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SOLITARY WAVES IN NONLINEAR DISPERSIVE SYSTEMS

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*This paper is dedicated to the memory of Chen Qi Zhuang,
Hongqiu's father and her first mathematics teacher.*

Abstract. Evolution equations that feature both nonlinear and dispersive effects often possess solitary-wave solutions. Exact theory for such waves has been developed and applied to single equations of Korteweg-de Vries type, Schrödinger-type and regularized long-wave-type for example. Much less common has been the analysis of solitary-wave solutions for systems of equations. The present paper is concerned with solitary travelling-wave solutions to systems of equations arising in fluid mechanics and other areas of science and engineering. The aim is to show that appropriate modification of the methods coming to the fore for single equations may be effectively applied to systems as well. This contention is demonstrated explicitly for the Gear-Grimshaw system modeling the interaction of internal waves and for the Boussinesq systems that arise in describing the two-way propagation of long-crested surface water waves.

1. **Introduction.** Solitary waves play an important and sometimes dominant role in the propagation of nonlinear, dispersive wave motion. Consequently, the investigation of the existence, stability and other properties of such waves by analytical, numerical and experimental means has been a focus of activity for more than three decades. Much of the effort concerned with existence theory of solitary waves has been focused either on model equations for the unidirectional propagation of waves in nonlinear, dispersive media, or with the full Euler equations for surface and internal waves, though with some notable exceptions, such as the work of Toland (1981, 1984) on the system of equations derived by Bona and Smith (1976) for surface water wave propagation and M. Chen's work (1998) on more general classes of Boussinesq systems.

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The focus of this contribution is to develop general techniques that may be used to establish existence of travelling-wave solutions of systems of evolution equations. Two strategies will be proposed. One is to utilize the concentration-compactness theory of P.-L. Lions (1984) which has been used in the context of solitary waves by Weinstein (1987), Albert (1996) and Chen and Bona (1998), for example. The other methodology to be developed here for systems is the positive operator theory of Krasnosell'skii (1964a, 1964b) which has been applied in the context of nonlinear, dispersive waves by Benjamin *et al.* (1990) and Chen and Bona (1998).

The general results will be shown to be effective on two example systems of equations that are of current interest. One is the system derived by Gear and Grimshaw (1984) to describe the interaction of internal waves having different vertical structures, but nearly identical phase speeds, and the other is the Boussinesq system for two-way propagation of small-amplitude, long-wavelength, long-crested disturbances on the surface of a lake or ocean or in a channel. These systems are now described in a little more detail.

In an idealized, incompressible, fluid system consisting of two horizontal layers of different densities with the heavier fluid below the lighter one, small-amplitude long waves propagating on the fluid interface, or pycnocline, may be approximately described by writing the vertical displacement η in the form

$$\eta(x,z,t) = u(x,t)\rho(z), \tag{1.1}$$

where x is the horizontal coordinate in the primary direction of the waves' propagation, z is the vertical coordinate in a standard Cartesian frame and t is elapsed time. Here, the waves are presumed to be long-crested, so not varying significantly in the horizontal direction y perpendicular to x. In the approximation afforded by the representation (1.1), the waves' vertical structure is determined by ρ which is in turn the solution of a Sturm-Liouville eigenvalue problem in which the eigenvalue is related to the primary speed of propagation (the speed of propagation of waves of extreme length). In certain circumstances, it may happen that waves with different vertical structure have practically the same speed of propagation, and in this case interesting interactions can occur between them. If u and v denote the horizontal variation of a pair of such waves, then their propagation is governed at the level of modelling displayed in (1.1) by the system

$$\begin{cases} u_t + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x = 0, \\ b_1 v_t + rv_x + vv_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 uu_x + b_2 a_1 (uv)_x = 0 \end{cases} \tag{1.2}$$

derived by Gear and Grimshaw (1984). Here a_1, a_2, a_3, b_1, b_2 and r are real constants with b_1, b_2 positive, determined by the densities of the fluid layers and their vertical extents. Once u and v are obtained from (1.2), the entire motion is approximately

$$u(x,t)\rho_1(z) + v(x,t)\rho_2(z),$$

where ρ_1 and ρ_2 are the vertical structure functions corresponding to u and v, respectively. The initial-value problem for (1.2) is known to comprise a well-posed problem (see Bona *et al.* 1992). Here, attention is given to travelling-wave solutions.

One of the more venerable of the systems arising in fluid mechanics is the original Boussinesq equations (Boussinesq 1871)

$$\begin{cases} \eta_t + u_x + (u\eta)_x = 0, \\ u_t + \eta_x + uu_x + \dfrac{1}{3}\eta_{xtt} = 0, \end{cases} \tag{1.3}$$

put forward as a description of waves in a channel or long-crested, long-wavelength water waves propagating on the surface of water. In Boussinesq's conception, η is a scaled version of the small displacement of the free surface whilst u is the depth-averaged horizontal velocity of the water at the spatial point x. Based on this original system, but taking a different view of the velocity variable, and using the lowest-order relationship (the linear wave equation) to systematically alter the dispersion terms leads to the family of formally equivalent systems

$$\begin{cases} \eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0, \\ u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \end{cases} \tag{1.4}$$

where

$$a = \frac{1}{2}(\theta^2 - \frac{1}{3})\lambda, \quad b = \frac{1}{2}(\theta^2 - \frac{1}{3})(1-\lambda),$$
$$c = \frac{1}{2}(1-\theta^2)\mu, \quad d = \frac{1}{2}(1-\theta^2)(1-\mu).$$

The parameter $\theta \in [0,1]$ specifies which velocity variable is used in the model whilst λ and μ are modelling parameters, (see Benjamin 1974, Whitham 1974, Bona and Smith 1976, Bona and Chen 1999, Bona, Chen and Saut 2000). The constants a, b, c, d are actually a three-parameter family satisfying the constraints

$$a + b = \frac{1}{2}(\theta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1-\theta^2) \geq 0, \quad a + b + c + d = \frac{1}{3}.$$

When $a = b = c = 0$ and $d = \frac{1}{3}$, the system reduces to a regularized version of the original Boussinesq system (1.3), when $a = 0$, $b = \frac{1}{6}$, $c = 0$, $d = \frac{1}{6}$, the system is the regularized Bosssinesq system studied by Bona and M. Chen (1998), when $a = 0$, $b = \frac{1}{3}$, $c = -\frac{1}{3}$, $d = \frac{1}{3}$, the system is the one put forward by Bona and Smith (1976) and studied by Toland (1981, 1984), whereas, when $a = c = \frac{1}{6}$ and $b = d = 0$, there obtains a coupled KdV-system.

Both the (1.2) and the (1.4) systems may be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + L\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}_x - N\begin{pmatrix} u \\ v \end{pmatrix}_t = 0, \tag{1.5}$$

where the operators L and N are 2×2 matrices, each entry of which is a Fourier multiplier operator associated with the dispersion relation, and f_1, f_2 are non-linear, smooth, real-valued functions defined on $\mathbb{R}^2$ satisfying $f_1(0,0) = f_2(0,0) = 0$. (In the specific systems described above, f_1 and f_2 are quadratic polynomials.)

Solitary waves in such systems are the focus of attention here. These waves have the form $u(x,t) = \phi(x - Ct), v(x,t) = \psi(x - Ct)$ where $C > 0$, say. Substituting this travelling-wave form into (1.5) and demanding ϕ and ψ vanish suitably at $\pm\infty$ reduces (1.5) to another system of equations, namely

$$(C - CN - L)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f_1(\phi,\psi) \\ f_2(\phi,\psi) \end{pmatrix}, \tag{1.6}$$

where $C - CN - L = CI - CN - L$ and I is the 2×2 identity matrix.

Once it has been appreciated that both the concrete systems just described have the form depicted in (1.5), it is natural to generalize the discussion to evolution equations of the form

$$\mathbf{u}_t + L\mathbf{u}_x + \mathbf{f}(\mathbf{u})_x - N\mathbf{u}_t = 0, \tag{1.7}$$

where the boldfaced letter $\mathbf{u} = (u_1, \cdots, u_m)$ stands for an $m-$dimensional vector of functions, each component u_j of which is a function of $x \in \mathbb{R}$ and time $t \in$

$[0,\infty)$, and whose transpose is denoted by $\mathbf{u}^t$, L and N are $m\times m$ matrices whose entries are Fourier multiplier operators, $\mathbf{f}=(f_1,\cdots,f_m)$, where for $1\le j,k\le m$, each component f_j is a real-valued, smooth function defined on $\mathbb{R}^m$ satisfying $f_j(0,\cdots,0)=0$ and $\partial_k f_j(0,\cdots,0)=0$. The study of solitary waves $\mathbf{u}(x,t)=\mathbf{u}(x-Ct)=\big(u_1(x-Ct),\cdots,u_m(x-Ct)\big)$ that vanish at $\pm\infty$ reduces the system (1.7) to the form

$$(C-CN-L)\mathbf{u}=\mathbf{f}(\mathbf{u}). \tag{1.8}$$

The system (1.6) is a special case of the system (1.8) where $m=2$.

