac{3}{2\nu} - 2 - \epsilon + 2\sigma\right\} \tag{3.12}$$

and $\epsilon > 0$ can be chosen to be arbitrarily small. For ν satisfying $0 < \nu < \frac{3}{4}$, it is readily seen that starting with $\sigma_0 = 0$, then

$$\sigma_{k+1} \geqslant \min\left\{2^k\left(\frac{3}{2\nu} - 2 - \epsilon\right), \frac{1}{2\nu}\right\}$$

and hence after a finite number $n = n(\nu)$ of iterations, it transpires that $\sigma_n = 1/2\nu$. These results are contained in the following proposition.

Proposition 3.2. *If $0 < \nu < \frac{3}{4}$ is given, then for initial data $u_0 \in L^2(R) \cap L^1(R)$, the solution u of (2.1) satisfies*

$$\sup_{t\geqslant 0}((1+t)^{1/2\nu}|u(\cdot,t)|_{L^2}^2) < \infty, \tag{3.13}$$

and, moreover, for any $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that the Fourier transform $\hat{u}$ of u satisfies,

$$|\hat{u}(k,t)| \leqslant \exp[-|k|^{2\nu}t]|\widehat{u_0}(k)| + c_\epsilon |k|^\rho \qquad \textit{for } |k| < \delta, \tag{3.14}$$

where $\rho = \rho(\nu) = \min\{1, 2 - 2\nu - \epsilon\}$ and $\epsilon > 0$ is chosen small enough that $\rho > \frac{1}{2}$. If $p = 2$ or 3, then ρ may be replaced throughout by 1.

Proof. From theorem 2.2 it is known that $u \in L^\infty(R^+; L^2)$, which is equivalent to (3.11) with $\sigma = \sigma_0 = 0$. Assuming (3.11) holds with $\sigma = \sigma_n$ satisfying $0 \leqslant \sigma_n < 1/2\nu$, it remains to verify the recursion relation (3.12). Using Hölder's inequality on the integral term in (3.5) with conjugate exponents $(r(\sigma_n), q(\sigma_n))$ given by

$$(r,q) = \begin{cases} (1,\infty) & \text{if } \sigma_n = 0, \\ \left(\dfrac{1}{1-\sigma_n+\epsilon'}, \dfrac{1}{\sigma_n - \epsilon'}\right) & \text{if } 0 < \sigma_n \leqslant 1, \\ (\infty, 1) & \text{if } 1 < \sigma_n, \end{cases}$$

where $\epsilon' = \epsilon'(\sigma_n)$ need only satisfy $0 < \epsilon' < \sigma_n$, there obtains

$$\begin{aligned} \int_0^t \exp[-|k|^{2\nu}(t-s)]|u(\cdot,s)|_{L^2}^2 \, ds &\leqslant |\exp[-|k|^{2\nu}(t-\cdot)]|_{L^r(0,t)}||u|_{L^2}^2(\cdot)|_{L^q(0,t)} \\ &\leqslant c_\epsilon |k|^{-2\nu/r}. \end{aligned} \tag{3.15}$$

The induction assumption (3.11) was used to bound $||u|_{L^2}^2(\cdot)|_{L^q(0,t)}$ independently of $t > 0$. Substituting this bound back into (3.5) yields the pointwise bounds

$$|\widehat{y}(k,t)| \leqslant \exp[-|k|^{2\nu}t]|\widehat{u_0}(k)| + c_\epsilon |k|^{1-2\nu/r}. \tag{3.16}$$

Integrating (3.16) with respect to k, and estimating in obvious ways gives the uniform bound

$$\begin{aligned} |y|_{L^2}^2 &\leqslant \int_{-\delta}^{\delta} (\exp[-|k|^{2\nu}t]|\widehat{u_0}(k)| + c_\epsilon |k|^{1-2\nu/r})^2 \, dk, \\ &\leqslant c\delta |u_0|_{L^1} + c_\epsilon \delta^{3-4\nu/r}. \end{aligned} \tag{3.17}$$

Alternately, we may bound $|y|_{L^2}^2$ in the t–δ-plane, for $t > 1$ via the estimate

$$|y|_{L^2}^2 \leqslant c_\epsilon \delta^{3-4\nu/r} + c\int_{-\infty}^{\infty} |\hat{u}_0|_{L^\infty} \exp[-2|k|^{2\nu}t]\,\mathrm{d}k \leqslant c_\epsilon\delta^{3-4\nu/r} + c(1+t)^{-1/2\nu}. \tag{3.18}$$

Using the uniform bounds (3.17) in the inequality (3.10) yields

$$\begin{aligned}|z(\cdot,t)|_{L^2} &\leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}t] + c_\epsilon\delta^{3/2}(\delta+\delta^{3-4\nu/r})\int_0^t \exp[-\tfrac{1}{2}\delta^{2\nu}(t-s)]\,\mathrm{d}s,\\ &\leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}t] + c_\epsilon\delta^{3/2-2\nu}(\delta+\delta^{3-4\nu/r}).\end{aligned} \tag{3.19}$$

The relation $|u|_{L^2}^2 = |y|_{L^2}^2 + |z|_{L^2}^2$ together with (3.18) and (3.19) implies that for all δ with $0 < \delta < \delta_0$ (keeping only leading-order terms in δ)

$$|u(\cdot,t)|_{L^2}^2 \leqslant c(\exp[-\delta^{2\nu}t] + (1+t)^{-1/2\nu}) + c_\epsilon\delta^{3-4\nu/r}. \tag{3.20}$$

Observe that on the curve

$$\delta = \delta(t) = (1+t)^{-1/2\nu}(\ln((1+t)^{-(3/2\nu-2/r)}))^{1/2\nu}$$

there obtains the particular bound

$$|u(\cdot,t)|_{L^2}^2 \leqslant c(1+t)^{-1/2\nu} + c_\epsilon((1+t)^{-1}\ln(1+t))^{3/2\nu-2/r}. \tag{3.21}$$

