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SIMILARITY SOLUTIONS OF THE GENERALIZED KORTEWEG-DE VRIES EQUATIONS

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ABSTRACT. Numerical simulations and group invariance considerations point to the existence of similarity solutions of the form

$$U(x,t) = \frac{1}{(t_* - t)^{2/3p}} \psi\left(\frac{x_* - x - c(t_* - t)^{1/3}}{(t_* - t)^{1/3}} \right) \tag{*}$$

of the generalized Korteweg-de Vries equation

$$U_t + U^p U_x + U_{xxx} = 0. \tag{**}$$

Here, x_*, t_* and c are real parameters, x and t are real variables with $t \neq t_*$, p is a positive integer and interest is focussed on the case where $p \geq 4$ for which solutions of the initial-value problem for (**) are not known to be always globally defined. It is shown that smooth solutions of (**) of the form appearing in (*) do indeed exist. Some detailed properties of the function ψ appearing in (*) are also obtained.

1. Introduction. Numerical simulation by high-speed computers of solutions of the generalized Korteweg-de Vries equation

$$U_t + U^p U_x + U_{xxx} = 0, \tag{1.1}$$

where p is a positive integer and subscripts connote partial differentiation, indicate that for $p \geq 4$, the initial-value problem is not necessarily globally well posed for

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other than small initial data. It is our purpose here to examine one aspect that arises from attempts to explain this apparent lack of global well-posedness.

Equation (1.1) with $p = 1$ is the well known Korteweg-de Vries equation (KdV-equation henceforth) written in scaled variables in a travelling frame of reference, that arose last century as a model for surface water waves. In more recent times, it has arisen as a model for waves in a variety of nonlinear dispersive media (cf. Benjamin 1974, Newell 1985). The case $p = 2$, the so-called modified KdV equation, also arises in physically interesting situations, for instance in modelling waves in a crystalline lattice. Larger values of p could arise in principle as indicated in Benjamin *et al.* (1972, §2), though this eventuality seems unlikely in practice. However, models of the form

$$U_t + U^p U_x - LU_x = 0 \tag{1.1a}$$

where L is a Fourier multiplier operator with symbol $\gamma(\xi)$, say, and $p = 1$ or 2 come to the fore in a wide range of physically interesting situations. The operator L given by $\widehat{LV}(\xi) = \gamma(\xi)\widehat{V}(\xi)$ is related to the linearized dispersion relation governing infinitesimal waves in the system in question, as explained by Benjamin *et al.* (1972, §2). In applications, γ is often not a polynomial (see Abdelouhab *et al.* 1989), and in such cases L is a non-local operator and various aspects of the analysis of (1.1a) are more challenging. Many of the dispersion relations γ that come up in practice have $\gamma(\xi) \sim |\xi|^\beta$ as $|\xi| \to \infty$ for some $\beta < 2$. In attempting to understand more fully the interaction between nonlinearity and dispersion, it has been tempting in regard to many theoretical aspects to shift attention from (1.1a) where the nonlinearity is fixed ($p = 1$, say) and the dispersion γ varies from problem to problem, to (1.1) where the dispersion is fixed ($\beta = 2$) and represented by a local operator and the nonlinearity has variable strength depending on p. The expectation, which is borne out in some respects (see Bona *et al.* 1987 for example), is that intuition gleaned from the study of (1.1) will aid in comprehending (1.1a).

The mathematical context in which our results acquire interest is now explained. Kato (1975, 1979, 1983) has shown that for all positive integers p, the initial-value problem for (1.1) wherein one specifies

$$U(x,0) = U_0(x) \tag{1.2}$$

for $x \in \mathbb{R}$, say, is locally well posed provided U_0 is smooth enough and decays suitably to zero at $\pm\infty$. That is, corresponding to any such U_0, there is a $T > 0$ depending on U_0 and a solution U of (1.1) satisfying (1.2) which is defined at least for $(x,t) \in \mathbb{R} \times (-T,T)$. Kato further showed, as did Schechter (1978) and Strauss (1974), that $T = +\infty$ if $p < 4$ or if $p \geq 4$ and U_0 is small in appropriate senses (if $p = 4$, U_0 not too large in L_2-norm while if $p > 4$, U_0 not too large in H^1-norm suffices). It remains open whether or not (1.1)–(1.2) is globally well posed for $p \geq 4$ and initial data unrestricted in size, though one sees easily that a solution of (1.1)–(1.2) may be extended to the unbounded range $0 \leq t < +\infty$ if for some $q > p/2$, its L_q-norm remains bounded on bounded temporal intervals (cf. Albert *et al.* 1988). In particular, a solution remains smooth and various Sobolev-space norms remain bounded on bounded time intervals if it remains uniformly bounded on $\mathbb{R} \times [0,T]$ for any finite T.

These observations together with what appear to be related questions connected with the instability of the solitary-wave solutions of (1.1) (see Bona *et al.* 1987, Pego & Weinstein 1993) led to the initiation of detailed numerical simulations of solutions of (1.1)–(1.2), (see Bona *et al.* 1986). These experiments, which featured high-order numerical schemes with weak stability limitations and having adaptive refinement in both the spatial and the temporal variable, show clear signs that the L_∞-norm of certain solutions of (1.1) corresponding to order-one-sized initial data do not remain bounded, but instead become infinite in finite time. More refined numerical studies (Bona *et al.* 1995, 1996) have indicated that the blow-up of solutions has a similarity form, namely

$$U(x,t) = \frac{1}{(t_* - t)^{2/3p}} \Psi\left(\frac{x_* - x - c(t_* - t)^{1/3}}{(t_* - t)^{1/3}}\right) + \text{ bounded term}, \tag{1.3}$$

where x_*, t_* and c are real parameters and the similarity profile Ψ is a smooth function which tends to zero at $\pm\infty$.

Assuming there is a solution of (1.1) having the form exhibited in (1.3) and ignoring the bounded remainder, it is readily determined that Ψ would have to satisfy the ordinary differential equation

$$\Psi''' + \Psi^p\Psi' - \frac{2}{3p}\Psi - \frac{1}{3}(z+c)\Psi' = 0, \tag{1.4}$$

where z connotes the similarity variable

$$z = \frac{x_* - x}{(t_* - t)^{1/3}} - c \tag{1.5}$$

and the primes denote differentiation with respect to z. If we let $s = z + c$ and define $\varphi(s)$ to be $\Psi(s-c)$, then φ would satisfy the equation

$$\varphi''' + \varphi^p\varphi' - \frac{2}{3p}\varphi - \frac{1}{3}s\varphi' = 0 \tag{1.6}$$

if Ψ is a solution of (1.4). Up to signs that reflect the choice of the independent and dependent variables, this equation has appeared a couple of times in the literature (cf. Blaha *et al.* 1989). It corresponds to the scaling law

$$u \longrightarrow \lambda^{2/p}u, \qquad x \longrightarrow \lambda x, \qquad t \longrightarrow \lambda^3 t$$

for the generalized KdV equation (1.1).

In the body of this paper, attention will be given to showing that (1.6) does indeed possess solutions that approach zero at infinity. Section 2 concentrates on the special case $p = 4$ where the analysis is more transparent and the results more detailed. The approach used for the case $p = 4$ is suggestive of the analysis developed in Section 3 that successfully treats the general case $p > 4$. The principal results of our study are enunciated here to provide a focus for the technical developments to follow in Sections 2 and 3.

Main Theorem. *(The case $p = 4$) There exists an infinite family of non-trivial, C^∞-solutions φ of (1.6) with $p = 4$, defined on the entire real axis and having the following properties:*

(i) $\varphi(s) > 0$ *for* $s > 0$ *and*

$$\varphi(s) = cs^{-1/2}e^{-2s^{3/2}/3\sqrt{3}}\left(1 - \frac{2}{3\sqrt{3}s^{3/2}} + o\left(\frac{1}{s^{3/2}}\right)\right) \tag{1.7}$$

as $s \to +\infty$, where c is a positive constant, and

(ii) *there exist real constants a and b, not both zero, such that*

$$\varphi(s) = (-s)^{-1/2}\left[a\cos\left(\frac{(-s)^{3/2}}{3\sqrt{3}}\right) + b\sin\left(\frac{(-s)^{3/2}}{3\sqrt{3}}\right) + O\left(\frac{1}{(-s)^{3/2}}\right)\right]^2 \tag{1.8}$$

as $s \to -\infty$. In particular, $\varphi(s)$ has an infinite number of zeroes on $(-\infty, 0]$.

(The case $p > 4$) Let $p > 4$ be an integer. For every $k > 0$, there exists a positive, C^∞-solution ψ of (1.6) defined on all of $\mathbb{R}$, having the following properties:

(iii)

$$\psi(s) = ks^{-(3/4)+(1/p)}\exp\left(-\frac{2s^{3/2}}{3\sqrt{3}}\right)\left(1 + O(s^{-3/2})\right) \tag{1.9}$$

as $s \to +\infty$, and

(iv) *there exist real numbers a, b, c, with $a \neq 0$, such that*

$$\psi(s) = a(-s)^{-2/p} + (-s)^{-(3/4)+(1/p)}\left\{b\cos\left(\frac{2(-s)^{3/2}}{3\sqrt{3}}\right) + c\sin\left(\frac{2(-s)^{3/2}}{3\sqrt{3}}\right)\right\} + O\left(|s|^{-(2/p)-(3/2)}\right)$$

as $s \to -\infty$.

2. Similarity Solutions: the Case p = 4. Consideration is given here to providing solutions of (1.6) in case $p = 4$, for which value the equation takes the special form

$$\varphi'''(s) - \frac{1}{3}s\varphi'(s) - \frac{1}{6}\varphi(s) + \varphi(s)^4\varphi'(s) = 0. \tag{2.1}$$

It is our intention to prove (2.1) has solutions and to provide rigorous error bounds on their asymptotics for large values of the independent variable.

We begin the analysis with a few simple observations that set the stage for the more exacting work to follow. The main reason the case $p = 4$ is significantly simpler to understand than the cases where $p > 4$ is that solutions of (2.1) satisfy the additional relation

$$\frac{d}{ds}\left\{\varphi\,\varphi'' - \frac{1}{2}(\varphi')^2 - \frac{1}{6}s\,\varphi^2 + \frac{1}{6}\varphi^6\right\} = 0 \tag{2.2}$$

obtained by multiplying (2.1) by φ. Attention will focus upon solutions of (2.1) that decay to zero at infinity. By imposing this condition in a form strong enough that the quantity in braces in (2.2) tends to zero at one end or the other of the real axis, one infers that the solutions of (2.1) of interest here satisfy

$$\varphi\,\varphi'' - \frac{1}{2}(\varphi')^2 - \frac{1}{6}s\,\varphi^2 + \frac{1}{6}\varphi^6 = 0. \tag{2.3}$$

On the other hand, a smooth solution of (2.3) that has only isolated zeroes is also a solution of (2.1). Solutions of (2.3) have a special property which will be useful presently.