This paper consists of five parts including the Introduction. Section 2 provides theory applicable to the Gear-Grimshaw system and to certain other two-dimensional members of the general class depicted in (1.8). These developments rely upon the concentration-compactness principle. An application of positive-operator theory to the system (1.6) with particular reference to the system (1.2) and to the family of systems (1.4) is the focus of Section 3. Sections 4 and 5 extend the theory to the systems as in (1.8). There is also an Appendix where some technical issues arising at several points in the development are settled.

It is worth pointing out explicitly that the theory resulting from the use of the concentration-compactness principle and that related to positive-operator theory overlap, as we show by using both on the Gear-Grimshaw system, but neither contains the other. Concentration-compactness methods rely on less structure in the linearized dispersion relation, but require homogeneity of the nonlinearity. Positive-operator theory only requires a superlinearity presumption on the nonlinearity, but demands more of the dispersion relation. Positive-operator theory has additional advantages when it is applicable because the theory rests on topological degree theory. In consequence, perturbation results and existence of periodic travelling-waves appear as corollaries. These latter points are not developed here, but they will be enunciated in a separate essay.

2. Solitary-Wave Solutions of the Gear-Grimshaw System and its Generalizations via the Concentration-Compactness Principle.

2.1. **Notation.** For $p\ge 1$, the Banach space $L_p=L_p(\mathbb{R})$ is the class of Lebesgue-measurable functions on the real line $\mathbb{R}$ that are p-th power integrable, with the usual modification if $p=\infty$. The norm of a function $f\in L_p$ is denoted by $|f|_p$. The Fourier transform of a function $f\in L_2$ is defined as $\widehat{f}(\xi)=\int e^{-i\xi x}f(x)\,dx$. An unadorned integral will always connote the integral over the real line $\mathbb{R}$. For a non-negative number s, the Sobolev space $H^s=H^s(\mathbb{R})=\{f\in L_2:(1+\xi^2)^{\frac{s}{2}}|\widehat{f}(\xi)|\in L_2\}$ carries the norm $\|\cdot\|_s$ defined by $\|f\|_s^2=\int(1+\xi^2)^s|\widehat{f}(\xi)|^2\,d\xi$. In particular, $H^0=L_2$, so $\|f\|_0=|f|_2$. In case $f\in H^k(\mathbb{R})$ for all $k\ge 0$, we write $f\in H^\infty(\mathbb{R})$. When $s=m$ is a positive integer, $\|f\|_s$ is equivalent to $\big\{|f|_2^2+|D_x^m f|_2^2\big\}^{\frac12}$; when s is not an integer, $s=m+\delta$, say, where m is non-negative integer, and $0<\delta<1$, $\|f\|_s$ is equivalent to

$$\Big\{\|f\|^2+\min\{1,m\}\|D_x^m f\|^2+\int\!\!\int\frac{|D_x^m f(x)-D_y^m f(y)|^2}{|x-y|^{1+2\delta}}\,dy\,dx\Big\}^{\frac12},$$

where $D_x^m f$ denotes the mth derivative of f with respect to its argument x, and in case $m=0$, the zero-th derivative $D_x^0 f$ of f stands for f. The product spaces $L_p\times L_p=\big\{(f,g): f,\ g\in L_p\big\}$ and $H^r\times H^s=\big\{(f,g): f\in H^r,\ g\in H^s\big\}$ are

Banach spaces with the norms $|(f,g)|_{p\times p}^p = |f|_p^p + |g|_p^p$ and $\|(f,g)\|_{r\times s}^2 = \|f\|_r^2 + \|g\|_s^2$, respectively.

2.2. **Solitary-Wave Solutions of the Gear-Grimshaw System.** The Gear-Grimshaw version of the travelling-wave system (1.6) is

$$\begin{cases} -C\phi + \frac{1}{2}\phi^2 + \phi'' + a_3\psi'' + \frac{1}{2}a_1\psi^2 + a_2\phi\psi = 0, \\ -b_1C\psi + r\psi + \frac{1}{2}\psi^2 + \psi'' + b_2a_3\phi'' + \frac{1}{2}b_2a_2\phi^2 + b_2a_1\phi\psi = 0, \end{cases}$$

or

$$\begin{cases} (C - D_x^2)\phi - a_3D_x^2\psi = \frac{1}{2}\phi^2 + \frac{1}{2}a_1\psi^2 + a_2\phi\psi, \\ -b_2a_3D_x^2\phi + (b_1C - r - D_x^2)\psi = \frac{1}{2}\psi^2 + \frac{1}{2}b_2a_2\phi^2 + b_2a_1\phi\psi. \end{cases} \tag{2.1}$$

Introduce the notation

$$\mathcal{L}_C = \begin{pmatrix} b_2(C - D_x^2) & -b_2a_3D_x^2 \\ -b_2a_3D_x^2 & (b_1C - r) - D_x^2 \end{pmatrix}$$

and

$$F(\phi,\psi) = \frac{b_2}{6}\phi^3 + \frac{a_2b_2}{2}\phi^2\psi + \frac{a_1b_2}{2}\phi\psi^2 + \frac{1}{6}\psi^3.$$

The system (2.1), with the first equation multiplied by b_2, may be written in the tidy form

$$\mathcal{L}_C(\phi,\psi)^t = \operatorname{grad} F(\phi,\psi). \tag{2.2}$$

Thus existence of solitary-wave solutions of the Gear-Grimshaw system is reduced to the existence of appropriate solutions of (2.2). Motivated by the ideas of Weinstein (1987), introduce the functional

$$\Lambda(f,g) = \frac{J(f,g)}{\left(\int F(f,g)\,dx\right)^{\frac{2}{3}}}$$

where

$$J(f,g) = \int (f,g)\mathcal{L}_C(f,g)^t\,dx$$

for $(f,g) \in H^1 \times H^1$. Elementary calculus shows that any non-zero critical point, in particular, any non-zero minimizer of Λ is, up to a rescaling, a non-trivial solution of (2.2). Furthermore, Λ is homogeneous of degree zero, which is to say that for any constant $\lambda \neq 0$, $\Lambda(\lambda f, \lambda g) = \Lambda(f,g)$. Hence instead of studying directly the minimization problem for Λ, we consider instead the variational problem

$$\Theta(1) = \inf\Big\{J(f,g) : f,g \in H^1, \int F(f(x),g(x))\,dx = 1\Big\}. \tag{2.3}$$

From its definition, the operator $\mathcal{L}_C$ is self-adjoint on $H^1 \times H^1$ and satisfies the inequalities

$$\underline{\gamma}(\|f\|_1^2 + \|g\|_1^2) \le J(f,g) \le \overline{\gamma}(\|f\|_1^2 + \|g\|_1^2) \tag{2.4}$$

where

$$\underline{\gamma} = \min\Big\{b_2C,\ b_1C - r,\ \frac{1 - b_2a_3^2}{2},\ \frac{b_2(1 - b_2a_3^2)}{2}\Big\}$$

and

$$\overline{\gamma} = \max\big\{b_2C,\ b_1C - r,\ b_2(1 + b_2a_3^2),\ 2\big\}.$$

In particular, $\mathcal{L}_C$ is positive definite on $H^1 \times H^1$ if $\underline{\gamma} > 0$. On the other hand, repeated use of Young's inequality shows that

$$|F(\phi,\psi)| \le \gamma_0(|\phi|^3 + |\psi|^3),$$

where $\gamma_0 = \max\{\frac{b_2}{6} + \frac{|a_2|b_2}{3} + \frac{|a_1|b_2}{6}, \quad \frac{|a_2|b_2}{6} + \frac{|a_1|b_2}{3} + \frac{1}{6}\}$. Since $H^1 \times H^1 \subset L_p \times L_p$ for any p in the range $2 \le p \le \infty$ with an embedding constant less than or equal to 1, it follows that $0 < \Theta(1) < \infty$ and that every minimizing sequence $\{(u_n, v_n)\}_{n=1}^\infty$ is bounded in $H^1 \times H^1$. Denote by ρ_n the quantity $|u_n|^2 + |u_n'|^2 + |v_n|^2 + |v_n'|^2$, and let $\int \rho_n(x)\,dx = \mu_n$. Then the sequence $\{\mu_n\}_{n=1}^\infty$ is bounded and furthermore,

$$\mu_n = \|(u_n, v_n)\|_1^2 \ge |(u_n, v_n)|_3^2 \ge \left(\gamma_0^{-1}\int F(u_n, v_n)\,dx\right)^{\frac{2}{3}} = \gamma_0^{-\frac{2}{3}}.$$

Without loss of generality, suppose $\mu_n \to \mu$ as $n \to \infty$, so that $\mu \ge \gamma_0^{-\frac{2}{3}}$. The concentration-compactness principle then yields the following result.