In light of the definition of $r = r(\sigma_n)$ above, (3.21) verifies the recurrence relation (3.12) and (3.13) is thereby established. The formula (3.14) comes from (3.16) evaluated with $r = r(\sigma_n = 1/2\nu)$. □

REMARK. We observe for future reference that (3.14) implies

$$|y(\cdot,t)|_{L^2} \leqslant c\delta^{1/2}, \tag{3.22}$$

while the optimal version of (3.19) for δ satisfying $0 < \delta < \delta_0$ is

$$|z(\cdot,t)|_{L^2} \leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}t] + c\delta^{5/2-2\nu}. \tag{3.23}$$

A referee kindly pointed out that if we were willing to restrict attention to $0 < \nu \leqslant \frac{1}{2}$, a more transparent proof of proposition 3.2 is available. We have preferred to keep the argument presented above because, as the referee is aware, it contributes to our knowledge in the well-studied quasilinear case $\nu > \frac{1}{2}$ by removing the hypothesis of small initial data.

The following result demonstrates that the H^1-norm of u decays at the same temporal rate as the L^2-norm of u. The proof relies on an exploitation of the balance between the conservation law structure in (2.1) and the orthogonal decomposition $u = y + z$. It will be shown in theorem 3.7 that $|u_x(\cdot,t)|_{L^2}$ decays more quickly than $|u(\cdot,t)|_{L^2}$, but the intermediate result below represents a crucial step towards this goal.

PROPOSITION 3.3. *For ν given in the range $0 < \nu < \frac{3}{4}$ and for initial data $u_0 \in H^1(\boldsymbol{R}) \cap L^1(\boldsymbol{R})$, the solution u of (2.1) satisfies*

$$\sup_{t\geqslant 0}((1+t)^{1/2\nu}|u(\cdot,t)|_{H^1}^2) < \infty. \tag{3.24}$$

Proof. Consider the term $|z|_{H^1}$, which in view of (3.22) controls the temporal decay of $|u|_{H^1}$. Take the L^2-inner product of (2.1) with the quantity $z - z_{xx} - Q_\delta u^2/2$ and observe that the obstreperous nonlinear term $(uu_x, Q_\delta(u^2)) = \frac{1}{2}((Q_\delta u^2)_x, Q_\delta u^2)$ vanishes and that there is also a cancellation associated to the balance between dispersion and nonlinearity. There remains the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{H^1}^2 - \frac{1}{2}\int_R z_t u^2\,\mathrm{d}x + |D^\nu z|_{H^1}^2 \leqslant \tfrac{1}{2}|(D^{2\nu}z, u^2)|. \tag{3.25}$$

The interesting term above is the second one on the left-hand side. Since we view δ as independent of t for the present, the decomposition $u_t = y_t + z_t$ holds, and this term may be rewritten as

$$\int_R z_t u^2\,\mathrm{d}x = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_R u^3\,\mathrm{d}x - \int_R y_t u^2\,\mathrm{d}x.$$

Moreover, applying the L^2-norm to the first equation in (3.2) and subsequently using the triangle inequality, lemma 3.1, and (3.22) leads to the inequality

$$\begin{aligned}|y_t|_{L^2} &\leqslant |D^{2\nu}y|_{L^2} + |y_{xxx}|_{L^2} + \tfrac{1}{2}|P_\delta(u^2)_x|_{L^2}\\ &\leqslant (\delta^{2\nu} + \delta^3)|y|_{L^2} + \delta|P_\delta u^2|_{L^2} \leqslant c\delta^{1/2}.\end{aligned} \tag{3.26}$$

The term on the right-hand side of (3.25) is bounded using lemma 2.3, (2.12 *b*) of lemma 2.4 with $\epsilon_2 = 1$, and Young's inequality in the following way:

$$|(D^{2\nu}z, u^2)| \leqslant |D^\nu z|_{L^2}|D^\nu u^2|_{L^2} \leqslant c|D^\nu z|_{L^2}|D^\nu u|_{H^1}|u|_{L^2} \leqslant \tfrac{1}{2}|D^\nu z|_{L^2}^2 + c|u|_{L^2}^2. \tag{3.27}$$

In light of these observations, (3.25) can be extended to the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\tfrac{1}{2}|z|_{H^1}^2 - \frac{1}{6}\int_R u^3\,\mathrm{d}x\right\} + \tfrac{1}{2}|D^\nu z|_{H^1}^2 \leqslant c(1 + |y_t|_{L^\infty})|u|_{L^2}^2, \tag{3.28}$$

and employing the estimates afforded by (3.26), proposition 3.2 and lemma 3.1, there obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}|z|_{H^1}^2 + \delta^{2\nu}|z|_{H^1}^2 \leqslant c(1+t)^{-1/2\nu} + \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int_R u^3\,\mathrm{d}x. \tag{3.29}$$

Integration from 0 to t yields

$$\begin{aligned}|z|_{H^1}^2 &\leqslant \exp[-\delta^{2\nu}t]|z_0|_{H^1}^2\\ &\quad + \int_0^t c\exp[-\delta^{2\nu}(t-s)]\left\{(1+s)^{-1/2\nu} + \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}s}\int_R u^3\,\mathrm{d}x\right\}\mathrm{d}s.\end{aligned} \tag{3.30}$$

There has arisen the need for the following elementary lemma.