Lemma 2.1. *Suppose φ to be a non-trivial, C^3-solution of (2.1) and (2.3) on a real interval I. Then either $\varphi(s) \geq 0$ for all $s \in I$ or $\varphi(s) \leq 0$ for all $s \in I$. Moreover, φ has isolated zeroes in I.*

Proof. Suppose $\varphi(s_0) = 0$ for some $s_0 \in I$. Since $\varphi''(s_0)$ is finite, equation (2.3) implies that $\varphi'(s_0) = 0$ also. If $\varphi''(s_0) = 0$ as well, then the standard uniqueness result applied to the initial-value problem for (2.1) shows that $\varphi \equiv 0$ on I, contrary to our hypothesis. Thus if φ is a non-trivial solution of (2.1) satisfying (2.3), then $\varphi(s_0) = 0$ implies $\varphi'(s_0) = 0$ and $\varphi''(s_0) \neq 0$. In consequence, φ cannot change signs on I.

If the zeroes of φ accumulate at s_0, say, then φ, φ' and φ'' all vanish at s_0, and it is again adduced that $\varphi \equiv 0$, a contradiction. □

An effective change of the dependent variable is now introduced, following the lead of Blaha *et al.* (1989) and, independently, Ghidaglia and Jaffard†. Notice that if φ is a solution of (2.1) or (2.3), so is $-\varphi$. Because of this and Lemma 2.1, we might as well restrict attention to non-negative solutions of (2.1)–(2.3). If φ is any such solution, define v by $v^2(s) = \varphi(s)$. Equation (2.3) for φ implies v to satisfy the equation

$$v''(s) - \frac{1}{12}sv(s) + \frac{1}{12}v(s)^9 = 0. \tag{2.4}$$

Conversely, if v is a non-trivial solution of (2.4), then by the usual local existence and uniqueness theory, v has isolated zeroes and $v^2(s)$ satisfies (2.3) and hence (2.1).

Attention is now focussed upon equation (2.4). In particular, consider the following initial-value problem for (2.4):

$$\begin{aligned} &v'' - \frac{1}{12}sv + \frac{1}{12}v^9 = 0 \qquad \text{for } s \geq 0, \\ &v(0) = \alpha, \; v'(0) = \beta, \end{aligned} \tag{2.5}$$

where $\alpha > 0$. In the next few lemmas, a shooting argument is used to show that appropriate choices of α and β lead to solutions of (2.5) that are positive for $s > 0$ and which decay to zero as s tends to $+\infty$. After this point is established, it will be shown that this solution of the initial-value problem (2.4) can be extended to $-\infty$ and that it decays to the zero state there as well, thus providing a solution of (2.1) that leads to a similarity solution in the form (1.3) of the partial differential equation (1.1) for the case $p = 4$.

Lemma 2.2. *For any β and for any $\alpha > 0$, the solution $v_{\alpha,\beta}$ of the initial-value problem in (2.5) exists on the entire positive real axis $\mathbb{R}^+ = [0, \infty)$.*

Proof. It suffices to show that $v(s)$ and $v'(s)$ remain bounded on any bounded interval $[0, A]$, say. To this end, remark that if

$$F(s) = F_{\alpha,\beta}(s) = \frac{1}{2}v'(s)^2 - \frac{1}{24}sv(s)^2 + \frac{1}{120}v(s)^{10}, \tag{2.6}$$

†Private Communication.

then

$$F'(s) = -\frac{1}{24}v(s)^2 \leq 0.$$

Therefore, for $0 \leq s \leq A$, it is seen that

$$\frac{1}{2}v'(s)^2 - \frac{A}{24}v(s)^2 + \frac{1}{120}v(s)^{10} \leq F(s) \leq F(0) = \frac{1}{2}\beta^2 + \frac{1}{120}\alpha^{10}.$$

This inequality in turn shows immediately that $v(s)$ and $v'(s)$ are bounded for $s \in [0, A]$. □

Remark 2.3. The functional F displayed in (2.6) will reappear several times in the subsequent developments.

It is convenient to define

$$z(s) = z_{\alpha,\beta}(s) = \frac{1}{\alpha}\, v_{\alpha,\beta}(\alpha^{-4}s),$$

where $v_{\alpha,\beta}$ is the solution on $\mathbb{R}^+$ of (2.5) whose existence is guaranteed by Lemma 2.2. Then $z_{\alpha,\beta}$ satisfies the initial-value problem

$$\begin{aligned} z''(s) - \frac{1}{12\alpha^{12}}sz(s) + \frac{1}{12}z(s)^9 = 0, \\ z(0) = 1, \qquad z'(0) = \frac{\beta}{\alpha^5}. \end{aligned} \tag{2.7}$$

Lemma 2.4. *There exists an $\epsilon > 0$ such that if $\alpha > 0$ and β are given with*

$$\frac{|\beta|}{\alpha^5} < \epsilon \qquad \textit{and} \qquad \frac{1}{12\alpha^{12}} < \epsilon, \tag{2.8}$$

then $v_{\alpha,\beta}$ takes negative values on $\mathbb{R}^+$.

Proof. Let $w(s)$ denote the solution of the initial-value problem obtained from (2.7) by dropping the variable-coefficient term and making the second initial condition homogeneous, namely

$$\begin{aligned} w''(s) + \frac{1}{12}w(s)^9 = 0, \\ w(0) = 1, \qquad w'(0) = 0. \end{aligned} \tag{2.9}$$

Equation (2.9) can be integrated once, and phase-plane analysis then confirms w to be a periodic function taking both positive and negative values. Let $\bar{s}$ be a value where $w(\bar{s}) < 0$. Because solutions vary continuously with respect to the initial data and the coefficients, there is an $\epsilon > 0$ such that if

$$\frac{|\beta|}{\alpha^{s}} < \epsilon \qquad \text{and} \qquad \frac{1}{12\alpha^{12}} < \epsilon,$$

then $z_{\alpha,\beta}(\bar{s})$ is still negative. Thus $v_{\alpha,\beta}(\bar{s}/\alpha^4) < 0$ and the result is proved. □

Corollary 2.5. *For any $\beta \in \mathbb{R}$, there is an $\alpha_0 = \alpha_0(\beta) > 0$ such that if $\alpha \geq \alpha_0$, then the solution $v_{\alpha,\beta}$ of (2.5) assumes negative values on $\mathbb{R}^+$.*

Lemma 2.6. *Let v be a non-trivial solution of (2.5) and let F be the associated functional defined in (2.6). If $s_0 > 0$ is such that $F(s_0) \leq 0$, then $v(s) \neq 0$ for all $s \geq s_0$.*

Proof. Since F is decreasing, $F(s) \leq 0$ for all $s \geq s_0$. On the other hand, if $v(s) = 0$, then $v'(s) \neq 0$ since v is non-trivial. Hence $F(s) = \frac{1}{2}v'(s)^2 > 0$, which is precluded if $s \geq s_0$. □

Lemma 2.7. *There exists an $\alpha_1 > 0$ such that if $0 < \alpha \leq \alpha_1$, then (i) $v(s) = v_{\alpha,0}(s) > 0$ for all $s \geq 0$, and (ii) there is an $s_1 > 0$ independent of $\alpha \in (0, \alpha_1]$ for which $F(s_1) = F_{\alpha,0}(s_1) < 0$, where F is the functional associated to v as in (2.6).*

Proof. Fix an $\alpha > 0$ and let $v = v_{\alpha,0}$. Since $v''(s) = \frac{1}{12}v(s)[s - v(s)^8]$, it follows that $v''(s) < 0$ as long as $v^8(s) > s \geq 0$. Because $v'(0) = 0$, it is also true that $v'(s) < 0$ as long as $v^8(s) > s$. Thus v is decreasing at least until $s \geq v^8(s)$. Let s_0 be the smallest positive value for which $s_0 = v^8(s_0)$. Since $v''(s) < 0$ on $[0, s_0)$ and $v'(0) = 0$, we have $v'(s_0) < 0$. It follows from the foregoing that

$$\begin{aligned} -v'(s_0) &= -\int_0^{s_0} v''(s)ds = \frac{1}{12}\int_0^{s_0} [v^9(s) - sv(s)]ds \\ &\leq \frac{1}{12}\int_0^{s_0} v^9(s)ds \leq \frac{1}{12}s_0\alpha^9 = \frac{1}{12}v^8(s_0)\alpha^9 \leq \frac{1}{12}\alpha^{17} \end{aligned}$$

and

$$v(s_0) = \alpha + \int_0^{s_0} v'(s)ds \geq \alpha + s_0 v'(s_0) \geq \alpha - \frac{1}{12} s_0 \alpha^{17}$$
$$= \alpha - \frac{1}{12} v^8(s_0)\alpha^{17} \geq \alpha - \frac{1}{12}\alpha^{25} = \alpha\left(1 - \frac{\alpha^{24}}{12}\right).$$

Hence one has

$$F(s_0) = \frac{1}{2}v'(s_0)^2 - \frac{1}{24}s_0 v(s_0)^2 + \frac{1}{120}v(s_0)^{10}$$
$$= \frac{1}{2}v'(s_0)^2 - \left(\frac{1}{24} - \frac{1}{120}\right) v(s_0)^{10}$$
$$\leq \frac{\alpha^{34}}{288} - \frac{1}{30}\alpha^{10}\left(1 - \frac{\alpha^{24}}{12}\right)^{10}.$$

It is clear from the last inequality that there is an α_1 such that if $0 < \alpha \leq \alpha_1$ then $F(s_0) < 0$. Of course, the value of s_0 depends on α. Then, for α in the interval $(0, \alpha_1]$, we see that $v(s) = v_{\alpha,0}(s) > 0$ for $0 < s \leq s_0$ and that $F(s_0) < 0$. Lemma 2.6 then implies that $v(s) > 0$ for all $s \geq s_0$ and (i) is shown to be valid. For part (ii), choose $s_1 = \alpha_1^8$. Then for $0 < \alpha \leq \alpha_1$, we have

$$s_0 = s_0(\alpha) = v(s_0)^8 \leq \alpha^8 \leq \alpha_1^8.$$

In consequence, since F is decreasing,

$$F(s_1) = F(\alpha_1^8) \leq F(s_0) < 0,$$

and s_1 is independent of α in $(0, \alpha_1]$. □

Corollary 2.8. *Let α_1 be as in the last lemma. There exists a $\beta_0 > 0$ such that if $|\beta| \leq \beta_0$ and $\alpha \in [\frac{1}{2}\alpha_1, \alpha_1]$, then $v_{\alpha,\beta}(s) > 0$ for all $s \geq 0$.*

Proof. It is known that if $\frac{1}{2}\alpha_1 \leq \alpha \leq \alpha_1$, then $v_{\alpha,0}(s) > 0$ for all $s > 0$ and there is an $s_1 > 0$ for which $F_{\alpha,0}(s_1) < 0$. Appealing again to standard continuous-dependence results, there is inferred the existence of a constant $\beta_0 > 0$ such that if $\frac{1}{2}\alpha_1 \leq \alpha \leq \alpha_1$, then $v_{\alpha,\beta}(s) > 0$ for all $s \in [0, s_1]$ and $|\beta| \leq \beta_0$, and $F_{\alpha,\beta}(s_1) < 0$ for the same range of (α, β). Resorting to Lemma 2.6 thus assures the advertised conclusion. □

We now fix a value of β with $|\beta| \leq \beta_0$, where β_0 is as in the last corollary. For this fixed value of β, define

$$A = \{\alpha > 0 : v_{\alpha,\beta}(s) > 0 \text{ for all } s \geq 0\}$$

and

$$B = \{\alpha > 0 : v_{\alpha,\beta}(s) < 0 \text{ for some } s > 0\}.$$

Corollary 2.5 implies B to be non-empty while Corollary 2.8 implies A is likewise non-empty. Since (2.5) is second-order, if $v(s) = 0$, then $v'(s) \neq 0$ if v is non-trivial, which is certainly the case if $\alpha > 0$. It follows that $A \cup B = (0, \infty)$. By continuous dependence, B is an open set and so A is closed. Moreover, B contains the entire interval $[\alpha_0, \infty)$, where $\alpha_0 = \alpha_0(\beta)$ as in Corollary 2.5. Let $\alpha_2 = \sup\{\alpha : \alpha \in A\}$ and set $v = v_{\alpha_2,\beta}$. The value α_2 lies in A since A is closed, whence $v(s) > 0$ for $s \geq 0$. Also, $\alpha_2 + \epsilon$ lies in B for any $\epsilon > 0$. Let $F(s) = F_{\alpha_2,\beta}(s)$.