Theorem 2.1. *If $b_2 a_3^2 < 1$, then for any wave speed $C > \frac{\tau}{b_1}$, every minimizing sequence for (2.3) is, up to spatial translates, relatively compact. Consequently, the system (2.1) or (2.2) has a nontrivial solution which lies in $H^\infty \times H^\infty$.*

Proof. By Lions' principle, if the theorem is not valid, then there is a subsequence $\{\rho_{n_k}\}_{k=1}^\infty$ of $\{\rho_n\}_{n=1}^\infty$ which satisfies either the Vanishing or the Dichotomy criterion. If Vanishing occurs, then for any $R > 0$,

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\le R} \rho_{n_k}(x)\,dx = 0,$$

whence

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{|x-y|\le R} (|u_{n_k}(x)|^2 + |v_{n_k}(x)^2|)\,dx = 0.$$

Since the derivatives u_{n_k}' and v_{n_k}' are bounded in L_2, Lemma 2.4 of Chen and Bona (1998) leads to the contradiction

$$1 = \int F(u_{n_k}, v_{n_k})\,dx \le \gamma_0 \int (|u_{n_k}(x)|^3 + |v_{n_k}(x)|^3)\,dx \to 0.$$

If Dichotomy occurs, there is a $\bar\mu \in (0,\mu)$ such that for any $\epsilon > 0$, there corresponds a k_0 and sequences $\{\rho_k^1\}_{k=1}^\infty, \{\rho_k^2\}_{k=1}^\infty \subset L_1$, $\rho_k^1, \rho_k^2 \ge 0$, such that for $k \ge k_0$,

$$\begin{cases} |\rho_{n_k} - (\rho_k^1 + \rho_k^2)|_1 \le \epsilon, \quad \left|\int \rho_k^1\,dx - \bar\mu\right| \le \epsilon, \quad \left|\int \rho_k^2\,dx - (\mu - \bar\mu)\right| \le \epsilon, \\ \text{supp}\,\rho_k^1 \cap \text{supp}\,\rho_k^2 = \emptyset \quad \text{and} \quad \lim_{k\to\infty} \text{dist}\{\text{supp}\,\rho_k^1, \text{supp}\,\rho_k^2\} = \infty. \end{cases} \tag{2.5}$$

In fact, the supports of ρ_k^1 and ρ_k^2 may be assumed to be separated as follows:

$$\text{supp}\,\rho_k^1 \subset (y_k - E_0, y_k + E_0), \quad \text{supp}\,\rho_k^2 \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, \infty),$$

for some fixed $E_0 > 0$, a sequence $\{y_k\}_{k=1}^\infty \subset \mathbb{R}$ and a sequence $\{R_k\}_{k=1}^\infty$ for which $R_k \to \infty$ (see again Chen and Bona 1998, p. 58). To obtain splitting functions u_k^1, u_k^2 and v_k^1, v_k^2 of u_{n_k} and v_{n_k}, respectively, $k = 1, 2, \cdots$, let $\zeta, \psi \in C_b^\infty$ with $0 \le \zeta, \psi \le 1$ be such that $\zeta(x) + \psi(x) = 1$ for all $x \in \mathbb{R}$, and

$$\zeta(x) = \begin{cases} 1, & \text{when } |x| \le 1, \\ 0, & \text{when } |x| \ge 2, \end{cases} \qquad \psi(x) = \begin{cases} 0, & \text{when } |x| \le 1, \\ 1, & \text{when } |x| \ge 2. \end{cases} \tag{2.6}$$

For $x \in \mathbb{R}$, denote by ζ_k and ψ_k the functions $\zeta_k(x) = \zeta(\frac{x-y_k}{E_1})$ and $\psi_k(x) = \psi(\frac{x-y_k}{R_k})$, where $E_1 > E_0$ is chosen large enough (in this case, it is required that $E_1 \geq \max_{x\in\mathbb{R}}|\zeta'(x)|$) that for $k \geq k_0$,

$$\left|\int \left(|\zeta_k u_{n_k}|^2 + |\zeta_k v_{n_k}|^2 + |D_x(\zeta_k u_{n_k})|^2 + |D_x(\zeta_k v_{n_k})|^2\right) - \rho_k^1\, dx\right| \leq 2\epsilon$$

and

$$\left|\int \left(|\psi_k u_{n_k}|^2 + |\psi_k v_{n_k}|^2 + |D_x(\psi_k u_{n_k})|^2 + |D_x(\psi_k v_{n_k})|^2\right) - \rho_k^2\, dx\right| \leq 2\epsilon.$$

To see this is possible, first note that from (2.5), for $k \geq k_0$,

$$\begin{aligned}\epsilon \geq |\rho_{n_k} - (\rho_k^1 + \rho_k^2)|_1 = &\int_{|x-y_k|\leq E_0} |\rho_{n_k} - \rho_k^1|\, dx \\ &+ \int_{|x-y_k|\geq 2R_k} |\rho_{n_k} - \rho_k^2|\, dx + \int_{E_0\leq|x-y_k|\leq 2R_k} \rho_{n_k}\, dx.\end{aligned}$$

In consequence of this relation, it transpires that for $k \geq k_0$,

$$\begin{aligned}&\left|\int \left(|\zeta_k u_{n_k}|^2 + |\zeta_k v_{n_k}|^2 + |D_x(\zeta_k u_{n_k})|^2 + |D_x(\zeta_k v_{n_k})|^2 - \rho_k^1\right) dx\right| \\ &= \left|\int_{|x-y_k|\leq 2E_1} \left(|\zeta_k u_{n_k}|^2 + |\zeta_k v_{n_k}|^2 + |D_x(\zeta_k u_{n_k})|^2 + |D_x(\zeta_k v_{n_k})|^2 - \rho_k^1\right) dx\right| \\ &\leq \int_{|x-y_k|\leq E_0} |\rho_{n_k} - \rho_k^1|\, dx + \max_{x\in\mathbb{R}}\{|\zeta_k(x)|^2 + |\zeta_k'(x)|^2\} \int_{E_0\leq|x-y_k|\leq 2E_1} \rho_{n_k} dx \\ &\leq \int_{|x-y_k|\leq E_0} \left|\rho_{n_k} - \rho_k^1\right| dx + \max_{x\in\mathbb{R}}\{1 + \frac{1}{E_1^2}|\zeta'(x)|^2\} \int_{E_0\leq|x-y_k|\leq 2E_1} \rho_{n_k} dx \\ &\leq \int_{|x-y_k|\leq E_0} \left|\rho_{n_k} - \rho_k^1\right| dx + 2\int_{E_0\leq|x-y_k|\leq 2R_k} \rho_{n_k} dx \;\leq\; 2\epsilon\end{aligned}$$

and

$$\begin{aligned}&\left|\int \left(|\psi_k u_{n_k}|^2 + |\psi_k v_{n_k}|^2 + |D_x(\psi_k u_{n_k})|^2 + |D_x(\psi_k v_{n_k})|^2 - \rho_k^2\right) dx\right| \\ &= \left|\int_{|x-y_k|\geq R_k} \left(|\psi_k u_{n_k}|^2 + |\psi_k v_{n_k}|^2 + |D_x(\psi_k u_{n_k})|^2 + |D_x(\psi_k v_{n_k})|^2 - \rho_k^2\right) dx\right| \\ &\leq \left|\int_{R_k\leq|x-y_k|\leq 2R_k} \left(|\psi_k u_{n_k}|^2 + |\psi_k v_{n_k}|^2\right.\right. \\ &\qquad \left.\left. + |D_x(\psi_k u_{n_k})|^2 + |D_x(\psi_k v_{n_k})|^2 - \rho_k^2\right) dx\right| + \int_{|x-y_k|>2R_k} \left|\rho_{n_k} - \rho_k^2\right| dx \\ &\leq \max_{x\in\mathbb{R}}\left\{|\psi(x)|^2 + \frac{1}{R_k^2}|\psi'(x)|^2\right\} \int_{R_k\leq|x-y_k|\leq 2R_k} \rho_{n_k}\, dx \\ &\qquad + \int_{|x-y_k|\geq 2R_k} |\rho_{n_k} - \rho_k^2|\, dx \\ &\leq 2\int_{E_0\leq|x-y_k|\leq 2R_k} \rho_{n_k}\, dx + \int_{|x-y_k|\geq 2R_k} |\rho_{n_k} - \rho_k^2|\, dx \;\leq\; 2\epsilon.\end{aligned}$$

Thus if we set $u_k^1 = \zeta_k u_{n_k}$, $v_k^1 = \zeta_k v_{n_k}$, $u_k^2 = \psi_k u_{n_k}$, $v_k^2 = \psi_k v_{n_k}$, and define w_k^u, w_k^v by $u_{n_k} = u_k^1 + u_k^2 + w_k^u$ and $v_{n_k} = v_k^1 + v_k^2 + w_k^v$, then $u_k^1, v_k^1, u_k^2, v_k^2, w_k^u, w_k^v \in H^1$ and