LEMMA 3.4. *For any $a, b > 0$ there exists a constant $c > 0$ such that for all δ and t positive satisfying*

$$\left(\frac{2b\log(1+t)}{t}\right)^{1/2a} \leqslant \delta,$$

we have the relation

$$\int_0^t \exp[-\delta^{2a}(t-s)](1+s)^{-b}\,\mathrm{d}s \leqslant c\delta^{-2a}(1+t)^{-b}. \tag{3.31}$$

Proof. For β satisfying $0 < \beta < 1$, divide the integral above into two parts and bound the term $(1+s)^{-b}$ by its sup-norm over each subinterval, thereby obtaining

$$\int_0^t \frac{\exp[-\delta^{2a}(t-s)]}{(1+s)^b}\,\mathrm{d}s \leqslant \int_0^{t(1-\beta)} \exp[-\delta^{2a}(t-s)]\,\mathrm{d}s + \int_{t(1-\beta)}^t \frac{\exp[-\delta^{2a}(t-s)]}{(1+t(1-\beta))^b}\,\mathrm{d}s$$
$$\leqslant \delta^{-2a}\exp[-\delta^{2a}\beta t] + (1+(1-\beta)t)^{-b}\delta^{-2a}.$$

If

$$\beta = \frac{b\log(1+t)}{t\delta^{2a}},$$

then the conditions on t and δ above guarantee that $0 < \beta \leqslant \frac{1}{2}$ and hence

$$\int_0^t \exp[-\delta^{2a}(t-s)](1+s)^{-b}\,\mathrm{d}s \leqslant (1+2^b)\delta^{-2\nu}(1+t)^{-b}.$$

The lemma is established. □

The estimation in (3.30) is now completed. For the last term in the integral, integration by parts, the inequality $|\int_R u^3\,\mathrm{d}x| \leqslant |u|_{L^\infty}|u|_{L^2}^2$, the previously obtained uniform bounds on $|u(\cdot,t)|_{L^\infty}$, and proposition 3.2 lead to

$$\left|\int_0^t \exp[-\delta^{2\nu}(t-s)]\frac{\mathrm{d}}{\mathrm{d}s}\int_R u^3\,\mathrm{d}x\,\mathrm{d}s\right|$$
$$\leqslant c\delta^{2\nu}\int_0^t \frac{\exp[-\delta^{2\nu}(t-s)]}{(1+s)^{1/2\nu}}\,\mathrm{d}s + c((1+t)^{-2\nu} - \exp[-\delta^{2\nu}t]). \quad (3.32)$$

Combining (3.32) with lemma 3.4 and (3.30), we obtain

$$|z|_{H^1}^2 \leqslant c(\exp[-\delta^{2\nu}t] + \delta^{-2\nu}(1+t)^{-1/2\nu}). \quad (3.33)$$

To complete the estimation of $|u|_{H^1}$, observe that

$$|y|_{H^1}^2 = |y|_{L^2}^2 + |y_x|_{L^2}^2 \leqslant (1+\delta^2)|y|_{L^2}^2 \leqslant c|u|_{L^2}^2 \leqslant c(1+t)^{-1/2\nu}. \quad (3.34)$$

Adding together the last two estimates leads to a family of inequalities for $|u(\cdot,t)|_{H^1}$ of the form

$$|u|_{H^1}^2 = |y|_{H^1}^2 + |z|_{H^1}^2 \leqslant c(\exp[-\delta^{2\nu}t] + \delta^{-2\nu}(1+t)^{-1/2\nu}), \quad (3.35)$$

which are valid in the subset of the t–δ-plane defined by the relation

$$\left(\frac{1}{t\nu}\log(1+t)\right)^{1/2\nu} \leqslant \delta.$$

In particular, (3.35) holds for t large enough and $\delta = \delta_0 < 1$, thus proving (3.24). □

The following enhancement of Schwartz's inequality provides a sharp bound on the interaction of the z-modes with the long wavelength y-modes, improving upon the standard Schwartz inequality by the small factor $(\exp[-\delta^{2\nu}t] + \delta^{\rho(\nu)})$.

LEMMA 3.5 (Enhanced Schwartz). *Let $y \in P_\delta L^2(\boldsymbol{R})$ and $z \in Q_\delta L^2(\boldsymbol{R})$ be given. If in addition $\widehat{y}$ satisfies the estimate (3.14), then for all $s \geqslant 0$ the following inequality holds:*

$$|(D^s y^2, z)| \leqslant c\delta^{s+3/2}(\exp[-\delta^{2\nu}t] + \delta^{\rho(\nu)})|z|_{L^2}. \quad (3.36)$$

Proof. By Plancherel's theorem, it is observed that

$$(D^s y^2, z) = \int_{R^2} |k_1+k_2|^s \hat{z}(k_1+k_2)\widehat{y}(k_1)\widehat{y}(k_2)\,\mathrm{d}k_1\,\mathrm{d}k_2.$$

The definition of the orthogonal projections P_δ and Q_δ shows that the latter integral is supported only on the region $\Delta = \{(k_1,k_2) : |k_1| \leqslant \delta, |k_2| \leqslant \delta, |k_1+k_2| \geqslant \delta\}$. Taking absolute values and applying the Schwartz inequality, we have

$$\begin{aligned} &|(D^s y^2, z)| \\ &\quad \leqslant \int_\Delta |\hat{z}(k_1+k_2)||k_1+k_2|^s|\widehat{y}(k_1)||\widehat{y}(k_2)|\,\mathrm{d}k_1\,\mathrm{d}k_2 \\ &\quad \leqslant \biggl(\int_\Delta |\hat{z}(k_1+k_2)|^2|k_1+k_2|^{2s}\,\mathrm{d}k_1\,\mathrm{d}k_2\biggr)^{1/2}\biggl(\int_\Delta |\widehat{y}(k_1)|^2|\widehat{y}(k_2)|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\biggr)^{1/2}. \end{aligned} \tag{3.37}$$