Lemma 2.9. *With the notation above, $F(s) > 0$ for all $s \geq 0$.*

Proof. If not, then since F is strictly decreasing it would follow that there is a point $\bar{s}$ such that $F(\bar{s}) < 0$. By continuous dependence again, there is an $\epsilon > 0$ such that if $|\alpha - \alpha_2| < \epsilon$, then $v_{\alpha,\beta}(s) > 0$ for $0 \leq s \leq \bar{s}$ and $F_{\alpha,\beta}(\bar{s}) < 0$. It follows from Lemma 2.6 that if $|\alpha - \alpha_2| < \epsilon$, then $v_{\alpha,\beta}(s) > 0$ for all $s \geq 0$. This is a contradiction because there are such values of α that lie in B. □

Corollary 2.10. *If $F = F_{\alpha_2,\beta}$ and $v = v_{\alpha_2,\beta}$, then $\lim_{s\to\infty} F(s) = F_\infty \geq 0$ and $v \in L_2(0,\infty)$.*

Proof. Since F is decreasing and non-negative, it has a non-negative limit at ∞. Also,

$$F'(s) = -\frac{1}{12}v^2(s),$$

so

$$\int_0^r v^2(s)ds = -12\int_0^r F'(s)ds = 12(F(0) - F(r)) \leq 12F(0),$$

whence $v \in L_2(0,\infty)$. □

Lemma 2.11. *With v as in the last result, there exists an $s_1 \geq 0$ such that $v'(s) < 0$ for all $s \geq s_1$. If $\beta \leq 0$, then $v'(s) \leq 0$ for all $s \geq 0$.*

Proof. Since $F(s) > 0$ for all $s \geq 0$, it follows that

$$\frac{1}{2}v'(s)^2 > \frac{1}{24}sv(s)^2 - \frac{1}{120}v(s)^{10} = \frac{v(s)^2}{120}[5s - v(s)^8].$$

Hence if $v(s)^8 < 5s$, $v'(s) \neq 0$. Thus if there is an s_1 with $v(s_1)^8 < 5s_1$ and $v'(s_1) < 0$, then $v'(s) < 0$ for all $s \geq s_1$. Such an s_1 must exist, for if not then for all $s \geq s_1$ either $v(s)^8 \geq 5s$ or $v'(s) \geq 0$. As $v > 0$ everywhere, such a function could not lie in $L_2(0, \infty)$.

If $\beta \leq 0$, then $v'(0) \leq 0$ and

$$v''(s) = \frac{1}{12}v(s)(s - v(s)^8).$$

Thus $v''(0) < 0$ and so $v'(s) < 0$ as long as $s < v(s)^8$. At the first value $\bar{s}$ where $\bar{s} = v(\bar{s})^8$, we still have $v(\bar{s})^8 < 5\bar{s}$ and $v'(\bar{s}) < 0$. It was just argued that for such an $\bar{s}$, $v'(s) < 0$ for $s \geq \bar{s}$. □

Corollary 2.12. *With v as above, $\lim_{s\to\infty} sv^2(s) = 0$, and in particular, $v(s) \to 0$ as $s \to \infty$.*

Proof. This follows since $v > 0$ is eventually decreasing and lies in $L_2(0, \infty)$. □

Corollary 2.13. *With the notation presently in force, we have $\lim_{s\to\infty} v'(s) = 0$ and $\lim_{s\to\infty} F(s) = 0$.*

Proof. Since $v(s) \to 0$ and $F(s) \to F_\infty$ as $s \to +\infty$, it follows that

$$\lim_{s\to\infty} v'(s)^2 = 2F_\infty.$$

If $F_\infty \neq 0$, then because $v'(s)$ is eventually negative, we would have $v'(s) \to -L$ where $L^2 = 2F_\infty$, $L > 0$. This is impossible if $v(s) \to 0$ as $s \to \infty$. □

We continue to assume $v = v_{\alpha_2,\beta}$ where α_2 and β are fixed as delineated previously. Similarly, it is assumed that $F = F_{\alpha_2,\beta}$. A detailed asymptotic analysis of $v(s)$ is now undertaken regarding its behavior as s becomes unboundedly large. At this point, it is known that $F(s) > 0$ for $s \geq 0$, that $F(s) \to 0$ as $s \to +\infty$, and that $v(s)$, $v'(s) \to 0$ as $s \to +\infty$.

Since $F'(s) = -v(s)^2/24$, it follows that

$$v'(s)^2 = \frac{sv(s)^2}{12} - \frac{v(s)^{10}}{60} + \frac{1}{12}\int_s^\infty v(r)^2 dr,$$

or, what is the same,

$$\frac{v'(s)^2}{sv(s)^2} = \frac{1}{12} - \frac{v(s)^8}{60s} + \frac{1}{12s\,v(s)^2}\int_s^\infty v(r)^2 dr. \tag{2.10}$$

The second term on the right-hand side of the last formula tends to zero as $s \to +\infty$, and consequently

$$\liminf_{s\to\infty} \frac{v'(s)^2}{s\,v(s)^2} \geq \frac{1}{12}.$$

In particular,

$$\liminf_{s\to\infty} -\frac{s\,v'(s)}{v(s)} = +\infty,$$

and it therefore follows by an application of l'Hôpital's rule that the last term in (2.10) vanishes in the limit as s approaches $+\infty$. It is thus concluded that

$$\lim_{s\to\infty} \frac{v'(s)^2}{sv(s)^2} = \frac{1}{12},$$

which is identical to

$$\lim_{s\to\infty} \frac{-v'(s)}{s^{1/2}v(s)} = \frac{1}{2\sqrt{3}} \tag{2.11}$$

since $v'(s) < 0$ and $v(s) > 0$ for s near $+\infty$. Another application of l'Hôpital's rule together with (2.11) shows that

$$\lim_{s\to\infty} \frac{\int_s^\infty v(r)^2 dr}{s^{-1/2}v(s)^2} = \sqrt{3}. \tag{2.12}$$

Now rewrite (2.10) in the form

$$\frac{s^{1/2}v'(s)^2}{v(s)^2} - \frac{s^{3/2}}{12} = -\frac{s^{1/2}v(s)^8}{60} + \frac{s^{1/2}}{12\,v(s)^2}\int_s^\infty v(r)^2 dr.$$

The first term on the right-hand side tends to zero because of Corollary 2.12, for example, and so (2.12) then implies that

$$\lim_{s\to\infty} \frac{s^{1/2}v'(s)^2}{v(s)^2} - \frac{s^{3/2}}{12} = \frac{\sqrt{3}}{12}. \tag{2.13}$$

Define

$$\theta(s) = 12\frac{s^{1/2}v'(s)^2}{v(s)^2} - s^{3/2} - \sqrt{3}, \tag{2.14}$$

so that $\theta(s) \to 0$ as $s \to \infty$ according to (2.13), and then rewrite (2.14) as

$$v'(s)^2 = \frac{sv(s)^2}{12}\left(1 + \frac{\sqrt{3}}{s^{3/2}} + \frac{\theta(s)}{s^{3/2}}\right).$$

Since $v'(s) < 0$ for large values of s and $v(s) > 0$, the last formula implies

$$-v'(s) = v(s)\left(\frac{s^{1/2}}{2\sqrt{3}} + \frac{1}{4s} + \frac{\theta_1(s)}{s}\right), \tag{2.15}$$

where $\theta_1(s)$ also tends to zero as s tends to infinity. Returning to (2.12), the use of l'Hôpital's rule and the results in hand yield

$$\lim_{s\to\infty} s^{3/2}\left[\frac{s^{1/2}\int_s^\infty v(r)^2 dr}{v(s)^2} - \sqrt{3}\right] = -3. \tag{2.16}$$

Formula (2.10) implies that

$$s^{3/2}\left[\frac{v'(s)^2}{sv(s)^2} - \frac{1}{12}\right] = -\frac{s^{1/2}v(s)^8}{60} + \frac{s^{1/2}}{12v(s)^2}\int_s^\infty v(r)^2 dr.$$

The first term on the right-hand side of the last equation tends to zero as s tends to infinity, while the second term tends to $\sqrt{3}/12$ on account of (2.12). One is thus led to consider the weighted difference

$$s^{3/2}\left[s^{3/2}\left(\frac{v'(s)^2}{sv(s)^2} - \frac{1}{12}\right) - \frac{\sqrt{3}}{12}\right],$$

which, according to (2.16), converges to $-\frac{1}{4}$ as s becomes unboundedly large. Let $\theta_2(s)$ be defined by

$$\theta_2(s) = 12s^{3/2}\left[s^{3/2}\left(\frac{v'(s)^2}{sv(s)^2} - \frac{1}{12}\right) - \frac{\sqrt{3}}{12}\right] + 3, \tag{2.17}$$

so that $\theta_2(s)$ tends to zero as s tends to infinity. Solving (2.17) for $v'(s)^2$ leads to the formula

$$v'(s)^2 = \frac{sv(s)^2}{12}\left[1 + \frac{\sqrt{3}}{s^{3/2}} - \frac{3}{s^3} + \frac{\theta_2(s)}{s^3}\right].$$

Again solving for v' and recalling that $v'(s) < 0$ for s large, there obtains

$$-v'(s) = \frac{s^{1/2}v(s)}{2\sqrt{3}}\left[1 + \frac{\sqrt{3}}{2s^3} - \frac{3}{2s^3} + \frac{\theta_3(s)}{s^3}\right],$$

where $\theta_3(s)$ tends to zero as s tends to infinity. The last formula may be rewritten as

$$\frac{d}{ds}\left[\log v(s) + \frac{s^{3/2}}{3\sqrt{3}} + \log s^{1/4}\right] = \frac{v'(s)}{v(s)} + \frac{s^{1/2}}{2\sqrt{3}} + \frac{1}{4s} = \frac{\sqrt{3}}{4s^{5/2}} - \frac{\theta_3(s)}{s^{5/2}}. \tag{2.18}$$

Since the right-hand side of (2.18) is integrable over the interval $[1, \infty)$, say, it follows that

$$\log v(s) + \frac{s^{3/2}}{3\sqrt{3}} + \log s^{1/4} \to k$$

as $s \to +\infty$, where k is a finite real constant. Indeed, since $\theta_3(s)$ tends to zero at infinity,

$$\log v(s) + \frac{s^{3/2}}{3\sqrt{3}} + \log s^{1/4} = k - \frac{1}{2\sqrt{3}s^{3/2}} + o\left(\frac{1}{s^{3/2}}\right)$$

as $s \to +\infty$. Exponentiating this relation leads to the conclusion that

$$v(s) = cs^{-1/4}e^{-s^{3/2}/3\sqrt{3}}\left(1 - \frac{1}{2\sqrt{3}s^{3/2}} + o\left(\frac{1}{s^{3/2}}\right)\right) \tag{2.19}$$

as $s \to +\infty$, where $c = e^k > 0$.