$\int |F(u_k^1, v_k^1)|\,dx$ and $\int |F(u_k^2, v_k^2)|\,dx$ are bounded. Moreover, there are subsequences of $\{(u_k^1, v_k^1)\}_{k=1}^\infty$ and $\{(u_k^2, v_k^2)\}_{k=1}^\infty$, still denoted by $\{(u_k^1, v_k^1)\}_{k=1}^\infty$ and $\{(u_k^2, v_k^2)\}_{k=1}^\infty$, respectively, for which there is a $k_1 > k_0$ and $\lambda \in \mathbb{R}$ such that for $k \geq k_1$,

$$\Big| \int F\big(u_k^1(x), v_k^1(x)\big)\,dx - \lambda \Big| \leq \epsilon, \quad \Big| \int F\big(u_k^2(x), v_k^2(x)\big)\,dx - (1-\lambda) \Big| \leq \epsilon,$$

$$\begin{aligned} &\|(w_k^u, w_k^v)\|_{1\times 1} = \|\,(1 - \zeta_k - \psi_k)\,(u_{n_k}, v_{n_k})\|_{1\times 1} \\ &\leq \max_{x\in\mathbb{R}} \Big\{ |1 - \zeta_k(x) - \psi_k(x)|,\, |\frac{\zeta'(x)}{E_1} + \frac{\psi'(x)}{R_k}| \Big\} \Big(\int_{E_1 \leq |x - y_k| \leq 2R_k} \rho_{n_k}\,dx \Big)^{\frac{1}{2}} \\ &\leq 2\epsilon \end{aligned}$$

and

$$\operatorname{supp}(u_k^1, v_k^1) \subset (-2E_1, 2E_1), \quad \operatorname{supp}(u_k^2, v_k^2) \subset (-\infty, -R_k) \cup (R_k, \infty),$$

where $R_k > E_1 > 0$ and $\lim_{k\to\infty} R_k = \infty$. A simple calculation shows that, as $k \to \infty$,

$$\begin{aligned} J(u_{n_k}, v_{n_k}) &= J\big(u_k^1 + u_k^2 + w_k^u, v_k^1 + v_k^2 + w_k^v\big) \\ &= J\big(u_k^1, v_k^1\big) + J\big(u_k^2, v_k^2\big) + J\big(w_k^u, w_k^v\big) + \sum_{j=1,2} \{J(u_k^j, w_k^v) + J(v_k^j, w_k^u)\} \\ &= J\big(u_k^1, v_k^1\big) + J\big(u_k^2, v_k^2\big) + \mathcal{R}, \end{aligned}$$

where the remainder $\mathcal{R}$ is bounded by a constant times ϵ. Thus, it is adduced that

$$\begin{aligned} \Theta(1) &= \lim_n J(u_n, v_n) = \lim_k J(u_{n_k}, v_{n_k}) \\ &= \lim_k J\Big((u_k^1, v_k^1) + (u_k^2, v_k^2) + (w_k^u, w_k^v) \Big) \\ &\geq \liminf_k J(u_k^1, v_k^1) + \liminf_k J(u_k^2, v_k^2) + \text{order}(\epsilon), \end{aligned}$$

where order(ϵ) stands for a remainder that is bounded by ϵ times a constant which is independent of k sufficiently large. If $\int F\big(u_k^1, v_k^1\big)\,dx \to \lambda = 0$, then from the boundedness of J,

$$\begin{aligned} \liminf_k J(u_k^1, v_k^1) &\geq \liminf_k \underline{\gamma} \|(u_k^1, v_k^1)\|_{1\times 1}^2 \geq \liminf_k \underline{\gamma} |\rho_k^1|_1 - \text{order}(\epsilon) \\ &\geq \underline{\gamma}\bar{\mu} - \text{order}(\epsilon), \end{aligned}$$

where $\bar{\mu} = \liminf_k |\rho_k^1|_1 > 0$ (since Vanishing has been ruled out). It follows that

$$\Theta(1) > \underline{\gamma}\bar{\mu} + \liminf_k J(u_k^2, v_k^2) - \text{order}(\epsilon).$$

Choosing $\epsilon > 0$ sufficiently small in the last relation leads to the contradiction

$$\Theta(1) > \frac{1}{2}\underline{\gamma}\bar{\mu} + \Theta(1) > \Theta(1).$$

If, on the other hand, $\int F\big(u_k^1, v_k^1\big)\,dx \to \lambda \neq 0$, then

$$\Theta(1) \geq \Theta(\lambda) + \Theta(1-\lambda) + \text{order}(\epsilon),$$

and letting $\epsilon \to 0$ gives

$$\Theta(1) \geq \Theta(\lambda) + \Theta(1-\lambda).$$

But according to the definition of Θ, $\Theta(\lambda) = |\lambda|^{\frac{2}{3}}\Theta(1)$, so the last inequality implies

$$\begin{aligned}\Theta(1) &\geq |\lambda|^{\frac{2}{3}}\Theta(1) + |1-\lambda|^{\frac{2}{3}}\Theta(1) \\ &= \left(|\lambda|^{\frac{2}{3}} + |1-\lambda|^{\frac{2}{3}}\right)\Theta(1) \\ &> \Theta(1) > 0,\end{aligned}$$

another contradiction. Thus Dichotomy is seen to be impossible.

Since Vanishing and Dichotomy have been ruled out, it is concluded that there is a sequence $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}$ so that for any $\epsilon > 0$, there are constants $R < \infty$ and $n_0 > 0$ such that for $n > n_0$,

$$\int_{|x-y_n|\leq R} \rho_n(x)\,dx \geq \mu - \epsilon, \qquad \int_{|x-y_n|\geq R} \rho_n(x)\,dx \leq \epsilon$$

and

$$\begin{aligned}&\Big|\int_{|x-y_n|\geq R} F(u_n, v_n)\,dx\Big| \leq \int_{|x-y_n|\geq R} |F(u_n,v_n)|\,dx \\ &\leq \gamma_0 \|(u_n,v_n)\|_{1\times 1} \int_{|x-y_n|\geq R} \rho_n(x)\,dx \\ &= \text{order}(\epsilon)\end{aligned}$$

as $n \to \infty$. It follows that for n large enough,

$$\Big|\int_{|x-y_n|\leq R} F(u_n,v_n)\,dx - 1\Big| \leq \epsilon.$$

Letting $\tilde{u}_n(x) = u_n(x-y_n), \tilde{v}_n(x) = v_n(x-y_n)$ for $x \in \mathbb{R}$, the above property means that $(\tilde{u}_n, \tilde{v}_n)$ (or a subsequence) converges weakly in $H^1 \times H^1$, almost everywhere on $\mathbb{R}$, and strongly in $L_3 \times L_3$ to some $H^1 \times H^1$-function $(\tilde{u}, \tilde{v})$, say, and that

$$\int F(\tilde{u},\tilde{v})\,dx = \lim_{n\to\infty} \int F(\tilde{u}_n, \tilde{v}_n)\,dx = 1.$$

Furthermore, it is seen that

$$\Theta(1) = \liminf_n J(\tilde{u}_n, \tilde{v}_n) \geq J(\tilde{u}, \tilde{v}).$$

Thus the function $(\tilde{u}, \tilde{v})$ solves the variational problem (2.3), and therefore if ϕ and ψ are defined by $\phi = \frac{\Theta(1)}{3}\tilde{u}, \psi = \frac{\Theta(1)}{3}\tilde{v}$, then (ϕ,ψ) solves the problem (2.2), which is to say, there is a non-trivial solitary-wave solution of the Gear-Grimshaw system propagating at speed C.

Since $(\phi,\psi) \in H^1 \times H^1$ and H^1 is an algebra, $\operatorname{grad} F(\phi,\psi) \in H^1 \times H^1$, whence,

$$(\phi,\psi) = \mathcal{L}_C{}^{-1} \operatorname{grad} F(\phi,\psi) \in H^3 \times H^3.$$

Arguing inductively, the advertised regularity result follows. □

Remark: By adapting the theory of Li and Bona (1996), Bona and Li (1997), it can be shown that the solitary-wave solution (ϕ,ψ) whose existence was just established is in fact the restriction to the real axis of a pair (Φ,Ψ) that is analytic in a complex strip $\{z : |z| < \sigma\}$ for some $\sigma > 0$. In particular, (ϕ,ψ) is real analytic and given locally by a Taylor series whose radius of convergence is bounded below, independently of the spatial point about which the expansion is made.

2.3. **More General Dispersion and Nonlinearity.** In this subsection, the goal is to extend the results exposed in the previous Subsection 2.2 to models that have more complex and possibly competing or non-local dispersion relations. For convenience, continue to denote the operator $C - CN - L$ in (1.6) by

$$\mathcal{L} = \mathcal{L}_C = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2.7}$$

Then equation (1.6) can be rewritten in the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f(\phi,\psi) \\ g(\phi,\psi) \end{pmatrix}, \tag{2.8}$$

where it is now supposed that the operators A_{ij} are Fourier multiplier operators with symbols a_{ij}, say, so that

$$\widehat{A_{ij}v}(\xi) = a_{ij}(\xi)\widehat{v}(\xi),$$

and f, g are smooth functions defined on $\mathbb{R}^2$.

The following hypotheses will be in force in the remainder of this subsection.

(H1) For $i,j = 1,2$, $a_{ij}(\xi) = a_{ji}(\xi) = \sum_k \alpha_{ij}^{(k)}|\xi|^{2r_k}$, where $r_k > 0$ are not necessarily integers and the series has only finitely many terms.