To estimate the first integral on the right-hand side of (3.37), observe that on Δ the bound $|k_1+k_2| \leqslant 2\delta$ holds, and hence

$$\begin{aligned} \biggl(\int_\Delta |\hat{z}(k_1+k_2)|^2|k_1+k_2|^{2s}\,\mathrm{d}k_1\,\mathrm{d}k_2\biggr)^{1/2} &\leqslant c\delta^s\biggl(\int_0^\delta\int_{\delta-k_2}^\delta |\hat{z}(k_1+k_2)|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\biggr)^{1/2} \\ &\leqslant c\delta^s|z|_{L^2}\biggl(\int_0^\delta \mathrm{d}k_2\biggr)^{1/2} \\ &\leqslant c\delta^{s+1/2}|z|_{L^2}. \end{aligned} \tag{3.38}$$

For the second integral, employ the estimate (3.14) for $|\widehat{y}(k,t)|$, the triangle inequality, and the fact that for $(k_1,k_2) \in \Delta$ the sum $|k_1|^{2\nu}+|k_2|^{2\nu} \geqslant |k_1+k_2|^{2\nu} \geqslant \delta^{2\nu}$ to find

$$\begin{aligned} &\biggl(\int_\Delta |\widehat{y}(k_1)|^2|\widehat{y}(k_2)|^2\biggr)^{1/2} \\ &\qquad \leqslant c\biggl(\int_\Delta \exp[-2(|k_1|^{2\nu}+|k_2|^{2\nu})t]\biggr)^{1/2} \\ &\qquad\quad + 2c_\epsilon\biggl(\int_\Delta \exp[-2|k_1|^{2\nu}t]|k_2|^{2\rho}\biggr)^{1/2} + c_\epsilon\biggl(\int_\Delta |k_1k_2|^{2\rho}\biggr)^{1/2} \\ &\qquad \leqslant c\exp[-\delta^{2\nu}t]\biggl(\int_\Delta 1\biggr)^{1/2} + c_\epsilon\biggl(\int_\Delta |k_2|^{2\rho}\biggr)^{1/2} + c_\epsilon\biggl(\int_\Delta |k_1k_2|^{2\rho}\biggr)^{1/2} \\ &\qquad \leqslant c\delta\exp[-\delta^{2\nu}t] + c_\epsilon\delta^{\rho+1}. \end{aligned} \tag{3.39}$$

Taken together, the inequalities (3.37)–(3.39) imply (3.36). □

The following bounds on $|z|_{H^k}$ are useful in the sequel.

PROPOSITION 3.6. *Let $u_0 \in H^k$ for some $k \geqslant 2$. There exist constants $t_0 = t_0(|u_0|_{H^k})$ and $\delta_0 = \delta_0(|u_0|_{H^k})$ such that for all $t \geqslant t_0$ and $0 < \delta < \delta_0$,*

$$|z(\cdot,t)|_{H^k} \leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}(t-t_0)] + c\delta^{5/2+\rho(\nu)-2\nu}, \tag{3.40}$$

where ρ is as given in proposition 3.2.

Proof. Taking the L^2-inner product of (2.1) with the quantity $z + D^{2k}z$ and integrating by parts, there results,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{H^k}^2 + |D^\nu z|_{H^k}^2 \leqslant |(uu_x, z)| + |(uu_x, D^{2k}z)|. \tag{3.41}$$

Expand u as $y + z$ in the first term on the right-hand side of (3.41), apply the enhanced Schwartz inequality (3.36) and inequality (3.22) to obtain

$$\begin{aligned} |(uu_x, z)| &\leqslant c|(yy_x, z)| + c|y_x|_{L^\infty}|z|_{L^2}^2 \\ &\leqslant c\delta^{5/2}(\exp[-\delta^{2\nu}t] + \delta^\rho)|z|_{H^k} + c\delta^{2-2\nu}|D^\nu z|_{H^k}^2. \end{aligned} \tag{3.42}$$

Consequently, there is a $\delta_0 = \delta_0(|u_0|_{H^k})$ such that for $\delta \leqslant \delta_0$, we have

$$|(uu_x, z)| \leqslant c\delta^{5/2}(\exp[-\delta^{2\nu}t] + \delta^\rho)|z|_{H^k} + \tfrac{1}{4}|D^\nu z|_{H^k}^2. \tag{3.43}$$

For the term $|(uu_x, D^{2k}z)|$, we consider only the case $k = 2$, as the cases $k > 2$ require similar estimates and an inductive argument. Expand u as $y + z$, collect terms linear, quadratic and cubic in z, integrate by parts, and apply Schwartz's inequality and lemma 3.1, to arrive at

$$\begin{aligned} |(uu_x, D^4z)| &\leqslant c\delta^{11/2}|y|_{L^2}^2|z|_{L^2} \\ &\quad + c\sum_{i=1}^{3}\delta^{i+1/2}|y|_{L^2}|D^{3-i}z|_{L^2}|D^2z|_{L^2} + c|Dz|_{L^\infty}|D^2z|_{L^2}^2. \end{aligned} \tag{3.44}$$

Using the inequality $|D^iz|_{L^2} \leqslant |D^\nu z|_{H^2}$ valid for $i = 1, 2$ when $0 \leqslant \nu \leqslant 1$, as well as the inequality (3.22) and the temporal bounds

$$|Dz|_{L^\infty} \leqslant c|Dz|_{L^2}^{1/2}|D^2z|_{L^2}^{1/2} \leqslant c(1+t)^{-1/8\nu}$$

afforded by proposition 3.3 and theorem 2.2, the expression (3.44) above reduces to

$$|(uu_x, D^4z)| \leqslant c\delta^{13/2}|z|_{H^2} + c(\delta^2 + (1+t)^{-1/8\nu})|D^\nu z|_{H^2}^2. \tag{3.45}$$

For $t \geqslant t_0 = t_0(|u_0|_{H^1})$ and $\delta \leqslant \delta_0(|u_0|_{H^2})$, we have $c((1+t)^{-1/8\nu} + \delta^2) \leqslant \frac{1}{4}$, say, and hence

$$|(uu_x, D^4z)| \leqslant c\delta^{13/2}|z|_{H^2} + \tfrac{1}{4}|D^\nu z|_{H^2}^2. \tag{3.46}$$