Attention is now turned to the behavior of solutions of the initial-value problem (2.5) as the independent variable s tends to $-\infty$. To study this issue, it is convenient to set $w(s) = v(-s)$ and consider the behavior of w as $s \to +\infty$. Of course, w satisfies the initial-value problem

$$w''(s) + \frac{1}{12}sw(s) + \frac{1}{12}w(s)^9 = 0, \qquad w(0) = \alpha, \qquad w'(0) = -\beta. \tag{2.20}$$

For $s \geq 0$, define the functional G in analogy with the F of (2.6) to be

$$G(s) = \frac{1}{2}w'(s)^2 + \frac{s}{24}w(s)^2 + \frac{1}{120}w(s)^{10}. \tag{2.21}$$

If w solves (2.20), then

$$G'(s) = \frac{1}{24}w(s)^2 \geq 0,$$

so G is increasing on $[0, \infty)$. Moreover, observe that

$$G'(s) = \frac{1}{24}w(s)^2 = \frac{1}{s}\frac{sw(s)^2}{24} \leq \frac{1}{s}G(s),$$

from which it follows that for any $s_0 > 0$,

$$\frac{G(s)}{s} \leq \frac{G(s_0)}{s_0}$$

for all $s \geq s_0$. Thus G grows at most linearly as s tends to $+\infty$. The latter *a priori* deduced bound together with standard local existence theory implies (2.20) to possess a globally defined solution. This state of affairs is worth formalizing.

Lemma 2.14. *For any $\alpha, \beta \in \mathbb{R}$, there exists a unique solution w of (2.20) which is defined for all positive values of the independent variable. Moreover, if α and β are not both zero and G is as defined in (2.21) where w is the solution of (2.20), then G is positive and strictly increasing on $[0, \infty)$ and there exists a constant C such that $G(s) \leq Cs$ for $s \geq 1$, say. Finally, there are constants C_0 and C_1 such that*

$$|w'(s)| \leq C_1 s^{1/2} \quad \text{and} \quad |w(s)| \leq C_0 \tag{2.22}$$

for $s \geq 1$.

Proof. Since $G(0) = \frac{1}{2}\beta^2 + \frac{1}{120}\alpha^{10}$, it is strictly positive unless both α and β are zero. Because $G'(s) = \frac{1}{24}w(s)^2$ and w has only isolated zeroes, G is strictly increasing. The two inequalities in (2.22) follow immediately from that fact that $G(s) \leq Cs$. □

Guided by formula (2.19), let u be defined by the relation $w(s) = s^{-1/4}u(s^{3/2}/3\sqrt{3})$. It follows that

$$u''(r) + u(r) + \frac{5}{36\,r^2}u(r) + \frac{1}{27\,r^2}u(r)^9 = 0. \tag{2.23}$$

We are interested in the asymptotic behavior of w at $+\infty$ which follows readily from the asymptotic behavior of u at $+\infty$. If u solves (2.23) on some interval $[a, \infty)$, say, define

$$H(r) = \frac{1}{2}u'(r)^2 + \frac{1}{2}u(r)^2 + \frac{5}{72\,r^2}u(r)^2 + \frac{1}{270\,r^2}u(r)^{10}.$$

A calculation reveals that

$$H'(r) = -\frac{5}{36\,r^3}u(r)^2 - \frac{1}{135\,r^3}u(r)^{10} \leq 0.$$

The following lemma follows immediately from this observation.

Lemma 2.15. *Let u solve (2.23) on an interval $[a, \infty)$. Then $u(r)$ and $u'(r)$ are bounded as $r \to +\infty$.*

Continuing the analysis of solutions of (2.23) near $+\infty$, write the equation as a system, viz.

$$\begin{aligned} u'(r) &= z(r) \\ z'(r) &= -u(r) - \frac{5}{36\,r^2}u(r) - \frac{1}{27\,r^2}u(r)^9 \\ &= -u(r) - f(r,u), \end{aligned}$$

or in matrix form

$$\frac{d}{dr}\begin{pmatrix} u(r) \\ z(r) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u(r) \\ z(r) \end{pmatrix} - \begin{pmatrix} 0 \\ f(r,u) \end{pmatrix}. \tag{2.24}$$

The variation of constants formula then leads to the representation

$$\begin{pmatrix} u(r_2) \\ z(r_2) \end{pmatrix} = T(r_2 - r_1)\begin{pmatrix} u(r_1) \\ z(r_1) \end{pmatrix} - \int_{r_1}^{r_2} T(r_2 - \rho)\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix} d\rho,$$

where

$$T(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

and $0 < r_1 < r_2$. The latter is equivalent to

$$T(-r)\begin{pmatrix} u(r) \\ z(r) \end{pmatrix} = T(-r_1)\begin{pmatrix} u(r_1) \\ z(r_1) \end{pmatrix} - \int_{r_1}^{r} T(-\rho)\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix} d\rho.$$

Lemma 2.16. *With notation as above, the limit*

$$\lim_{r\to+\infty} T(-r)\begin{pmatrix} u(r) \\ z(r) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.25}$$

exists.

Proof. Since $T(-r)$ is a unitary matrix, it follows that

$$\begin{aligned} \left|T(-\rho)\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix}\right| &\le \left|\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix}\right| \\ &\le \frac{C}{\rho^2}(|u(\rho)| + |u(\rho)|^9) \le \frac{C}{\rho^2} \end{aligned} \tag{2.26}$$

since u is bounded from the results of Lemma 2.15. (The norm of a vector in $\mathbb{R}^2$ is the standard Euclidean norm.) Thus,

$$\lim_{r\to+\infty} \int_{r_1}^{r} T(-\rho)\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix} d\rho$$

exists, and the proof is complete. □

Express the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ in (2.25) as

$$\begin{pmatrix} a \\ b \end{pmatrix} = T(-r_1)\begin{pmatrix} u(r_1) \\ z(r_1) \end{pmatrix} - \int_{r_1}^{\infty} T(-\rho)\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix} d\rho,$$

replace r_1 by r and apply $T(r)$ to obtain

$$\begin{pmatrix} u(r) \\ z(r) \end{pmatrix} = T(r)\begin{pmatrix} a \\ b \end{pmatrix} + T(r)\int_{r}^{\infty} T(-\rho)\begin{pmatrix} 0 \\ f(\rho, u(\rho)) \end{pmatrix} d\rho. \tag{2.27}$$

Because of the bound expressed in (2.26), the last representation presents a conclusion which is recorded in the following lemma.

Lemma 2.17. *With the above notation,*

$$\left|\begin{pmatrix} u(r) \\ z(r) \end{pmatrix} - T(r)\begin{pmatrix} a \\ b \end{pmatrix}\right| = 0\left(\frac{1}{r}\right)$$

as $r \to +\infty$.

Corollary 2.18. *If the limiting vector* $\begin{pmatrix} a \\ b \end{pmatrix}$ *is the zero vector in* $\mathbb{R}^2$, *then* $\left|\begin{pmatrix} u(r) \\ z(r) \end{pmatrix}\right| = 0(r^{-m})$ *as* $r \to \infty$ *for all values* $m \geq 0$.

Proof. If $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, Lemma 2.17 implies that

$$\left|\begin{pmatrix} u(r) \\ z(r) \end{pmatrix}\right| = 0\left(\frac{1}{r}\right) \quad \text{as} \quad r \to +\infty.$$

This in turn implies that $|f(r, u(r))| \leq C/r^3$ as $r \to +\infty$. Putting this inequality into the representation (2.27) leads to the conclusion

$$\left|\begin{pmatrix} u(r) \\ z(r) \end{pmatrix}\right| = 0\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to +\infty.$$

It follows that $|f(r, u(r))| = 0(1/r^4)$, whence (2.27) implies.

$$\left|\begin{pmatrix} u(r) \\ z(r) \end{pmatrix}\right| = 0\left(\frac{1}{r^3}\right) \quad \text{as} \quad r \to +\infty.$$

An obvious induction argument may be used to complete the proof. □

It thus transpires that if u is a solution of (2.23) on $[0, \infty)$, then there are constants $a, b \in \mathbb{R}$ such that

$$\left.\begin{aligned} u(r) &= a\cos(r) + b\sin(r) + 0\left(\frac{1}{r}\right) \\ u'(r) &= -a\sin(r) + b\cos(r) + 0\left(\frac{1}{r}\right) \end{aligned}\right\} \text{ as } r \to +\infty.$$

Translating this into information about w leads to the following result.

Proposition 2.19. *Let w be any solution of the initial-value problem (2.20). Then there exist real constants a and b such that*

$$w(s) = s^{-1/4}\left[a\cos\left(\frac{s^{3/2}}{3\sqrt{3}}\right) + b\sin\left(\frac{s^{3/2}}{3\sqrt{3}}\right)\right] + 0\left(\frac{1}{s^{7/4}}\right)$$

and

$$w'(s) = \frac{s^{1/4}}{2\sqrt{3}}\left[-a\sin\left(\frac{s^{3/2}}{3\sqrt{3}}\right) + b\cos\left(\frac{s^{3/2}}{3\sqrt{3}}\right)\right] + 0\left(\frac{1}{s^{5/4}}\right)$$

as $s \to +\infty$. Moreover, $a = b = 0$ if and only if $w(s) \equiv 0$.

Proof. The only aspect not already covered is the last statement. If $w \equiv 0$, then plainly $a = b = 0$. Suppose w is not identically zero. If $a = b = 0$, then according to Corollary 2.18 $u(r)$ and $u'(r)$ decay faster than any inverse power of r. Thus, $w(s)$ and $w'(s)$ decay faster than any inverse power of s. It follows that $G(s)$ defined in (2.21) tends to zero as s tends to infinity. This contradicts Lemma 2.14 unless $G \equiv 0$. But then $w(s) = w'(s) = 0$, whence $w \equiv 0$. □

This concludes the developments for the case $p = 4$. All the conclusions stated in the first part of the Main Theorem, including the asymptotic formulas (1.7) and (1.8) for φ, have been established in the theory set forth in this section.

In the next section, attention is turned to the second part of the Main Theorem which corresponds to the case $p > 4$ in (1.6).

3. Similarity Solutions: the Case $\mathbf{p > 4}$.

In this section we study solutions of equation (1.6) in case $p > 4$. Roughly speaking, we will show the existence of a one-parameter family of solutions which are positive and have exponential decay as $s \to +\infty$, but decay algebraically with oscillations as $s \to -\infty$.

The proof in case $p > 4$ is much more technical than in the case $p = 4$, because we have been unable to reduce the problem to the analysis of a second-order equation. Having to deal directly with the third-order equation (1.6) appears to require a fundamental change in strategy, since a standard shooting argument is no longer feasible. Indeed, the shooting argument used in Section 2 depends on the fact that a non-trivial solution of equation (2.4) must have non-vanishing derivative at any point where it equals zero. This is generally false for a third-order equation.