(H2) The operator $\mathcal{L}$ is elliptic in the sense that there are two positive indices r and s and positive numbers $\underline{\gamma}$ and $\bar{\gamma}$ such that, for any (x,y), (x_1,y_1), $(x_2,y_2) \in \mathbb{R}^2$,

$$(x,y)\left(a_{ij}(\xi)\right)\begin{pmatrix} x \\ y \end{pmatrix} \geq \underline{\gamma}\{(1+|\xi|^2)^r x^2 + (1+|\xi|^2)^s y^2\}$$

$$(x_1,y_1)\left(a_{ij}(\xi)\right)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \leq \bar{\gamma}\prod_{j=1}^{2}\{(1+|\xi|^2)^r x_j^2 + (1+|\xi|^2)^s y_j^2\}^{\frac{1}{2}}.$$

(H3) There is a homogeneous polynomial F of order p defined on $\mathbb{R}^2$ and a positive number γ_0 such that

$$(f,g)^t = \operatorname{grad} F,$$

where $p \geq 3$ is an integer, and

$$|F(x_1,x_2)| \leq \gamma_0(x_1^p + x_2^p).$$

As in Subsection 2.1, we have the following result.

Theorem 2.2. *If the conditions (H1)-(H3) are valid and $r, s \geq \frac{1}{2} - \frac{1}{p}$, then there is a non-trivial solution $(\phi,\psi) \in H^r \times H^s$ of (2.8). Moreover if $r, s > \frac{1}{2} - \frac{1}{p}$, then $(\phi,\psi) \in H^\infty \times H^\infty$.*

Following the steps laid out in Subsection 2.1, let

$$\Theta(1) = \inf\left\{J(u,v) : (u,v) \in H^r \times H^s, \int F(u,v)\,dx = 1\right\}, \tag{2.9}$$

where J is as before the functional

$$J(u,v) = \int (u,v)\mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} dx.$$

The hypotheses (H1) and (H2) guarantee the operator $\mathcal{L}$ is a positive definite operator defined on $H^r \times H^s$ and there are two positive constants $\underline{\gamma}$ and $\bar{\gamma}$ such that for any $(u,v) \in H^r \times H^s$,

$$\underline{\gamma}\|(u,v)\|_{r\times s}^2 \leq J(u,v) \leq \bar{\gamma}\|(u,v)\|_{r\times s}^2.$$

Moreover, since $H^r \times H^s \subset L_p \times L_p$, the hypothesis (H3) ensures that $0 < \Theta(1) < \infty$ and that any minimizing sequence $\{(u_n, v_n)\}_{n=1}^{\infty}$ for $\Theta(1)$ is bounded in $H^r \times H^s$.

To prove the theorem, the crucial point is to demonstrate that any minimizing sequence $\{(u_n, v_n)\}_{n=1}^{\infty}$ for $\Theta(1)$ is, up to underlying spatial translates, relatively compact. For this purpose, a sequence, $\{\rho_n\}_{n=1}^{\infty}$ related to $\{(u_n, v_n)\}_{n=1}^{\infty}$ is defined and the concentration-compactness principle is applied to obtain the desired conclusion. To define the sequence $\{\rho_n\}_{n=1}^{\infty}$, cases where one or both of r and s are integers are handled separately.

Case I: If both $r, s > 0$ are integers, define

$$\rho_n(x) = |u_n(x)|^2 + |D_x^r u_n(x)|^2 + |v_n(x)|^2 + |D_x^s v_n(x)|^2.$$

Case II: If one of $r, s > 0$ is not an integer, say $r = m + \delta$ with m a non-negative integer and $0 < \delta < 1$, but s is an integer, then define

$$\begin{aligned}\rho_n(x) =& |u_n(x)|^2 + \min\{1, m\}|D_x^m u_n(x)|^2 + \int_{-\infty}^{\infty} \frac{|D_x^m u_n(x) - D_y^m u_n(y)|^2}{|x-y|^{1+2\delta}}\, dy \\ &+ |v_n(x)|^2 + |D_x^s v_n(x)|^2.\end{aligned}$$

Case III: If both r, s are non-integers, say $r = m_1 + \delta_1$, $s = m_2 + \delta_2$ with m_1, m_2 non-negative integers and $0 < \delta_1, \delta_2 < 1$, let

$$\begin{aligned}\rho_n(x) = |u_n(x)|^2 + \min\{1, m_1\}|D_x^{m_1} u_n(x)|^2 + \int_{-\infty}^{\infty} \frac{|D_x^{m_1} u_n(x) - D_y^{m_1} u_n(y)|^2}{|x-y|^{1+2\delta_1}}\, dy \\ + |v_n(x)|^2 + \min\{1, m_2\}|D_x^{m_2} v_n(x)|^2 + \int_{-\infty}^{\infty} \frac{|D_x^{m_2} v_n(x) - D_y^{m_2} v_n(y)|^2}{|x-y|^{1+2\delta_2}}\, dy.\end{aligned}$$

In all cases, $\rho_n \geq 0$ and $\rho_n \in L_1$, $n = 1, 2 \cdots$. Moreover, the L_1-norm $|\rho_n|_1$ is equivalent to the $H^r \times H^s$-norm $\|(u_n, v_n)\|_{r\times s}^2$. Let $\mu_n = \int \rho_n(x)\, dx = |\rho_n|_1$, so that μ_n is bounded and $\mu = \liminf_{n\to\infty} \mu_n > 0$. By extracting a subsequence, it may be supposed that $\lim_{n\to\infty} \mu_n = \mu$. The stage is now set to effect a proof of Theorem 2.2.

Proof. The proof is made by contradiction. Suppose there exists a subsequence $\{\rho_{n_k}\}_{k=1}^{\infty}$ of $\{\rho_n\}_{n=1}^{\infty}$ for which Vanishing or Dichotomy occurs.

If Vanishing happens to $\{\rho_{n_k}\}_{k=1}^{\infty}$, which is to say, for any $R > 0$,

$$\lim_{k\to\infty} \sup_{|x-y|\leq R} \int \rho_{n_k}(x)\, dx = 0,$$

then,

$$\lim_{k\to\infty} \sup_{|x-y|\leq R} \int \left(u_{n_k}^2(x) + v_{n_k}^2(x)\right) dx = 0.$$

In Case I, $\{(u'_{n_k}, v'_{n_k})\}_{k=1}^{\infty}$ is bounded in $L_2 \times L_2$, and Lemma 2.4 of Chen and Bona (1998) implies

$$\lim_{k\to\infty} \int \left(|u_{n_k}(x)|^p + |v_{n_k}(x)|^p\right) dx = 0.$$

In Cases II and III, the modified Lemma 6.3 (Chen and Bona, 1998) implies the same conclusion. In all events, this contradicts Hypothesis (H3) which implies that $\int |u_{n_k}(x)|^p + |v_{n_k}(x)|^p\, dx \geq \gamma_0^{-1} \int F(u_{n_k}, v_{n_k})\, dx = \gamma_0^{-1}$.

If Dichotomy occurs, there is a $\bar{\mu} \in (0, \mu)$ such that for any $\epsilon > 0$ there corresponds a positive integer k_0 and $\rho_k^1, \rho_k^2 \in L_1$, $\rho_k^1, \rho_k^2 \geq 0$, such that for $k \geq k_0$,

$$\begin{cases} \left|\rho_{n_k} - (\rho_k^1 + \rho_k^2)\right|_1 \leq \epsilon, \quad \left|\int \rho_k^1\, dx - \bar{\mu}\right| \leq \epsilon, \quad \left|\int \rho_k^2\, dx - (\mu - \bar{\mu})\right| \leq \epsilon, \\ \text{supp } \rho_k^1 \cap \text{supp}\, \rho_k^2 = \emptyset \text{ and } \lim_{k\to\infty} \text{dist}\{\text{supp}\, \rho_k^1, \text{ supp}\, \rho_k^2\} = \infty. \end{cases} \tag{2.10}$$

Without loss of generality, it may be further supposed that the supports of ρ_k^1 and ρ_k^2 are separated as follows:

$$\text{supp } \rho_k^1 \subset (y_k - E_0, y_k + E_0), \quad \text{supp } \rho_k^2 \subset (-\infty, y_k - 2R_k) \cup (y_k + 2R_k, \infty)$$

for some fixed $E_0 > 0$, a sequence $\{y_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and $R_k \to \infty$ (see P.-L. Lions, 1984). The conditions in (2.10) imply

$$\int_{|x-y_k|\leq E_0} |\rho_{n_k} - \rho_k^1|\, dx + \int_{E_0 \leq |x-y_k| \leq 2R_k} \rho_{n_k}\, dx + \int_{|x-y_k|\geq 2R_k} |\rho_{n_k} - \rho_k^2|\, dx \leq \epsilon.$$

Appropriate splittings of u_{n_k}, $k = 1, 2, \cdots$, are obtained as follows. Let ζ and ψ be as described in (2.6). For $x \in \mathbb{R}$ and $E_1 > E_0$ large enough, define

$$\zeta_k(x) = \zeta(\frac{x - y_k}{E_1}) \qquad \text{and} \qquad \psi_k(x) = \psi(\frac{x - y_k}{R_k})$$

as before, and let

$$\eta_k^1(x) = \begin{cases} -\zeta(\frac{x-y_k}{E_1}) + \zeta(\frac{x-y_k}{R_k}), & \text{if } \; y_k + E_1 \leq x \leq y_k + 2R_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta_k^2(x) = \begin{cases} -\zeta(\frac{x-y_k}{E_1}) + \zeta(\frac{x-y_k}{R_k}), & \text{if } y_k - 2R_k \leq x \leq y_k - E_1, \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in \mathbb{R}$, it follows by construction that

$$\zeta_k(x) + \psi_k(x) + \eta_k^1(x) + \eta_k^2(x) = 1$$

and

$$\text{supp}\, \zeta_k \subset (y_k - 2E_1, y_k + 2E_1), \quad \text{supp}\, \psi_k \subset (-\infty, y_k - R_k) \cup (y_k + R_k, \infty),$$
$$\text{supp}\, \eta_k^1 \subset (y_k - 2R_k, y_k - E_1) \quad \text{and} \quad \text{supp}\, \eta_k^2 \subset (y_k + E_1, y_k + 2R_k).$$