Combining (3.41), (3.43), and (3.45), there obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|z|_{H^k}^2 + \tfrac{1}{2}|D^\nu z|_{H^2}^2 \leqslant c\delta^{5/2}(\exp[-\delta^{2\nu}t] + \delta^\rho)|z|_{H^2}. \tag{3.47}$$

Applying lemma 3.1 to bound $|D^\nu z|_{H^2}^2$ from below by $\delta^{2\nu}|z|_{H^2}^2$ in (3.47), dividing by $|z|_{H^2}$, and integrating from t_0 to t leads to

$$\begin{aligned} |z(\cdot, t)|_{H^2} &\leqslant c(1 + t\delta^{5/2})\exp[-\delta^{2\nu}(t - t_0)] + c\delta^{5/2+\rho-2\nu}, \\ &\leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}(t - t_0)](\sup_{t>t_0}(1 + t\delta^{5/2})\exp[-\tfrac{1}{2}\delta^{2\nu}(t - t_0)]) + \delta^{5/2+\rho-2\nu}. \end{aligned} \tag{3.48}$$

A simple calculation shows

$$\sup_{t>0}\{(1 + t\delta^{5/2})\exp[-\tfrac{1}{2}\delta^{2\nu}(t - t_0)]\} \leqslant c(1 + \delta^{5/2-2\nu}) \leqslant c$$

and the result (3.40) follows. □

We are now prepared to state and prove our principal decay estimate.

THEOREM 3.7. *Let $0 < \nu < \frac{3}{4}$ be given. For initial data $u_0 \in H^k(\boldsymbol{R}) \cap L^1(\boldsymbol{R})$, for some $k \geqslant 2$, the solution u of (2.1) satisfies,*

$$\sup_{t\geqslant 0}((1+t)^{(2j+1)/2\nu}|D^j u(\cdot,t)|_{L^2}^2) < \infty, \qquad \text{for } j = 0,\dots,k. \tag{3.49}$$

Moreover, for any $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ and for all $0 < \delta < \delta_0(|u_0|_{H^k})$ there are times $\tau_j = \tau_j(\delta)$, $j = 0,\dots,k$ such that for $t \geqslant \tau_j$, the projection $z = Q_\delta u$ satisfies

$$|D^j z(\cdot,t)|_{L^2} \leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}(t-\tau_j)] + c_\epsilon \delta^{j+\gamma(\nu,\epsilon)}, \qquad \text{for } j = 0,\dots,k. \tag{3.50}$$

The times τ_j are given by $\tau_j(\delta) = t_0 - c_j\delta^{-2\nu}\log(\delta)$, where t_0 and c_j depend only upon $|u_0|_{H^j}$, and the exponent γ is $\gamma(\nu,\epsilon) = \frac{5}{2} + \rho(\nu,\epsilon) - 2\nu$, where ρ is as given in proposition 3.2.

Proof. Fix $k \geqslant 2$. Inequality (3.50) is proven first, and inequality (3.49) follows from (3.50) and (3.22). For $j = 0$, return to (3.6) and follow the subsequent arguments with the exception of using the enhanced Schwartz inequality to bound $|(yy_x, z)|$. In place of (3.9), there obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}|z(\cdot,t)|_{L^2} + \tfrac{1}{2}\delta^{2\nu}|z(\cdot,t)|_{L^2} \leqslant c\delta^{5/2}(c\exp[-\delta^{2\nu}t] + c_\epsilon\delta^\rho). \tag{3.51}$$

Define $\tau_0 = -\delta^{-1/2\nu}\rho\log(\delta)$, so that for $t \geqslant \tau_0$, one has $\exp[-\delta^{2\nu}t] \leqslant \delta^\rho$. Integrating (3.51) from τ_0 to t yields

$$|z(\cdot,t)|_{L^2} \leqslant c\exp[-\tfrac{1}{2}\delta^{2\nu}(t-\tau_0)] + c_\epsilon\delta^\gamma, \tag{3.52}$$

which is exactly (3.50) for $j = 0$. Arguing inductively, assume (3.50) is valid for $j < n$ and aim to prove it for $j = n$. Taking the L^2-inner product of the nth spatial derivative of (2.1) with the quantity $D^n z$ and integrating by parts, there results the equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|D^n z|_{L^2}^2 + |D^{n+\nu}z|_{L^2}^2 = \tfrac{1}{2}(D^{n+1}u^2, D^n z). \tag{3.53}$$

Write $u = y + z$ on the right-hand side of (3.53), distribute the derivatives, and collect terms linear, quadratic, and cubic in z to reach the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}|D^n z|_{L^2}^2 + |D^{n+\nu}z|_{L^2}^2 \leqslant c\Big(|(D^{n+1}y^2, D^n z)| + \sum_{j=1}^{n+1}|(D^j y D^{n+1-j}z, D^n z)| + \sum_{j=1}^{[n/2]}|(D^j z D^{n+1-j}z, D^n z)|\Big), \tag{3.54}$$

where $[x]$ represents the least integer greater than x. Apply the enhanced Schwartz inequality (3.36) and lemma 3.1 to the linear and quadratic terms of (3.54), respectively, to deduce

$$\left.\begin{aligned} |(D^{n+1}y^2, D^n z)| &\leqslant c\delta^{n+5/2}(\exp[-\delta^{2\nu}t] + c_\epsilon\delta^\rho)|D^n z|_{L^2}, \\ |((D^j y)(D^{n+1-j}z), D^n z)| &\leqslant |D^j y|_{L^\infty}|D^{n+1-j}z|_{L^2}|D^n z|_{L^2} \\ &\leqslant c\delta^{2-2\nu}|D^{n+\nu}z|_{L^2}^2. \end{aligned}\right\} \tag{3.55}$$