Instead of looking at the initial-value problem associated with (1.6), or some related equation, at some finite value $s_0 \in \mathbb{R}$, and then studying separately the behavior of solutions as $s \to \pm\infty$, we directly construct solutions defined in a neighborhood of $+\infty$ which have the correct exponential decay. Then our task is to show that such solutions can be continued to $-\infty$, and that they have an appropriate asymptotic behavior as $s \to -\infty$.†

The study commences as with the case $p = 4$ by multiplying (1.6) by φ, thereby obtaining a modified version of (2.2), namely

$$\frac{d}{ds}F(s) = -\frac{p-4}{6p}\varphi(s)^2, \tag{3.1}$$

where

$$F(s) = \varphi''(s)\varphi(s) - \frac{\varphi'(s)^2}{2} - \frac{s\varphi(s)^2}{6} + \frac{\varphi(s)^{p+2}}{p+2}. \tag{3.2}$$

Since a non-trivial solution of (1.6) has only isolated zeroes, it follows that $F(s)$ is everywhere strictly decreasing. In particular, it follows that φ and its derivatives can not all decay rapidly both as $s \to +\infty$ and $s \to -\infty$, for then $\lim_{s\to\pm\infty} F(s) = 0$. Other helpful information follows directly from (3.1) and (3.2).

Lemma 3.1. *Let φ be a nontrivial solution of (1.6) with $p > 4$ defined on a suitable unbounded interval in $\mathbb{R}$.*

(a) *If $\lim_{s\to+\infty} F(s) = 0$, then φ has no zeroes.*

(b) *If φ is not in $L^2(-\infty, 0)$, then φ can have at most finitely many zeroes on $(-\infty, 0)$.*

Proof. For part (a) is suffices to remark that if $\lim_{s\to+\infty} F(s) = 0$, then, since F is strictly decreasing, $F(s) > 0$ for all $s \in \mathbb{R}$, and that if $\varphi(s) = 0$, then from (3.2)

†Thanks go to M. Weinstein for suggesting this approach.

$F(s) \le 0$. Part (b) follows upon noting that if $s_m \to -\infty$ is a sequence of zeroes of φ, then $F(s_m) \le 0$. This implies that $\lim_{s\to-\infty} F(s) \le 0$, which in turn implies, by integrating (3.1), that $\varphi \in L^2(-\infty, 0)$. □

It follows that a solution which decays exponentially to zero, along with its first few derivatives, as $s \to +\infty$ and which is positive in a neighborhood of $+\infty$ must remain positive throughout its interval of existence.

Guided by the results for $p = 4$, we are led to define

$$\varphi(s) = (-s)^{-\gamma} u\left(\frac{2(-s)^{3/2}}{3\sqrt{3}}\right) \tag{3.3}$$

for $s < 0$. It transpires that for a certain choice of γ, the function u satisfies an equation which, if terms which should decay more rapidly are discarded, can be solved explicitly. This enables one to guess the asymptotic state of φ as $s \to -\infty$. The analogous transformation can be made for $s > 0$, again allowing one to guess the precise type of exponential decay the desired solution of (1.6) should have. In both cases the useful value of γ is

$$\gamma = \frac{3}{4} - \frac{1}{p}. \tag{3.4}$$

In fact, the line of argument is a little more exacting that just described, but the gist of an effective proof is encompassed in our remark.

In what follows, the first step is to construct a family of solutions of (1.6) which are positive and have exponential decay as $s \to +\infty$. Then, it is shown that these solutions can be continued "backwards" onto the whole real line and that they respect certain helpful *a priori* estimates as $s \to -\infty$. These estimates are used in the investigation of the detailed asymptotic structure of this family of solutions as $s \to -\infty$.

For the rest of this section, we assume p is an integer greater than 4.

3.1. Exponential decay as $s \to +\infty$.

To construct solutions of (1.6) which decay exponentially as $s \to +\infty$, a change of variables is used which is more elaborate than (3.3) and which incorporates more completely the expected asymptotic behavior. Define the function u by the transformation

$$\varphi(s) = s^{-\gamma} u(r), \tag{3.5}$$

where γ is given in (3.4) and

$$r = \exp\left(\frac{-2s^{3/2}}{3\sqrt{3}}\right) \tag{3.6}$$

or, what is the same,

$$-\log r = \frac{2s^{3/2}}{3\sqrt{3}}.$$

Note that $s > 0$ corresponds to $0 < r < 1$, and that $s \to +\infty$ as $r \to 0^+$. In these variables, a search is initiated for a solution φ of (1.6) such that $u(r) = kr + o(r)$ as $r \to 0^+$, where k is some fixed positive number.

A straightforward, but tedious calculation shows that φ is a solution of equation (1.6) for $s > 0$ if and only if u is a solution of the following ordinary differential equation:

$$\begin{aligned} r^3 u'''(r) + \left(3r^2 + \frac{\delta r^2}{(-\log r)}\right) u''(r) + \frac{\delta}{(-\log r)}[ru'(r) - u(r)] \\ + \frac{Aru'(r)}{(-\log r)^2} + \frac{Bu(r)}{(-\log r)^3} + \frac{Cru(r)^p u'(r)}{(-\log r)^{p/2}} + \frac{Du(r)^{p+1}}{(-\log r)^{(p+2)/2}} = 0, \end{aligned} \tag{3.7}$$

where

$$\delta = \frac{1}{2} - \frac{2}{p} > 0 \tag{3.8}$$

and A, B, C, and D are constants depending on p, whose values are determined explicitly from equation (1.6) and the transformation (3.5) and (3.6). The only one of the four values which is needed explicitly is

$$A = \frac{4}{3p^2} - \frac{8}{3p} + \frac{41}{36}. \tag{3.9}$$

Note that $A > 0$ for any $p > 4$. It is worth remarking that if the transformation defined by (3.5) and (3.6) is carried out for values of γ other than the value in (3.4), an equation similar to (3.7) will obtain with some different coefficients and the powers of $(-\log r)$ in the last two terms will also be different. The crucial change is that the terms $ru'(r)$ and $-u(r)$ don't appear in exactly the combination $ru'(r) - u(r)$, but rather with different coefficients. In analyzing (3.7), the term featuring $ru'(r) - u(r)$ is the most delicate to estimate; use will be made of the fact that its derivative is exactly $ru''(r)$.

For a reader actually making the computation leading from (1.6) to (3.7), it seems easier to reach the correct formula in two stages, letting first $\varphi(s) = s^{-\gamma}z(t)$ where $t = \frac{2}{3\sqrt{3}}s^{3/2}$, obtaining an equation for z, and then setting $z(t) = u(r)$ where $r = e^{-t}$ to derive an equation for u. Also, it will be worth retaining the calculations to help with verifying statement (b) of Proposition 3.4. The first stage of calculation just suggested will reappear in Subsection 3.3.

Equation (3.7) can be rewritten in the equivalent form

$$\begin{aligned}\frac{d}{dr}&\left\{\frac{r^3u''(r)}{(-\log r)^{\delta}} + \frac{Aru(r)}{(-\log r)^{\delta+2}} + \frac{C}{p+1}\frac{ru(r)^{p+1}}{(-\log r)^{\delta+(p/2)}}\right\}\\ &= \frac{\delta}{(-\log r)^{\delta+1}}[u(r) - ru'(r)] + \frac{u(r)}{(-\log r)^{\delta+2}}[f(r) + u(r)^p g(r)],\end{aligned} \tag{3.10}$$

where

$$f(r) = A + \frac{(\delta+2)A - B}{(-\log r)} \quad \text{and} \quad g(r) = \frac{C(-\log r) + C(\delta + (p/2)) - D(p+1)}{(p+1)(-\log r)^{(p-2)/2}}.$$

Since $p > 4$, if $0 < R < 1$, then $f, g \in C([0, R])$, with $f(0) = A$ and $g(0) = 0$.

Our goal is to construct solutions u of (3.7) on $[0, R]$ with $u(0) = 0$ and $u'(0) = k \neq 0$. The matter is complicated by the fact that such solutions will not be in $C^2([0, R])$. To prove the existence of solutions, we transform (3.10) into an integral equation. It will not be immediately obvious that the integral equation is equivalent to (3.10), or even that it makes sense. These facts will be proved in the course of establishing existence of appropriate solutions, by the use of a very particular function space. The integral equation in question, written in a way that is convenient for subsequent calculations rather than its most compact form, is

$$\begin{aligned} u'(r) &= k - A\int_0^r \frac{1}{\tau(-\log\tau)^2}\,\frac{u(\tau)}{\tau}\,d\tau - \frac{C}{p+1}\int_0^r \frac{1}{\tau(-\log\tau)^{p/2}}\,\frac{u(\tau)^{p+1}}{\tau}\,d\tau \\ &+ \int_0^r \frac{1}{\tau(-\log\tau)^2}\left\{\frac{(-\log\tau)^{\delta+2}}{\tau^2}\int_0^\tau \frac{u(\sigma)}{(-\log\sigma)^{\delta+2}}\,[f(\sigma)+u(\sigma)^p g(\sigma)]\,d\sigma\right\}d\tau \qquad (3.11)\\ &+ \delta\int_0^r \frac{1}{\tau(-\log\tau)^{3-\varepsilon}}\left\{\frac{(-\log\tau)^{\delta+3-\varepsilon}}{\tau^2}\int_0^\tau \frac{u(\sigma)-\sigma u'(\sigma)}{(-\log\sigma)^{\delta+1}}\,d\sigma\right\}d\tau. \end{aligned}$$

If $0 < \varepsilon < 2$ and $0 < R < 1$, let $M_{\varepsilon,R}$ be the set of functions $u \in C^1([0,R]) \cap C^2((0,R])$ such that $u(0) = 0$ and $r(-\log r)^{2-\varepsilon}u''(r)$ is bounded on $(0,R]$. (Unfortunately, the positive value of ε is needed. Without it, this last condition appears too strong to carry out the contraction-mapping argument to appear below. The reason for considering ε close to 2 is to provide a larger uniqueness class for solutions.) The space $M_{\varepsilon,R}$ will carry the norm

$$|||\,u\,||| = |||\,u\,|||_{\varepsilon,R} = \sup_{0\le r\le R} |u'(r)| + \sup_{0<r\le R} |r(-\log r)^{2-\varepsilon}u''(r)|.$$

Proposition 3.2. *If a function u lies in $M_{\varepsilon,R}$ for some ε in $(0,2)$ and R in $(0,1)$, then all the integrals in (3.11) are absolutely convergent. If, in addition, u is a solution of (3.11), then $u \in C^1([0,R]) \cap C^3((0,R])$, $u(0) = 0$, $u'(0) = k$, and u is a solution of the differential equation (3.7) on $(0,R]$. Furthermore,*

$$\lim_{r\to 0} r(-\log r)^2 u''(r) = -kA/2,$$

where A is given by (3.9).

Proposition 3.3. *Let $k \in \mathbb{R}$ and $\varepsilon \in (0,2)$. There exists $R \in (0,1)$ such that (3.11) has a unique solution in $M_{\varepsilon,R}$.*

Proposition 3.4. *Let $k \in \mathbb{R}$, $k \neq 0$. There exists $s_0 > 0$ and a nontrivial C^3-solution φ of (1.6) on the interval (s_0, ∞) having the properties:*

(a) $\varphi(s) = k\, s^{-(3/4)+(1/p)} \exp\left(\frac{-2s^{3/2}}{3\sqrt{3}}\right)\left(1 - \frac{9As^{-3/2}}{4\sqrt{3}} + o(s^{-3/2})\right)$ *as $s \to \infty$,*

(b) *and $F(s) \to 0$ as $s \to \infty$.*

The uniqueness part of Proposition 3.3 translates into a uniqueness class for the solutions of (1.6) described in Proposition 3.4. However, translating the condition $u \in M_{\varepsilon,R}$ into a condition on φ using the transformation (3.5) seems rather cumbersome and would involve φ' and φ'' as well as φ. This condition has therefore not been made explicit.