The sequence $\{(u_{n_k}, v_{n_k})\}_{k=1}^{\infty}$ may be decomposed in the form

$$u_{n_k} = \zeta_k u_{n_k} + \psi_k u_{n_k} + \eta_k^1 u_{n_k} + \eta_k^2 u_{n_k},$$

$$v_{n_k} = \zeta_k v_{n_k} + \psi_k v_{n_k} + \eta_k^1 v_{n_k} + \eta_k^2 v_{n_k}.$$

Make the definitions

$$u_k^1 = \zeta_k u_{n_k}, \qquad u_k^2 = \psi_k u_{n_k}, \qquad v_k^1 = \zeta_k v_{n_k}, \qquad v_k^2 = \psi_k u_{n_k}$$

and

$$w_k^u = \eta_k^1 u_{n_k} + \eta_k^2 u_{n_k}, \qquad w_k^v = \eta_k^1 v_n + \eta_k^2 v_{n_k}.$$

Notice their construction implies that

$$\begin{aligned}
&u_k^1, u_k^2, w_k^u \in H^r, \qquad v_k^1, v_k^2, w_k^v \in H^s,\\
&\operatorname{supp} u_k^1,\ \operatorname{supp} v_k^1 \subset (y_k - 2E_1, y_k + 2E_1),\\
&\operatorname{supp} u_k^2,\ \operatorname{supp} v_k^2 \subset (-\infty, y_k - R_k) \cup (y_k + R_k, \infty),\\
&\operatorname{supp} w_k^u,\ \operatorname{supp} w_k^v \subset (y_k - 2R_k, y_k - E_1) \cup (y_k + E_1, y_k + 2R_k),
\end{aligned}$$

and, from Lemma 6.2 in the Appendix, if E_1 is chosen large enough, then when $k > k_0$,

$$\|(w_k^u, w_k^v)\|_{r\times s} \le C'\epsilon,$$

$$-C''\epsilon \le \|(u_{n_k}, v_{n_k})\|_{r\times s}^2 - \{\|(u_k^1, v_k^1)\|_{r\times s}^2 + \|(u_k^2, v_k^2)\|_{r\times s}^2\} \le C''\epsilon,$$

where C' and C'' are constants which depend only on μ. Hence, there exist subsequences of $\{(u_k^1, v_k^1)\}_{k=1}^\infty$ and $\{(u_k^2, v_k^2)\}_{k=1}^\infty$, still denoted by $\{(u_k^1, v_k^1)\}_{k=1}^\infty$ and $\{(u_k^2, v_k^2)\}_{k=1}^\infty$, respectively, and a number λ so that, for any $\epsilon > 0$, there is a $k_1 \ge k_0$ such that when $k > k_1$,

$$\Big|\int F(u_k^1, v_k^1)\,dx - \lambda\Big| \le \epsilon, \qquad \Big|\int F(u_k^2, v_k^2)\,dx - (1-\lambda)\Big| \le \epsilon$$

and

$$\begin{aligned}
J(u_{n_k}, v_{n_k}) &= J\big((u_k^1, v_k^1) + (u_k^2, v_k^2) + (w_k^u, w_k^v)\big)\\
&= J(u_k^1, v_k^1) + J(u_k^2, v_k^2) + J(w_k^u, w_k^v) + 2\int (u_k^1, v_k^1)\mathcal{L}(u_k^2, v_k^2)\,dx\\
&\quad + 2\int (u_k^1, v_k^1)\mathcal{L}(w_k^u, w_k^v) + 2\int (u_k^2, v_k^2)\mathcal{L}(w_k^u, w_k^v)\\
&= J(u_k^1, v_k^1) + J(u_k^2, v_k^2) + 2\int (u_k^1, v_k^1)\mathcal{L}(u_k^2, v_k^2)\,dx + \operatorname{order}(\epsilon),
\end{aligned}$$

because $\|(w_k^u, w_k^v)\|_{r\times s} \le C'\epsilon$ and $(u_k^1, v_k^1), (u_k^2, v_k^2)$ are bounded in $H^r \times H^s$, independently of k. Notice that

$$\{\operatorname{supp} u_k^1 \cup \operatorname{supp} v_k^1\} \cap \{\operatorname{supp} u_k^2 \cup \operatorname{supp} v_k^2\} = \emptyset$$

and

$$\operatorname{dist}\{\operatorname{supp} u_k^1 \cup \operatorname{supp} v_k^1, \operatorname{supp} u_k^2 \cup \operatorname{supp} v_k^2\} \ge R_k - 2E_1 \to \infty \text{ as } k \to \infty.$$

Lemma 6.1 in the Appendix implies $\lim_{k\to\infty} \int (u_k^1, v_k^1)\mathcal{L}(u_k^2, v_k^2)\,dx = 0$ under these circumstances. It follows as previously that

$$\Theta(1) \ge \Theta(\lambda) + \Theta(1-\lambda) > \Theta(1) > 0.$$

This contradiction rules out Dichotomy.

Since both Vanishing and Dichotomy have been excluded, it is concluded that there is a sequence $\{y_n\}_{n=1}^\infty \subset \mathbb{R}$ such that for any $\epsilon > 0$, there is an $R > 0$ so that for n sufficiently large,

$$\int_{|x-y_n|<R} \rho_n(x)\,dx > \mu - \epsilon,$$

or, what is the same,

$$\int_{|x-y_n|\ge R} \big(|u_n(x)|^p + |v_n(x)|^p\big)\,dx \le \epsilon.$$

The last inequality implies that

$$\Big|\int_{|x-y_n|\le R} F\big(u_n(x), v_n(x)\big)\, dx - 1\Big| \le \epsilon$$

for n sufficiently large. According to the definition of ρ_n, this amounts to

$$\int_{|x-y_n|\ge R} \rho_n(x)\, dx \le \epsilon.$$

Denote by $\tilde{u}_n(\cdot) = u_n(\cdot - y_n)$, $\tilde{v}_n(\cdot) = v_n(\cdot - y_n)$ the pair (u_n, v_n) translated by y_n. Then the above estimates mean that the sequence $\{(\tilde{u}_n, \tilde{v}_n)\}_{n=1}^{\infty}$ converges weakly in $H^r \times H^s$, strongly in $L_p \times L_p$ to some $(\tilde{u}, \tilde{v}) \in H^r \times H^s$, and in consequence of the latter fact,

$$\int F(\tilde{u}, \tilde{v})\, dx = \lim_{n\to\infty} \int F(\tilde{u}_n, \tilde{v}_n)\, dx = 1.$$

Furthermore,

$$\Theta(1) = \liminf_{n\to\infty} \int J(u_n, v_n)\, dx \ge \int J(\tilde{u}, \tilde{v})\, dx.$$

Thus the limiting pair $(\tilde{u}, \tilde{v})$ solves the variational problem (2.9), and therefore if $\phi = \big(\frac{\Theta(1)}{p}\big)^{\frac{1}{p-2}} \tilde{u}$ and $\psi = \big(\frac{\Theta(1)}{p}\big)^{\frac{1}{p-2}} \tilde{v}$, then (ϕ, ψ) solves the problem (2.8).

To prove the regularity result, if $s_0 = \min\{r, s\} > \frac{1}{2}$, then $H^r \times H^s$ is an algebra, so $(\phi, \psi) \in H^r \times H^s$ implies $\big(f(\phi, \psi), g(\phi, \psi)\big) \in H^{s_0} \times H^{s_0}$, and consequently $(\phi, \psi) = \mathcal{L}^{-1}\big(f(\phi, \psi), g(\phi, \psi)\big) \in H^{3s_0} \times H^{3s_0}$. It follows by induction that $(\phi, \psi) \in H^\infty \times H^\infty$. If $\frac{1}{2} \ge s_0 = \min\{r, s\} > \frac{1}{2} - \frac{1}{p}$, then $\big(f(\phi, \psi), g(\phi, \psi)\big) \in H^{s_0 - (\frac{1}{2} - \frac{1}{p})} \times H^{s_0 - (\frac{1}{2} - \frac{1}{p})}$, so $(\phi, \psi) = \mathcal{L}^{-1}\big(f(\phi, \psi), g(\phi, \psi)\big) \in H^{3s_0 - (\frac{1}{2} - \frac{1}{p})} \times H^{3s_0 - (\frac{1}{2} - \frac{1}{p})}$. Another induction establishes the regularity result since $3s_0 - (\frac{1}{2} - \frac{1}{p}) > s_0$. The proof of Theorem 2.2 is complete. □

3. Application of Positive Operator Theory to Existence of Solitary-Wave Solutions.

3.1. Preliminary Review of Positive Operators. In this subsection, a brief review is provided of ideas in the paper of Benjamin *et al* (1990).