For the cubic terms, there are two cases. When $n > [n/2] + 1$ (that is $n \geqslant 4$), proceed as follows:

$$\begin{aligned}|((D^j z)(D^{n+1-j} z), D^n z)| &\leqslant c|D^j z|_{L^\infty} |D^{n+1-j} z|_{L^2} |D^n z|_{L^2} \\ &\leqslant c\delta^{-(2\nu+j-1)}(|D^j z|_{L^2} |D^{j+1} z|_{L^2})^{1/2} |D^{n+\nu} z|_{L^2}^2. \quad (3.56)\end{aligned}$$

Since $j \leqslant [n/2] < n-1$, the inductive hypothesis (3.50) holds at orders j and $j+1$ and for $t > \tau_n = \tau_{[n/2]+1} - ([n/2]+1)\delta^{-2\nu} \log \delta$, one has $\exp[-\delta^{2\nu}(t-\tau_l)] \leqslant c\delta^{l+\gamma}$ for $l = j, j+1$. It follows that

$$(|D^j z|_{L^2} |D^{j+1} z|_{L^2})^{1/2} \leqslant c_\epsilon \delta^{j+1/2+\gamma}, \tag{3.57}$$

which in light of (3.56) implies

$$|((D^j z)(D^{n+1-j} z), D^n z)| \leqslant c\delta^{\gamma+3/2-2\nu} |D^{n+\nu} z|_{L^2}^2. \tag{3.58}$$

Thus for δ small enough, the cubic and quadratic terms in (3.54) may be absorbed into the term $|D^{n+\nu} z|_{L^2}^2$ on the right-hand side, thereby obtaining the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} |D^n z|_{L^2} + \tfrac{1}{2}\delta^{2\nu} |D^n z|_{L^2} \leqslant c\delta^{n+5/2}(\exp[-\delta^{2\nu} t] + c_\epsilon \delta^\rho), \tag{3.59}$$

valid for $t \geqslant \tau_n$. For the special cases $n = 1, 2$, and 3, the quadratic terms can be handled as above by taking δ small enough. After integration by parts the cubic terms reduce, in each of the three cases at hand, to a single term of the form $c|(DzD^n z, D^n z)|$. This may be estimated using Schwartz's inequality, the bound $|Dz|_{L^\infty} \leqslant (|Dz|_{L^2} |D^2 z|_{L^2})^{1/2}$, and proposition 3.6, resulting in

$$\begin{aligned}|(DzD^n z, D^n z)| &\leqslant c\delta^{-2\nu} |Dz|_{L^\infty} |D^{n+\nu} z|_{L^2}^2 \\ &\leqslant c\delta^{-2\nu}(\exp[-\tfrac{1}{2}\delta^{2\nu}(t-t_0)] + c_\epsilon \delta^{2\rho+1/2}) |D^{n+\nu} z|_{L^2}^2. \quad (3.60)\end{aligned}$$

Thus for $t > \tau_n = t_0 - 2\delta^{-2\nu}(2\rho + \frac{1}{2}) \log(\delta)$, it is adduced that

$$\exp[-\tfrac{1}{2}\delta^{2\nu}(t-t_0)] \leqslant c\delta^{2\rho+1/2},$$

and the analysis may proceed as in the previous case, for δ small enough, by absorbing the cubic terms into the dissipative term $|D^{n+\nu} z|_{L^2}^2$ on the left-hand side of (3.54).

Returning to equation (3.59), observe that in all cases $\exp[-\delta^{2\nu} t] \leqslant c\delta^\rho$ for $t > \tau_n$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} |D^n z|_{L^2} + \tfrac{1}{2}\delta^{2\nu} |D^n z|_{L^2} \leqslant c_\epsilon \delta^{n+5/2+\rho}. \tag{3.61}$$

Integration from τ_n to $t > \tau_n$ yields exactly (3.50).

The temporal decay of $|D^j u(\cdot, t)|_{L^2}$, follows readily from the identity $|D^j u|_{L^2}^2 = |D^j y|_{L^2}^2 + |D^j z|_{L^2}^2$, and the estimates on y and z previously derived. Applying Parseval's formula and (3.14), it transpires that

$$|D^j y(\cdot, t)|_{L^2}^2 \leqslant c \int_{-\infty}^{\infty} |k|^{2j} \exp[-2|k|^{2\nu} t] \, \mathrm{d}k + c_\epsilon \int_{-\delta}^{\delta} |k|^{2j+2\rho}, \tag{3.62}$$

which, arguing as in (2.5), can be extended to

$$|D^j y(\cdot, t)|_{L^2}^2 \leqslant c(1+t)^{-(2j+1)/2\nu} + c_\epsilon \delta^{2j+2\rho+1}. \tag{3.63}$$

Combining this estimate and (3.50) leads to the bound

$$|D^j u(\cdot,t)|_{L^2}^2 \leqslant c(1+t)^{-(2j+1)/2\nu} + c_\epsilon \delta^{2j+2\rho+1} + c(\exp[-\tfrac{1}{2}\delta^{2\nu}(t-\tau_j)] + c_\epsilon \delta^{j+\gamma})^2. \tag{3.64}$$

For times t satisfying

$$t \geqslant \tau_j(\delta) - 2\delta^{-2\nu}(j+\gamma)\log(\delta), \tag{3.65}$$

it thus transpires that

$$|D^j u(\cdot,t)|_{L^2}^2 \leqslant c(1+t)^{-(2j+1)/2\nu} + c_\epsilon \delta^{2j+2\rho+1} + c_\epsilon \delta^{2j+2\gamma}. \tag{3.66}$$

Denote by $\beta_j = \beta_j(\epsilon)$ the quantity

$$\max\left\{\frac{2j+1}{2j+2\rho+1}, \frac{2j+1}{2j+2\gamma}\right\}$$

and observe that $\rho > \frac{1}{2}$ implies $\beta_j < 1$, so the curve $\delta = (1+t)^{-\beta_j/2\nu}$ lies within the region of the t–δ-plane defined by (3.65) as long as $t \geqslant t_0 + c(1+t)^{\beta_j}\log(1+t)$, i.e. for t large enough depending only upon ν, the order j of the derivative, and the norm $|u_0|_{H^j}$. On this curve the bounds (3.66) reduce to

$$|D^j u(\cdot,t)|_{L^2}^2 \leqslant c'(1+t)^{-(2j+1)/2\nu}. \tag{3.67}$$

Since $u \in L^\infty(R^+; H^j)$, the supremum in (3.49) has force only for t large enough, and the proof of (3.49) is complete. □

The following is an immediate consequence of theorem 3.7.