The rest of this sub-section is devoted to proving these propositions. The line of reasoning starts with an elementary consequence of l'Hôpital's rule.

Lemma 3.5. *Let $a > -1$ and $b \in \mathbb{R}$. If h is a continuous function on $[0, R]$, then*

$$\lim_{t \to 0+} \frac{(-\log \tau)^b}{\tau^{a+1}} \int_0^\tau \frac{\sigma^a h(\sigma)}{(-\log \sigma)^b}\, d\sigma = \frac{h(0)}{a+1}\,.$$

Proof. This follows easily from an application of l'Hôpital's rule. □

Attention is now turned to the proof of Proposition 3.2. Denote the four integrals in (3.11) by I_1 through I_4 in the order in which they appear. Suppose that $u \in M_{\varepsilon,R}$, for some ε in $(0,2)$ and R in $(0,1)$. The functions

$$\frac{1}{\tau(-\log\tau)^2}, \quad \frac{1}{\tau(-\log\tau)^{p/2}} \quad \text{and} \quad \frac{1}{\tau(-\log\tau)^{3-\varepsilon}}$$

are all in $L^1([0,R])$. (Here is where the condition $\varepsilon < 2$ is used.) Since $u(0) = 0$, it follows that

$$|u(\tau)| \leq \tau \|u'\|_\infty \leq \tau |||\, u \,||| \tag{3.12}$$

for all $\tau \in (0, R]$. One immediately concludes that I_1 and I_2 are absolutely convergent and that

$$|I_1| \leq G(R) |||\, u \,|||\,,$$

$$|I_2| \leq G(R) |||\, u \,|||^{p+1}\,,$$

where here and below, G denotes a continuous function defined on $(0,1)$ such that $G(\rho) \to 0$ as $\rho \to 0^+$. The notation G will not necessarily connote the same function from line to line. For example, in the estimate for I_1, we may choose $G(\rho) = (-\log\rho)^{-1}$. As for I_3, Lemma 3.5 (with $a = 1$ and $b = \delta + 2$) and the estimate (3.12) immediately imply that I_3 is absolutely convergent and that

$$|I_3| \le G(R)|||\,u\,|||\{\|f\|_\infty + R^p\,|||\,u\,|||^p\,\|g\|_\infty\}.$$

Finally, to estimate I_4, note that $(u(\sigma) - \sigma u'(\sigma))' = -\sigma u''(\sigma)$. This plus the facts that $u(0) = 0$ and $|su''(s)| \le (-\log s)^{-2+\varepsilon}|||\,u\,|||$ imply that

$$\begin{aligned} \frac{|u(\sigma) - \sigma u'(\sigma)|}{(-\log\sigma)^{\delta+1}} &\le \frac{1}{(-\log\sigma)^{\delta+1}} \int_0^\sigma |su''(s)|\,ds \\ &\le \frac{\sigma}{(-\log\sigma)^{\delta+3-\varepsilon}} \frac{(-\log\sigma)^{2-\varepsilon}}{\sigma} \int_0^\sigma \frac{1}{(-\log\sigma)^{2-\varepsilon}}\,d\sigma\;|||\,u\,|||. \end{aligned} \tag{3.13}$$

Two applications of Lemma 3.5, first with $a = 0$ and $b = 2 - \varepsilon$, then with $a = 1$ and $b = \delta + 3 - \varepsilon$, show that I_4 is absolutely convergent and that

$$|I_4| \le G(R)\;|||\,u\,|||.$$

This proves the first part of Proposition 3.2. (So far, Lemma 3.5 has only been used to show that certain functions are bounded near 0. Also, we have not yet seen why $\varepsilon > 0$ is required.)

Now suppose that $u \in M_{\varepsilon,R}$ verifies equation (3.11). It is clear that $u \in C^2((0,R])$. Differentiating (3.11) and subsequently multiplying by $r^3(-\log r)^{-\delta}$, one obtains an integrated version of equation (3.10), which immediately implies (3.10). Thus u verifies (3.7).

To prove the last statement in Proposition 3.2, suppose first that $\varepsilon < 1$. Calculate $r(-\log r)^2 u''(r)$ by multiplying the differentiated version of (3.11) by $r(-\log r)^2$. This yields

$$\begin{aligned} r(-\log r)^2 u''(r) = -A\frac{u(r)}{r} &- \frac{C}{p+1}\frac{1}{(-\log r)^{(p/2)-2}}\frac{u(r)^{p+1}}{r} \\ &+ \frac{(-\log r)^{\delta+2}}{r^2}\int_0^r \frac{u(\sigma)}{(-\log\sigma)^{\delta+2}}\left[f(\sigma)+u(\sigma)^p g(\sigma)\right] d\sigma \\ &+ \frac{\delta}{(-\log r)^{1-\varepsilon}}\left\{\frac{(-\log r)^{\delta+3-\varepsilon}}{r^2}\int_0^r \frac{u(\sigma)-\sigma u'(\sigma)}{(-\log\sigma)^{\delta+1}} d\sigma\right\}. \end{aligned} \tag{3.14}$$

Since $p/2 > 2$ and $\varepsilon < 1$, it is clear from (3.13) and (3.14) that

$$\begin{aligned} &\lim_{r\to 0+} r(-\log r)^2 u''(r) \\ &= -A \lim_{r\to 0+}\frac{u(r)}{r} + \lim_{r\to 0+}\frac{(-\log r)^{\delta+2}}{r^2}\int_0^r \frac{\sigma}{(-\log\sigma)^{\delta+2}}\frac{u(\sigma)}{\sigma}\left[f(\sigma)+u(\sigma)^p g(\sigma)\right] d\sigma \\ &= -Ak + \frac{Ak}{2} = -\frac{Ak}{2}, \end{aligned}$$

where Lemma 3.5 has been applied with $a = 1$ and $h(\sigma) = \frac{u(\sigma)}{\sigma}\left[f(\sigma)+u(\sigma)^p g(\sigma)\right]$.

If, on the other hand, $1 \le \varepsilon < 2$, a bootstrap argument comes to the fore. Multiply the differentiated version of (3.11) by $r(-\log r)^{3-\varepsilon-\varepsilon'}$ for some small $\varepsilon' > 0$. Since $1 \le \varepsilon < 2$, it is clear that $3-\varepsilon-\varepsilon' < 2$. It then readily follows that $u \in M_{\varepsilon+\varepsilon'-1,R}$, and the previous argument may be applied.

This completes the proof of Proposition 3.2. □

Consideration is given to a proof of Proposition 3.3. Let $k \in \mathbb{R}$ and $\varepsilon \in (0,2)$ be fixed. Let $R \in (0,1)$ be arbitrary for the moment. Define an iteration on $M_{\varepsilon,R}$ as follows. If $u \in M_{\epsilon,R}$, let Ψu be the continuous function defined on $[0,R]$ such that $(\Psi u)(0) = 0$ and $(\Psi u)'(r)$ is given by the right-hand side of equation (3.11). It is clear from the estimates of I_1 through I_4 obtained while proving Proposition 3.2 that $\Psi u \in C^1([0,R])$ and

$$\|(\Psi u)'\|_\infty \le |k| + G(R)|||\,u\,|||\,[1 + |||\,u\,|||^p], \tag{3.15}$$

where G is a function of R such that $G(R) \to 0$ as $R \to 0^+$ (G depends on ε in that it incorporates the estimate of I_4).

Now calculate $r(-\log r)^{2-\varepsilon}(\Psi u)''(r)$ by differentiating (3.11) and multiplying the result by $r(-\log r)^{2-\varepsilon}$. This leads to

$$\begin{aligned} r(-\log r)^{2-\varepsilon}(\Psi u)''(r) &= -A\frac{1}{(-\log r)^{\varepsilon}}\frac{u(r)}{r} - \frac{C}{p+1}\frac{1}{(-\log \tau)^{(p/2)-2+\varepsilon}}\frac{u(\tau)^{p+1}}{\tau} \\ &+ \frac{1}{(-\log r)^{\varepsilon}}\left\{\frac{(-\log r)^{\delta+2}}{r^2}\int_0^r \frac{u(\sigma)}{(-\log\sigma)^{\delta+2}}\,[f(\sigma)+u(\sigma)^p g(\sigma)]\,d\sigma\right\} \\ &+ \frac{\delta}{-\log r}\left\{\frac{(-\log r)^{\delta+3-\varepsilon}}{r^2}\int_0^r \frac{u(\sigma)-\sigma u'(\sigma)}{(-\log\sigma)^{\delta+1}}\,d\sigma\right\}. \end{aligned} \tag{3.16}$$

It is clear from formulas (3.12) and (3.13) and Lemma 3.5 that

$$\sup_{0<r\leq R} |r(-\log r)^{2-\varepsilon}(\Psi u)''(r)| \leq G(R)|||\,u\,|||[1+|||u|||^p]. \tag{3.17}$$

By the way, one can now see why $\varepsilon > 0$ is necessary. If we had defined the norm $|||\cdot|||$ with $\varepsilon = 0$, then the first and third terms in (3.16) would lack the factor of $G(R)$. Indeed, the $\varepsilon = 0$ case is the "critical" norm, and proving local existence with critical norms is delicate, and not always possible.

It is clear from (3.15) and (3.17) that if $u \in M_{\varepsilon,R}$, then also $\Psi u \in M_{\varepsilon,R}$. Let X_ρ denote the closed ball of radius ρ in $M_{\varepsilon,R}$ with respect to the norm $|||\cdot|||_{\varepsilon,R}$. It also follows immediately from (3.15) and (3.16) that if $|k| < \rho$ and if R is sufficiently small, then Ψ maps X_ρ into itself. It is now routine to show that if R is sufficiently small, then Ψ is a strict contraction on X_ρ, which is to say there exists $L < 1$ such that

$$|||\Psi u - \Psi v||| \leq L|||\,u - v\,|||,$$

for all $u,\, v \in X_\rho$. The only additional estimate needed is

$$|u^{p+1} - v^{p+1}| \leq (p+1)\max[|u|^p, |v|^p]\,|u-v|.$$

Thus, Ψ has a unique fixed point in X_ρ.