Recall that a vector space X over $\mathbb{R}$ (or $\mathbb{C}$) is a *Fréchet space* if there is a sequence $\{p_n\}_{n=1}^{\infty}$ of semi-norms defined on it which are increasing (i.e. $p_{n+1}(x) \ge p_n(x)$ for every $x \in X$ and every $n = 1, 2, 3, \cdots$) and such that the translation invariant metric

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{p_j(x-y)}{1 + p_j(x-y)}, \quad \text{for } x, y \in X, \tag{3.1}$$

on X is complete. If the ball of radius $r > 0$ centered at the origin for this metric is denoted

$$\mathcal{B}_r = \{x \in X : d(x, 0) < r\},$$

then $\mathcal{B}_1 = X$.

A closed subset $\mathbb{K}$ of a real topological linear space X is called a *cone* if

(i) $\mathbb{K} + \mathbb{K} = \{u + v : u, v \in \mathbb{K}\} \subset \mathbb{K}$,

(ii) $\lambda\mathbb{K} = \{\lambda u : u \in \mathbb{K}\} \subset \mathbb{K}$ for all $\lambda \ge 0$, and

(iii) $\mathbb{K} \cap (-\mathbb{K}) = \{0\}$.

Let $\mathbb{K}$ be a cone in a Fréchet space X with metric as in (3.1). An operator A defined on $\mathbb{K}$ is said to be a *positive operator* if $A\mathbb{K} \subset \mathbb{K}$. A is $\mathbb{K}$-*compact* if $A(\mathbb{K} \cap \mathcal{B}_r)$ has a compact closure for each $r \in [0, 1)$.

Henceforth in this subsection, it is assumed that $\mathbb{K}$ is a cone in a Fréchet space X, $\{p_n\}_{n=1}^{\infty}$ is a generating family of semi-norms for the metric d on X defined in (3.1), and $A : \mathbb{K} \to \mathbb{K}$ is continuous and $\mathbb{K}$-compact. The symbol $i(\mathbb{K}, A, \mathbb{K}_R)$ denotes the fixed-point index of the positive operator A defined on $\mathbb{K}$ over the subset $\mathbb{K}_R = \mathbb{K} \cap \mathcal{B}_R$. When this index is non-zero, there is at least one fixed point of A in $\mathbb{K}_R$. If $0 \le r < R < 1$, denote the cone segment $\{x \in \mathbb{K} : r < d(x, 0) < R\}$ by $\mathbb{K}_r^R$. Here are some basic lemmas from Benjamin *et al.* (1990).

Lemma 3.1. *Suppose that* $0 < \rho < 1$ *and that either*

$$Ax - x \notin \mathbb{K} \text{ for all } x \in \mathbb{K} \cap \partial\mathcal{B}_\rho$$

or

$$tAx \neq x \text{ for all } x \in \mathbb{K} \cap \partial\mathcal{B}_\rho \text{ and all } t \in [0, 1].$$

Then

$$i(\mathbb{K}, A, \mathbb{K} \cap \mathcal{B}_\rho) = 1.$$

Lemma 3.2. *Suppose that* $0 < \rho < 1$ *and that either*

$$x - Ax \notin \mathbb{K} \text{ for all } x \in \mathbb{K} \cap \partial\mathcal{B}_\rho$$

or there exists $u^* \in \mathbb{K}$ *with* $p_1(u^*) > 0$ *such that* $x - Ax \neq au^*$ *for all* $x \in \mathbb{K} \cap \partial\mathcal{B}_\rho$ *and all* $a \ge 0$. *Then*

$$i(\mathbb{K}, A, \mathbb{K}_\rho) = 0.$$

3.2. **Application of Positive Operator Theory.** In this subsection, attention is first given to the abstract system (1.6) introduced in the beginning of this paper with regard to the prospect that it possesses solitary travelling waves. Then the Gear-Grimshaw system (1.2) and the family of Boussinesq systems (1.4) are considered as examples of our general theory. For simplicity, the nonlinear effects are assumed to be quadratic, so f_1 and f_2 in (1.6) are taken to be quadratic homogeneous polynomials defined on $\mathbb{R}^2$. As will be clear from the development, the arguments are easily adapted to any superlinear nonlinearities f_1 and f_2 that are positive for positive values of their arguments, whether or not they are homogeneous. It is further assumed that $C - CN - L$ is invertible. Applying $(C - CN - L)^{-1}$ to (1.6), there obtains the equivalent system of integral equations

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = A \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{cases} k_{11} * \phi^2 + k_{12} * \phi\psi + k_{22} * \psi^2, \\ m_{11} * \phi^2 + m_{12} * \phi\psi + m_{22} * \psi^2, \end{cases} \tag{3.2}$$

where $f * g(x)$ connotes the convolution $\int f(x - y)g(y)\, dy$ as usual. Let $\mathbb{C} = \mathbb{C}(\mathbb{R})$ be the class of continuous real-valued functions defined on $\mathbb{R}$, and let X be the Fréchet space

$$X = \mathbb{C} \times \mathbb{C} = \Big\{ (f, g) : f, g \in \mathbb{C} \Big\},$$

with generating family of semi-norms

$$p_j(f, g) = \max_{-jl \le x \le jl} \{ \, |f(x)|, \; |g(x)| \, \},$$

where $l > 0$ is to be determined later, $j = 1, 2, \cdots$. The cone in X that will occupy our attention is

$$\mathbb{K} = \big\{(f,g) \in X : f(x) = f(-x) \geq 0,\ g(x) = g(-x) \geq 0,\ f, g \text{ are non-increasing on } (0,\infty)\big\}.$$

Note that, if $(f,g) \in \mathbb{K}$, then the distance of (f,g) to the origin 0 is

$$d\big((f,g),0\big) = \frac{\max\{f(0), g(0)\}}{1 + \max\{f(0), g(0)\}},$$

or in other words, $d\big((f,g),0\big) = r$ implies that $\max\{f(0), g(0)\} = \frac{r}{1-r}$.

The following hypotheses will be in force.

(S1) The kernels k_{ij} and m_{ij} satisfy $k_{ij}(x) = k_{ij}(-x)$, $m_{ij}(x) = m_{ij}(-x)$, $k_{ij}, m_{ij} \in L_1 \cap \mathbb{C}$, $k_{ij} \geq 0, m_{ij} \geq 0$ are monotone decreasing on $(0, \infty)$, there is a number $\lambda \geq 0$ such that $k_{ij}(x)$ and $m_{ij}(x)$ are convex for $x \geq \lambda$ and either $k_{12} + m_{12}$ or both $k_{11} + m_{11}$ and $k_{22} + m_{22}$ are strictly convex when $x \geq \lambda$.

(S2) There are only finitely many fixed points of A in the cone $\mathbb{K}$ which are constant functions. Furthermore, if $(u,v) \in \mathbb{K}$ is a fixed point of A, then $u = 0$ implies that $v = 0$, and vice versa.

(S3) Not too many of the k_{ij} and m_{ij} vanish. More precisely, $k_{12} + k_{22} \neq 0$ and $m_{11} + m_{12} \neq 0$. For $1 \leq i, j \leq 2$, let $\kappa_{ij} = \int_0^2 k_{ij}(x)\,dx$ and $\mu_{ij} = \int_0^2 m_{ij}(x)\,dx$. Then for $a \geq 0$, the system of inequalities

$$\begin{cases} a + \kappa_{11} \displaystyle\int_0^1 u^2(x)\,dx + \kappa_{12} \int_0^1 u(x)v(x)\,dx + \kappa_{22} \int_0^1 v^2(x)\,dx \\ \qquad \leq \Big(\displaystyle\int_0^1 u^2(x)\,dx\Big)^{\frac{1}{2}} \\ a + \mu_{11} \displaystyle\int_0^1 u^2(x)\,dx + \mu_{12} \int_0^1 u(x)v(x)\,dx + \mu_{22} \int_0^1 v^2(x)\,dx \\ \qquad \leq \Big(\displaystyle\int_0^1 v^2(x)\,dx\Big)^{\frac{1}{2}} \end{cases}$$

implies that each term on the left-hand side is bounded and these bounds are only dependent on the quantities κ_{ij} and μ_{ij}, $1 \leq i, j \leq 2$.

Remark: The assumptions $k_{ij}(-\xi) = k_{ij}(\xi)$, $m_{ij}(-\xi) = m_{ij}(\xi)$ in (S1) are virtually universal in dispersive wave equations that arise in practice. The further hypotheses in (S1) are satisfied by an interesting set of examples as we show later. The condition (S2) is natural as will appear presently, because of the interaction between the two dependent variables ϕ and ψ. The first part of (S3) implies that if (u_0, v) (or (u, v_0)) satisfies (3.2), where $u_0 > 0$ (or $v_0 > 0$) is a constant function, then v (or u) must be also a positive constant function. Otherwise, suppose (u_0, v) satisfies (3.2), where $u_0 \equiv$ constant > 0 and $v \geq 0$ is not constant. Then the monotonicity of v implies v can be written in form $v = v_1 + v_0$, where $v_0 = v(\infty) \geq 0$, and $v_1 \geq 0$ is not identically zero. Substituting $(u_0, v_1 + v_0)$ into the first equation of (3.2) and simplifying by using the fact that $(u_0, v_0) = A(u_0, v_0)$ leads to the relation $0 = 2u_0 k_{12} * v_1 + k_{22} * (2v_0 v_1 + v_1^2)$. Because of positivity, v_1 must be zero, which is to say, v must be identical to some non-negative constant. In a similar fashion, it can be shown that if $v = v_0 > 0$ is a constant function, then u must be also a non-negative constant function. This excludes the possibility of trivial solutions in the form of (u_0, v) or (u, v_0), where $u_0, v_0 > 0$ are constants and

$u \geq 0, v \geq 0$ are non-constant functions. The first part of (S3) means that there is no equation in the system where only one unknown function u or v appears. The second part of (S3) indicates that the nonlinearity is not dominated just by the interaction term uv. The full import of Hypothesis (S3) will be clearer after we develop its consequences and apply it to concrete systems.