COROLLARY 3.8. *With the same hypothesis as those appearing in the statement of theorem 3.7, the solution u of (2.1) satisfies*

$$\sup_{t\geqslant 0}((1+t)^{(j+1)/2\nu}|D^j u(\cdot,t)|_{L^\infty}) < \infty, \qquad \text{for } j = 0,\dots,k-1. \tag{3.68}$$

4. Universal asymptotics

Theorems 2.1, 2.2 and 3.7 show that the nonlinear equation (2.1) and the linearized version (2.3) share the same regularity and asymptotic decay rates. It is now shown that these decay rates are sharp, and in fact the two equations share the same universal, self-similar asymptotic form governing the final stages of their decay. To facilitate the discussion, recall the asymptotic form f^* given in Fourier transformed variables as

$$\widehat{f^*}(k) = \exp[-|k|^{2\nu}]. \tag{4.1}$$

More specifically, for initial data u_0 from the weighted space $L^1(q)$, where $q(x) = \sqrt{1+x^2}$, with non-zero total mass $A = \int_{-\infty}^{\infty} u_0(x)\,dx = \widehat{u_0}(0)$, the associated solution u of (2.1) decays at exactly the rate prescribed by theorems 2.1 and 3.7. Moreover, the long-time asymptotics of the solutions are given, to leading order, by the universal asymptotic indicated below, which depends on the initial data only through the single parameter A.

The next proposition summarizes the result in view for the linearized problem (2.3).

PROPOSITION 4.1. *Given ν satisfying $0 < \nu \leqslant 1$, an integer $k \geqslant 2$, and initial data $u_0 \in H^k \cap L^1(q)$, there are positive constants $c_1, \dots, c_k$ depending only upon $|u_0|_{L^1(q)}$ such that the solution v of (2.3) corresponding to initial data u_0 satisfies*

$$\left| D^j \left\{ v(\cdot, t) - \frac{A}{t^{1/2\nu}} f^* \left(\frac{\cdot}{t^{1/2\nu}} \right) \right\} \right|_{L^2} \leqslant c_j (1+t)^{-(3+2j)/4\nu}, \qquad \textit{for } j = 0, \dots, k, \tag{4.2}$$

where A denotes the total mass of the initial data.

Proof. Observe that if $u_0 \in L^1(q)$, then $xu_0(x) \in L^1$, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}k} \widehat{u_0} \in L^\infty.$$

As a result, the following inequalities hold:

$$|\widehat{u_0}(k) - \widehat{u_0}(0)| \leqslant |k| |\frac{\mathrm{d}}{\mathrm{d}k} \widehat{u_0}|_{L^\infty} \leqslant c|k| |u_0|_{L^1(q)}. \tag{4.3}$$

Recalling that the solution of (2.3) is given in Fourier transformed variables by

$$\hat{v}(k, t) = \exp[-(|k|^{2\nu} + \mathrm{i}k^3)t] \widehat{u_0}(k), \tag{4.4}$$

one may write

$$\begin{aligned} \hat{v}(k, t) = A \exp[-|k|^{2\nu} t] + A(\exp[-(|k|^{2\nu} + \mathrm{i}k^3)t] - \exp[-|k|^{2\nu} t]) \\ + \exp[-(|k|^{2\nu} + \mathrm{i}k^3)t](\widehat{u_0}(k) - A). \end{aligned}$$

Subtract the quantity $A \exp[-|k|^{2\nu} t]$ from both sides, take the absolute value of the resulting expressions, and employ the estimate (4.3), thereby deriving the inequalities

$$\begin{aligned} |\hat{v}(k, t) - A \exp[-|k|^{2\nu} t]| &\leqslant (A|1 - \exp[-\mathrm{i}k^3 t]| + |k| |u_0|_{L^1(q)}) \exp[-|k|^{2\nu} t] \\ &\leqslant c(|k|^3 t + |k|) \exp[-|k|^{2\nu} t]. \end{aligned} \tag{4.5}$$

Multiply (4.5) by the quantity $|k|^j$ and take the L^2-norm of both sides. Following the arguments exposed in theorem 2.1 leads to (4.2). Note that the inverse Fourier transform of $A \exp[-|k|^{2\nu} t]$ is exactly the self-similar rescaling of f^* which appears on the left-hand side of (4.2). □

It is now demonstrated that $|D^j(u-v)|_{L^2}$ decays more quickly than either $|D^j u|_{L^2}$ or $|D^j v|_{L^2}$, and hence a simple application of the triangle inequality implies that u possesses the same universal asymptotics as evidenced by v in (4.2). These results are contained in the following theorem.