This completes the proof of Proposition 3.3. □

Finally, we prove Proposition 3.4. Let φ be given by the transformation (3.5), where u is the solution of (3.7) guaranteed by Propositions 3.2 and 3.3. That $F(s) \to 0$ as $s \to \infty$ follows easily by expressing the derivatives of φ in terms of the derivatives of u and using the known properties of u. To prove the asymptotic formula, consider in more detail the behavior of u near 0. Taylor's formula with integral remainder is

$$u(r) = u(0) + ru'(0) + \int_0^r (r-s)u''(s)\,ds.$$

This formula is clearly valid if u is C^1, and u'' is continuous for $r > 0$ and integrable at the origin, which is the case here. Also, we know $u(0) = 0$ and $u'(0) = k$. By Proposition 3.2, the second derivative u'' has the form

$$u''(s) = \left(-\frac{kA}{2} + \mu(s)\right)\frac{1}{s(-\log s)^2}, \tag{3.18}$$

where $\mu(s) \to 0$ as $s \to 0$, and so

$$u(r) = kr + \left(-\frac{kA}{2} + \mu(r)\right)\int_0^r \frac{r-s}{s(-\log s)^2}\,ds.$$

On the other hand, we have

$$\begin{aligned}\int_0^r \frac{r-s}{s(-\log s)^2}\,ds &= r\int_0^r \frac{1}{s(-\log s)^2}\,ds - \int_0^r \frac{1}{(-\log s)^2}\,ds \\ &= \frac{r}{-\log r} + O\left(\frac{r}{(-\log r)^2}\right).\end{aligned}$$

In consequence, it transpires that as $r \to 0^+$,

$$u(r) = kr\left(1 - \frac{A}{2(-\log r)} + o\left(\frac{1}{-\log r}\right)\right). \tag{3.19}$$

This last relation can be improved by the following bootstrap argument. Substituting (3.19) back into the first term on the right side of (3.14), and using $\varepsilon = 0$ in (3.13) and the last term of (3.14), one concludes that $\mu(s) = O(-\frac{1}{\log s})$ in (3.18). Carrying this estimate through in the above reasoning leads to the following improvement of (3.19):

$$u(r) = kr\left(1 - \frac{A}{2(-\log r)} + O\left(\frac{1}{(-\log r)^2}\right)\right) \tag{3.20}$$

as $r \to 0^+$. Evidently, with more effort, one could explicitly calculate the subsequent terms in this development.

Finally, formula (3.20) translates immediately via the transformation (3.5) to the development given in Proposition 3.4.

3.2. Continuation of the solution and a priori estimates

The purpose of this sub-section is to prove that the solutions constructed in Proposition 3.4 (with $k > 0$) can be continued as solutions to the entire real line, and to obtain estimates on their decay as $s \to -\infty$. The first result is the following.

Proposition 3.5. *Let $k > 0$ and let φ be the C^3-solution of (1.6) constructed in Proposition 3.4. Then the maximal interval of existence of φ is the whole real line.*

Proof. Recall that by Lemma 3.1, the function φ is strictly positive on its maximal interval of existence. Since it is easier to think of *forward* continuation of solutions, set

$$\eta(s) = \varphi(-s).$$

Equation (1.6) for φ is thus equivalent to the equation

$$\eta'''(s) + \eta(s)^p\eta'(s) + \frac{2}{3p}\eta(s) + \frac{1}{3}s\eta'(s) = 0 \tag{3.21}$$

for η.

To prove the proposition, it suffices to show that if η is a positive, regular solution of (3.21) on some interval $[s_0, s_1)$, where $s_1 < \infty$, then $\eta(s)$, $\eta'(s)$, and $\eta''(s)$ are bounded on $[s_0, s_1)$. Equation (3.21) can be re-written as

$$\frac{d}{ds}\left[\eta''(s) + \frac{2}{3p}s\eta(s) + \frac{\eta(s)^{p+1}}{p+1}\right] = \frac{2-p}{3p}s\eta'(s). \tag{3.22}$$

Multiplying (3.22) by $\left[\eta''(s) + \frac{2}{3p}s\eta(s) + \frac{\eta(s)^{p+1}}{p+1}\right]$ and re-writing the resulting right-hand-side, one obtains

$$K'(s) = \frac{p-2}{3p}\left[\frac{\eta'(s)^2}{2} + \frac{4}{3p}\frac{s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right] \tag{3.23}$$

where

$$K(s) = \frac{1}{2}K_1(s)^2 + sK_2(s)$$

and

$$K_1(s) = \eta''(s) + \frac{2}{3p}s\eta(s) + \frac{\eta(s)^{p+1}}{p+1},$$

$$K_2(s) = \frac{(p-2)}{3p}\left[\frac{\eta'(s)^2}{2} + \frac{2}{3p}\frac{s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right].$$

Suppose first that $s_1 > 0$ (the simplest case). It is clear from (3.23) that for all $s > 0$,

$$K'(s) \le 2K_2(s) \le \frac{2}{s}K(s), \tag{3.24}$$

from which it is concluded that K is bounded on $[s_0, s_1)$. Since $K_2(s) > 0$ for all $s > 0$, it follows immediately that $\eta(s)$, $\eta'(s)$, and $\eta''(s)$ are all bounded on $[s_0, s_1)$.

Suppose next that $s_1 \le 0$, and so $s < 0$ for all $s \in [s_0, s_1)$. It follows from (3.23) that

$$K'(s) \ge \frac{p-2}{3p}\left[\frac{\eta'(s)^2}{2} - \frac{4}{3p}\frac{|s_0|\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right],$$

whence K' is seen to be bounded below on $[s_0, s_1)$. Thus, K is also bounded below on $[s_0, s_1)$, from which it follows that

$$-sK_2(s) \le C + \frac{1}{2}K_1(s)^2 \tag{3.25}$$

on $[s_0, s_1)$. On the other hand, formula (3.22) is the same as

$$K_1'(s) = \frac{2-p}{3p}s\eta'(s),$$

which integrates to

$$K_1(s) - K_1(s_0) = \frac{p-2}{3p}\left(\int_{s_0}^{s}\eta(s)ds + s_0\eta(s_0) - s\eta(s)\right).$$

It follows that

$$K_1(s)^2 \le C\left(1 + s^2\eta(s)^2 + ||\eta||^2_{L^1(s_0,s)}\right). \tag{3.26}$$

This estimate, combined with (3.25) and the fact that K_2 is bounded below on $[s_0, s_1)$, implies that

$$\begin{aligned} |sK_2(s)| &\le C\left(1 + s^2\eta(s)^2 + ||\eta||^2_{L^1(s_0,s)}\right) \\ &\le C\left(1 + ||\eta||^2_{L^\infty(s_0,s)}\right) \\ &\le C\left(1 + ||K_2||_{L^\infty(s_0,s)}\right)^{2/(p+2)}, \end{aligned} \tag{3.27}$$

where the constant C might be different from line to line, but is always independent of $s \in [s_0, s_1)$.

If $s_1 < 0$, it follows immediately that K_2 is bounded above on $[s_0, s_1)$. Thus, η and η' are likewise bounded on $[s_0, s_1)$. Formula (3.26) then implies that K_1, and therefore η'', are bounded on $[s_0, s_1)$.

Finally, suppose $s_1 = 0$. Formula (3.27) implies that for $s \in [s_0, 0)$,

$$(1 + |K_2(s)|) \le \frac{C}{|s|}\left(1 + ||K_2||_{L^\infty(s_0,s)}\right)^{2/(p+2)},$$

from which it follows that

$$\left(1 + ||K_2||_{L^\infty(s_0,s)}\right) \le \frac{C}{|s|}\left(1 + ||K_2||_{L^\infty(s_0,s)}\right)^{2/(p+2)},$$

whence

$$\left(1 + ||K_2||_{L^\infty(s_0,s)}\right)^{p/(p+2)} \le \frac{C}{|s|}.$$

This last inequality implies that η' is integrable on $[s_0, 0)$, and so η is bounded on that interval. It now follows from (3.26) that $K_1(s)$ is bounded on $[s_0, 0)$, and therefore so also are η'' and η'.

This concludes the proof of the proposition. □

We now turn to the more delicate matter of finding bounds on η and η' as $s \to \infty$. From the analysis above, in particular formula (3.24), one concludes immediately that $K(s)$ is bounded by $C(1 + s^2)$ for large $s > 0$. It follows that η is bounded on all of $\mathbb{R}$, and that $|\eta'|$ is bounded by $C(1 + s^{1/2})$ for large $s > 0$. These estimates are not strong enough to prove the precise asymptotic behavior announced in the Main Theorem in the Introduction. Indeed, η tends to 0 in an oscillatory fashion, and to prove this precise behavior, we need first to obtain a decay estimate for $\eta(s)$ and sharper bounds on the growth of $\eta'(s)$ as $s \to \infty$.

The approach to these issues is to prove a finer version of the estimate (3.24). To this end, write (3.21) in the equivalent form

$$\frac{d}{ds}\left[\eta''(s) + \alpha s\eta(s) + \frac{\eta(s)^{p+1}}{p+1}\right] + \left(\frac{1}{3} - \alpha\right) s\eta'(s) + \left(\frac{2}{3p} - \alpha\right)\eta(s), \tag{3.28}$$

where $\alpha > 0$ is a parameter to be chosen presently. Formula (3.22) corresponds to $\alpha = 2/3p$. Multiplying (3.28) by $[\eta''(s) + \alpha s\eta(s) + \frac{\eta(s)^{p+1}}{p+1}]$ and further simplifying, one obtains the following generalization of (3.23):

$$\begin{aligned} L'_\alpha(s) = &\left(\frac{p+4}{3p} - 3\alpha\right)\frac{\eta'(s)^2}{2} + \left(\frac{2(p-2)}{3p}\right)\frac{\alpha s\eta(s)^2}{2} \\ &+ \left(\alpha(p+1) - \frac{p+4}{3p}\right)\frac{\eta(s)^{p+2}}{(p+1)(p+2)}, \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} L_\alpha(s) = &\frac{1}{2}\left[\eta''(s) + \alpha s\eta(s) + \frac{\eta(s)^{p+1}}{p+1}\right]^2 \\ &+ \left(\frac{1}{3} - \alpha\right) s\left[\frac{\eta'(s)^2}{2} + \frac{\alpha s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right] + \left(\frac{2}{3p} - \alpha\right)\eta(s)\eta'(s). \end{aligned}$$

The analysis of $L_\alpha(s)$ is technically complicated by the term with $\eta(s)\eta'(s)$. As a first step, it follows from (3.29) that for $s > 0$,

$$L'_\alpha(s) \le M(\alpha, p)\left[\frac{\eta'(s)^2}{2} + \frac{\alpha s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right],$$

where

$$M(\alpha, p) = \max\left[\left(\frac{p+4}{3p} - 3\alpha\right), \left(\frac{2(p-2)}{3p}\right), \left(\alpha(p+1) - \frac{p+4}{3p}\right)\right].$$

Suppose $0 < \alpha < \frac{1}{3}$, and let $\varepsilon > 0$ be small. Then it is observed that

$$\begin{aligned} L'_\alpha(s) &\leq (M(\alpha,p)+\varepsilon)\left[\frac{\eta'(s)^2}{2} + \frac{\alpha s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right] \\ &\quad -\varepsilon\left[\frac{\eta'(s)^2}{2} + \frac{\alpha s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right] \\ &\leq \frac{M(\alpha,p)+\varepsilon}{s\left(\frac{1}{3}-\alpha\right)}\left[L_\alpha(s) - \left(\frac{2}{3p}-\alpha\right)\eta(s)\eta'(s)\right] \\ &\quad -\varepsilon\left[\frac{\eta'(s)^2}{2} + \frac{\alpha s\eta(s)^2}{2} + \frac{\eta(s)^{p+2}}{(p+1)(p+2)}\right] \\ &\leq \frac{M(\alpha,p)+\varepsilon}{\left(\frac{1}{3}-\alpha\right)}\frac{L_\alpha(s)}{s}. \end{aligned} \tag{3.30}$$

The last inequality holds for sufficiently large s since $\frac{|\eta(s)\eta'(s)|}{s}$ is dominated for large s by $\varepsilon\left(\eta'(s)^2 + s\eta(s)^2\right)$, for any fixed $\varepsilon > 0$.