THEOREM 4.2. *Let the parameters p and ν of equation (2.1) satisfy $p = 1$ and $0 < \nu < \frac{3}{4}$ and let $\epsilon > 0$ be small. Then for initial data u_0 lying in the space $H^k \cap L^1(q)$ with q as above and $k \geqslant 2$, there are positive constants $c_1(\epsilon), \dots, c_k(\epsilon)$ such that the solution u of (2.1) has the universal asymptotic behaviour*

$$\left| D^j \left\{ u(\cdot, t) - \frac{A}{t^{1/2\nu}} f^* \left(\frac{\cdot}{t^{1/2\nu}} \right) \right\} \right|_{L^2} \leqslant c_j(\epsilon) \left(\frac{(1+t)}{\ln(1+t)} \right)^{-(2j+1+2\rho(\epsilon))/4\nu}, \qquad \textit{for } j = 0, \dots, k. \tag{4.6}$$

Here the universal form f^* is given by (4.1), the constant A denotes the total mass of the initial data, and the exponent $\rho(\epsilon)$ is as given in proposition 3.2. In the case $p = 2$ or 3 the same result holds with the exponent ρ replaced by 1.

Proof. The function u satisfies (2.1) with initial data u_0, while v satisfies (2.3) with the same initial data. It is natural to introduce the quantity $w = u - v$, which is a solution of

$$\left.\begin{aligned} w_t + Mw + w_{xxx} &= -u^p u_x, \qquad x \in R, \quad t > 0, \\ w(x,0) &= 0, \qquad x \in R. \end{aligned}\right\} \tag{4.7}$$

Fixing our attention on the case $p = 1$ for definiteness, the nonlinear term uu_x may be regarded as a forcing term, and w may be solved for explicitly in the form

$$\hat{w}(k,t) = -\frac{ik}{2}\int_0^t \exp[-(t-s)(|k|^{2\nu} + ik^3)]\hat{u} * \hat{u}(k,s)\,ds. \tag{4.8}$$

The convolution inequality $|f * g|_{L^\infty} \leqslant |f|_{L^2}|g|_{L^2}$ applied to $\hat{u} * \hat{u}$ in the integral above, the decay estimates (3.13), and Hölder's inequality applied as in the arguments preceding (3.15) yields, for any $\epsilon > 0$,

$$|\hat{w}(k,t)| \leqslant c_\epsilon |k|^\rho, \tag{4.9}$$

where $\rho = \rho(\nu, \epsilon)$ is as given in proposition 3.2. This readily provides a bound on $|D^j P_\delta w|_{L^2}$ via lemma 3.1, namely

$$|D^j P_\delta w|_{L^2}^2 = \int_{-\delta}^{\delta} |k|^{2j} |\hat{w}(k,t)|^2\,dk \leqslant c_\epsilon \delta^{2j+2\rho+1}. \tag{4.10}$$

To bound the Sobolev-norms $|D^j Q_\delta w|_{L^2}$, use the triangle inequality to conclude

$$|D^j Q_\delta w|_{L^2} = |D^j Q_\delta(u - v)|_{L^2} \leqslant |D^j z|_{L^2} + |D^j Q_\delta v|_{L^2}.$$

It is easily seen using (4.4) that

$$|D^j Q_\delta v|_{L^2} \leqslant c\exp[-\delta^{2\nu} t],$$

where the constant c depends only upon the quantity $|u_0|_{H^j}$. This estimate and the bounds on z afforded by (3.50) of theorem 3.7 imply

$$|D^j Q_\delta w|_{L^2} \leqslant c\exp[-\delta^{2\nu}(t - \tau_j)] + c_\epsilon \delta^{j+\gamma}. \tag{4.11}$$

Since $\rho + \frac{1}{2} < \gamma$, to leading order in δ, $|D^j w|_{L^2}$ may be bounded in the manner

$$|D^j w|_{L^2} \leqslant |D^j P_\delta w|_{L^2} + |D^j Q_\delta w|_{L^2} \leqslant c_\epsilon \delta^{j+\rho+1/2} + c\exp[-\delta^{2\nu}(t - \tau_j)], \tag{4.12}$$

valid for δ small enough and $t > \tau_j$. Writing $t = t_+ + \tau_j(\delta)$, and viewing t_+ as a fixed quantity, we optimize (4.12) by choosing

$$\delta = t_+^{-1/2\nu}\left(\frac{j + \rho + 1/2}{2\nu}\log(1 + t_+)\right)^{1/2\nu},$$

so obtaining

$$|D^j w|_{L^2} \leqslant c_j(\epsilon)\left(\frac{\log(1 + t_+)}{1 + t_+}\right)^{(j+\rho+1/2)/2\nu}. \tag{4.13}$$

For this choice of δ,

$$\tau_j = t_0 + c\frac{t_+}{\log(1+t_+)}\log\left(\frac{1+t_+}{\log(1+t_+)}\right) \leqslant t_0 + ct_+.$$

That is, $1/t_+ = O(1/t)$ and hence t_+ may be replaced with t in (4.13) so long as the constant $c_j(\epsilon)$ is properly adjusted. The triangle inequality implies that

$$\left|D^j\left\{u(\cdot,t) - \frac{A}{t^{1/2\nu}}f^*\left(\frac{\cdot}{t^{1/2\nu}}\right)\right\}\right|_{L^2} \leqslant |D^j w(\cdot,t)|_{L^2} + \left|D^j\left\{v(\cdot,t) - \frac{A}{t^{1/2\nu}}f^*\left(\frac{\cdot}{t^{1/2\nu}}\right)\right\}\right|_{L^2}, \quad (4.14)$$

and (4.6) then follows from (4.13) and (4.2). This completes the proof in the case $p = 1$; the cases $p = 2, 3$ are similar and omitted. □

5. Discussion

Introducing a family of orthogonal Fourier projections, we have shown them to be an effective tool in the analysis of the long-time evolution of model equations of the form (2.1). This methodology facilitates the extraction of dispersive smoothing effects and provides a natural mathematical truncation which reduces the short-wavelength component of the flow to a linear regime, forced by the long wavelengths. The asymptotic results of theorem 4.2 can be extended to include higher-order terms. Moreover, the foregoing analysis does not rely upon a small-data assumption, lending currency to the idea that a hybrid matched asymptotic could be developed connecting short-time perturbative formulae to the long-time asymptotic results presented here.

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