The idea now is to choose α so that the coefficient on the right-hand side of (3.30) is as small as possible. It is helpful to consider separately the cases $p \geq 8$ and $p < 8$. Suppose first that $p \geq 8$. If $\alpha > 0$ is sufficiently small, then $M(\alpha,p) = 2(p-2)/3p$. In this case, (3.30) implies that for any $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$L'_\alpha(s) \leq \left(\frac{2(p-2)}{p} + \varepsilon\right)\frac{L_\alpha(s)}{s}$$

for all sufficiently large $s > 0$. The following proposition is now a straightforward consequence of the last estimate.

Proposition 3.6. *If $p \geq 8$, then for all $\varepsilon > 0$,*

$$\limsup_{s\to\infty} s^{[(2/p)-\varepsilon]}\eta(s) < \infty,$$

$$\limsup_{s\to\infty} s^{[(2/p)-(1/2)-\varepsilon]}|\eta'(s)| < \infty.$$

Suppose next that $p < 8$, (i.e. $p = 5, 6$, or 7). In this case, a good choice is

$$\alpha = \frac{8-p}{9p}.$$

This is the value of α for which the first two coefficients on the right-hand side of (3.29) are equal. This choice of α gives

$$\frac{M(\alpha,p)}{(\frac{1}{3}-\alpha)} = \frac{3}{2},$$

and so for all $\varepsilon > 0$,

$$L'_\alpha(s) \leq \left(\frac{3}{2}+\varepsilon\right)\frac{L_\alpha(s)}{s}, \tag{3.31}$$

for $s > 0$ sufficiently large. We immediately obtain the following proposition.

Proposition 3.7. *If $p = 5, 6,$ or 7, then for all $\varepsilon > 0$,*

$$\limsup_{s\to\infty} s^{[(1/4)-\varepsilon]}\eta(s) < \infty,$$

$$\limsup_{s\to\infty} s^{[-(1/4)-\varepsilon]}|\eta'(s)| < \infty.$$

The estimates for $|\eta'|$ in Proposition 3.6 can be improved by developing this method more fully, but the results in hand suffice for the present purposes.

3.3. Precise asymptotics.

We continue as in the previous sub-section with $\eta(s) = \varphi(-s)$, where φ is the solution of (1.6) whose existence is guaranteed by Proposition 3.4 with $k > 0$. The function η is a positive, regular solution of (3.21) on the whole real line and has the asymptotic properties described in Propositions 3.6 and 3.7.

For $s > 0$, define w by

$$\eta(s) = s^{-\gamma}w(r), \tag{3.32}$$

where γ is given in (3.4) and

$$r = \frac{2s^{3/2}}{3\sqrt{3}}.$$

Propositions 3.6 and 3.7 readily translate in terms of w and w' to the following result.

Proposition 3.8. *If $p \geq 8$, then for all $\varepsilon > 0$,*

$$\limsup_{r \to \infty} r^{[(2/p)-(1/2)-\varepsilon]} (w(r) + |w'(r)|) < \infty.$$

If $p = 5, 6,$ or 7, then for all $\varepsilon > 0$,

$$\limsup_{r \to \infty} r^{[(2/3p)-(1/3)-\varepsilon]} (w(r) + |w'(r)|) < \infty.$$

A calculation shows that η is a solution of (3.21) if and only if w is a solution of the ordinary differential equation

$$\begin{gathered} w'''(r) + w'(r) - \frac{\delta}{r}[w''(r) + w(r)] = \\ \frac{Aw'(r)}{r^2} + \frac{Bw(r)}{r^3} + \frac{Cw(r)^p w'(r)}{r^{p/2}} + \frac{Dw(r)^{p+1}}{r^{(p+2)/2}}, \end{gathered} \tag{3.33}$$

where δ is given by (3.8) and A, B, C, and D are constants depending on p, whose values can be determined explicitly from equation (3.21) and the transformation (3.32). (These are not the same values as in equation (3.7).) Re-write equation (3.33) as a first-order semilinear system of equations as follows:

$$W'(r) = H(r)W(r) + G(r, W(r)), \tag{3.34}$$

where,

$$W(r) = \begin{pmatrix} w(r) \\ w'(r) \\ w''(r) \end{pmatrix}, \quad H(r) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\delta}{r} & -1 & \frac{\delta}{r} \end{pmatrix}, \quad G(r, W) = \begin{pmatrix} 0 \\ 0 \\ g(r, w, w') \end{pmatrix}, \tag{3.35}$$

and

$$g(r, w. w') = \frac{Aw'}{r^2} + \frac{Bw}{r^3} + \frac{Cw^p w'(r)}{r^{p/2}} + \frac{Dw^{p+1}}{r^{(p+2)/2}}. \tag{3.36}$$

The following is a straightforward consequence of Proposition 3.8, and will be important in proving the asymptotic behavior of solutions of the system (3.34) is governed by its linear part.

Lemma 3.9. *There exists $\beta > 0$ such that* $\limsup_{r\to\infty} r^{(1+\beta)}|g(r, w(r), w'(r))| < \infty$.

Now let $Q(r,s)$ be the propagator matrix generated by the family $H(r)$, which is to say $Q(r,r) = I$ the 3×3 identity matrix, for any $r > 0$, and

$$\frac{\partial}{\partial r} Q(r,s) = H(r)Q(r,s). \tag{3.37}$$

Since $Q(r,s)Q(s,r) = I$, one also deduces that

$$\frac{\partial}{\partial s} Q(r,s) = -Q(r,s)H(s). \tag{3.38}$$

If $W(s)$ is a solution of system (3.34), then

$$\frac{d}{ds} Q(r,s)W(s) = Q(r,s)W'(s) - Q(r,s)H(s)W(s) = Q(r,s)G(s, W(s)),$$

or in integrated form,

$$Q(r,s)W(s) = W(r) + \int_r^s Q(r,\sigma)G(\sigma, W(\sigma))d\sigma. \tag{3.39}$$

It will now be shown that $Q(r,s)W(s)$ converges as s tends to ∞. The key fact making this demostration possible is that $Q(r,s)$ is known explicitly. Indeed, equation (3.33), but with the right side replaced by zero, can be solved in closed form. Here are the nine elements $Q_{ij}(r,s)$ of the matrix $Q(r,s)$:

$$Q_{1,1}(r,s) = \cos(r-s) + s^{-\delta} \int_s^r t^\delta \sin(r-t)dt,$$

$$Q_{2,1}(r,s) = -\sin(r-s) + s^{-\delta} \int_s^r t^\delta \cos(r-t)dt,$$

$$Q_{3,1}(r,s) = -\cos(r-s) - s^{-\delta} \int_s^r t^\delta \sin(r-t)dt + \left(\frac{r}{s}\right)^\delta,$$

$$Q_{1,2}(r,s) = \sin(r-s),$$

$$Q_{2,2}(r,s) = \cos(r-s),$$

$$Q_{3,2}(r,s) = -\sin(r-s),$$

$$Q_{1,3}(r,s) = s^{-\delta} \int_s^r t^\delta \sin(r-t)dt,$$

$$Q_{2,3}(r,s) = s^{-\delta} \int_s^r t^\delta \cos(r-t)dt,$$

$$Q_{3,3}(r,s) = -s^{-\delta} \int_s^r t^\delta \sin(r-t)dt + \left(\frac{r}{s}\right)^\delta.$$

It is immediate to check that $Q(r,s)$, so-defined, verifies $Q(r,r) = I$ and equation (3.37). Also, one easily verifies that for any fixed $r > 0$,

$$\sup_{s \geq r} |Q_{ij}(r,s)| < \infty,$$

for $1 \leq i,j \leq 3$. In other words, the backwards propagators $Q(r,s)$, with $s > r$, are bounded as s tends to ∞. (On the other hand, this is false for the forward propagators $Q(r,s)$, with $r > s$, as r tends to ∞.) This estimate coupled with Lemma 3.9 implies that for a fixed $r \in \mathbb{R}$, the limit

$$\lim_{s\to\infty} \int_r^s Q(r,\sigma)G(\sigma, W(\sigma))d\sigma$$

exists and, therefore, by (3.39), that

$$\lim_{s\to\infty} Q(r,s)W(s) = L(r)$$

exists in $\mathbb{R}^3$. Letting s tend to ∞ in (3.39) leads to

$$L(r) = W(r) + \int_r^\infty Q(r,\sigma)G(\sigma, W(\sigma))d\sigma.$$

Replacing $W(r)$ by the expression given in (3.39) leads to the conclusion that

$$L(r) = Q(r,s)W(s) + \int_s^\infty Q(r,\sigma)G(\sigma, W(\sigma))d\sigma,$$

and so

$$Q(s,r)L(r) = W(s) + \int_s^\infty Q(s,\sigma)G(\sigma, W(\sigma))d\sigma.$$

Interchanging the roles of s and r and replacing s by s_0, the last formula may be re-written as

$$W(r) = Q(r,s_0)L(s_0) - Q(r,s_0)\int_r^\infty Q(s_0,\sigma)G(\sigma, W(\sigma))d\sigma. \tag{3.40}$$

Successive integrations by parts show that for a fixed s,

$$\int_s^r t^\delta \sin(r-t)dt = r^\delta + c_1\cos r + c_2 \sin r + O(r^{\delta-2}) \tag{3.41}$$

and

$$\int_s^r t^\delta \cos(r-t)dt = d_1 \cos r + d_2 \sin r + \delta r^{\delta-1} + O(r^{\delta-2}) \tag{3.42}$$

as $r \to \infty$. With these relations in hand, it follows from (3.40) that

$$w(r) = O(r^\delta) \quad \text{and} \quad |w'(r)| \leq O(1)$$

as $r \to \infty$, which in turn gives an improvement of Lemma 3.9, specifically

$$|g(r, w(r), w'(r))| = O(r^{-2}) \tag{3.43}$$

as $r \to \infty$. It now follows from (3.40), (3.41), and (3.42) that for some a, b, and $c \in \mathbb{R}$,

$$\begin{aligned} w(r) - \left[ar^\delta + b\cos r + c \sin r\right] &= O(r^{\delta-1}), \\ w'(r) - \left[a\delta r^{\delta-1} + c\cos r - b\sin r\right] &= O(r^{-1}), \end{aligned} \tag{3.44}$$

as $r \to \infty$. The first of these estimates implies that there exist a', b' and c' such that

$$\eta(s) - \left[a's^{-2/p} + s^{-(3/4)+(1/p)}\left(b'\cos\left(\frac{2s^{3/2}}{3\sqrt{3}}\right) + c'\sin\left(\frac{2s^{3/2}}{3\sqrt{3}}\right)\right)\right]$$

as $s \to \infty$.

To complete the proof of the Main Theorem for $p > 4$, it remains only to show that $a' \neq 0$ in the previous formula. Indeed, suppose $a' = 0$. Then $b' = c' = 0$, for if not, then $\eta(s) = \varphi(-s)$ has infinitely many zeroes, which is impossible since $\eta(s) > 0$ for all real s. The condition $a' = b' = c' = 0$ implies that $a = b = c = 0$ in (3.44). This means that $L(s_0) = 0$ in (3.40). An obvious bootstrap argument applied to (3.40) and starting with the estimate (3.43) then shows that $w(r), w'(r), w''(r)$ all decay as $r \to \infty$ faster than any inverse power of r. This in turn implies, by (3.12), that $F(s) \to 0$ as $s \to -\infty$, which is impossible since $F(s) \to 0$ as $s \to +\infty$ and $F'(s) < 0$ (by formula (3.1)). Thus, $a' \neq 0$.

This concludes the proof of the Main Theorem. □

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