Theorem 3.3. *Under the assumptions (S1) to (S3), the system (3.2) has a non-trivial solution* $(\phi, \psi) \in \mathbb{K}$.

The proof of this theorem is broken into three lemmas and one proposition.

Lemma 3.4. *The operator A defined in (3.2) is a continuous and $\mathbb{K}$-compact mapping of $\mathbb{K}$ into itself.*

Proof. This result follows exactly as in Lemma 3.1 of Benjamin *et al.* (1990). □

For convenience, the L_1-norms of the functions k_{ij} and m_{ij} are denoted by $K_{ij} = |k_{ij}|_1$ and $M_{ij} = |m_{ij}|_1$, respectively. Since the k_{ij} and m_{ij} are non-negative, these quantities may also be expressed as $K_{ij} = \widehat{k}_{ij}(0)$ and $M_{ij} = \widehat{m}_{ij}(0)$, where as before, the circumflex surmounting a function connotes that function's Fourier transform.

Lemma 3.5. *If* $r_0 = \max\{K_{11} + K_{12} + K_{22},\ M_{11} + M_{12} + M_{22}\}$, *then*

(a) for $0 < r < \frac{1}{1+r_0}$, *there is no* $(u, v) \in \mathbb{K} \cap \partial\mathcal{B}_r$ *and* $t \in [0, 1]$ *such that*

$$\begin{pmatrix} u \\ v \end{pmatrix} = tA \begin{pmatrix} u \\ v \end{pmatrix}.$$

(b) If r_0 is as above and $\frac{1}{1+r_0} < R < 1$ *and R is sufficiently close to* 1, *then there is no* $(u, v) \in \mathbb{K} \cap \partial\mathcal{B}_R$ *and no constant* $a > 0$ *such that*

$$\begin{pmatrix} u \\ v \end{pmatrix} - A \begin{pmatrix} u \\ v \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where 1 *is the constant function that maps every point in* $\mathbb{R}$ *to* 1.

Proof. Part (a). Arguing by contradiction, suppose that there is a $(u, v) \in \mathbb{K} \cap \partial\mathcal{B}_r$ such that $\begin{pmatrix} u \\ v \end{pmatrix} = tA \begin{pmatrix} u \\ v \end{pmatrix}$, which is to say

$$\begin{cases} u = t\big(k_{11} * u^2 + k_{12} * (uv) + k_{22} * v^2\big), \\ v = t\big(m_{11} * u^2 + m_{12} * (uv) + m_{22} * v^2\big). \end{cases}$$

Evaluating these latter equations at $x = 0$ and using positivity of the kernels and the fact that $0 \leq u(x) \leq u(0)$ and $0 \leq v(x) \leq v(0)$ for all x yields

$$\begin{cases} \frac{1}{t} u(0) \leq K_{11} u(0)^2 + K_{12} u(0) v(0) + K_{22} v(0)^2, \\ \frac{1}{t} v(0) \leq M_{11} u(0)^2 + M_{12} u(0) v(0) + M_{22} v(0)^2. \end{cases}$$

Since $(u, v) \in \mathbb{K} \cap \partial\mathcal{B}_r$, $\max\{u(0), v(0)\} = \frac{r}{1-r}$, so these last inequalities imply

$$\begin{cases} \frac{1}{t} u(0) \leq (K_{11} + K_{12} + K_{22})(\frac{r}{1-r})^2 \leq r_0(\frac{r}{1-r})^2, \\ \frac{1}{t} v(0) \leq (M_{11} + M_{12} + M_{22})(\frac{r}{1-r})^2 \leq r_0(\frac{r}{1-r})^2. \end{cases}$$

Using again the relation $\max\{u(0), v(0)\} = \frac{r}{1-r}$, it is adduced that

$$\frac{1}{t}\frac{r}{1-r} \le r_0\left(\frac{r}{1-r}\right)^2,$$

whence,

$$r \ge \frac{1}{1+tr_0} \ge \frac{1}{1+r_0},$$

and this contradicts the presumption $r < \frac{1}{1+r_0}$.

Part (b). Arguing by contradiction again, suppose that there is $(u, v) \in \mathbb{K} \cap \partial\mathcal{B}_R$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} - A\begin{pmatrix} u \\ v \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

or, what is the same,

$$\begin{cases} u(x) = a + k_{11} * u^2(x) + k_{12} * (uv)(x) + k_{22} * v^2(x), \\ v(x) = a + m_{11} * u^2(x) + m_{12} * (uv)(x) + m_{22} * v^2(x). \end{cases} \tag{3.3}$$

Denote $k_{11} * u^2 + k_{12} * (uv) + k_{22} * v^2$ and $m_{11} * u^2 + m_{12} * (uv) + m_{22} * v^2$ by $[A(u, v)]_1$ and $[A(u, v)]_2$, respectively and integrate both sides of (3.3) with respect to x over $[0, 1]$ to reach

$$\begin{cases} a + \int_0^1 [A(u,v)]_1\, dx = \int_0^1 u(x)\, dx \le \left(\int_0^1 u(x)^2\, dx\right)^{\frac{1}{2}}, \\ a + \int_0^1 [A(u,v)]_2\, dx = \int_0^1 v(x)\, dx \le \left(\int_0^1 v(x)^2\, dx\right)^{\frac{1}{2}}. \end{cases} \tag{3.4}$$

Because of the symmetry, positivity and monotonicity of u and v and the kernels k_{ij} and m_{ij}, $1 \le i, j \le 2$, it follows that

$$\begin{aligned} &\int_0^1 [A(u,v)]_1\, dx = \frac{1}{2}\int_{-1}^1 [A(u,v)]_1\, dx \\ &\ge \frac{1}{2}\int_{-1}^1\int_{-1}^1 \left[k_{11}(x-y)u^2(y) + k_{12}(x-y)u(y)v(y) + k_{22}(x-y)v^2(y)\right] dy\, dx \\ &= \frac{1}{2}\int_{-1}^1\int_{y-1}^{y+1} \left[k_{11}(z)u^2(y) + k_{12}(z)u(y)v(y) + k_{22}(z)v^2(y)\right] dz\, dy \\ &\ge \frac{1}{2}\kappa_{11}\int_{-1}^1 u^2(y)\, dy + \frac{1}{2}\kappa_{12}\int_{-1}^1 u(y)v(y)\, dy + \frac{1}{2}\kappa_{22}\int_{-1}^1 v^2(y)\, dy \\ &= \kappa_{11}\int_0^1 u^2(y)\, dy + \kappa_{12}\int_0^1 u(y)v(y)\, dy + \kappa_{22}\int_0^1 v^2(y)\, dy, \end{aligned}$$

where $\kappa_{ij} = \int_0^2 k_{ij}(x)\, dx$ for $i, j = 1, 2$, as mentioned already in (S3); similarly,

$$\int_0^1 [A(u,v)]_2\, dx \ge \mu_{11}\int_0^1 u^2(x)\, dx + \mu_{12}\int_0^1 u(x)v(x)\, dx + \mu_{22}\int_0^1 v^2(x)\, dx,$$

where $\mu_{ij} = \int_0^2 m_{ij}(x)\,dx$ for $i, j = 1, 2$. Substituting these two estimates into (3.4), there appears

$$\begin{cases} a+\kappa_{11}\displaystyle\int_0^1 u^2(x)\,dx + \kappa_{12}\int_0^1 u(x)v(x)\,dx + \kappa_{22}\int_0^1 v^2(x)\,dx \le \Big(\int_0^1 u^2(x)\,dx\Big)^{\frac{1}{2}}, \\ a+\mu_{11}\displaystyle\int_0^1 u^2(x)\,dx + \mu_{12}\int_0^1 u(x)v(x)\,dx + \mu_{22}\int_0^1 v^2(x)\,dx \le \Big(\int_0^1 v^2(x)\,dx\Big)^{\frac{1}{2}}. \end{cases} \tag{3.5}$$

Because of (S1), $k_{ii} + m_{ii} \neq 0$ is equivalent to $\kappa_{ii} + \mu_{ii} > 0$ for $i = 1, 2$, and $k_{12} + m_{12} \neq 0$ is equivalent to $\kappa_{12} + \mu_{12} > 0$. Hence Hypothesis (S3) guarantees that the inequalities (3.5) imply both $\int_0^1 u^2(x)\,dx$ and $\int_0^1 v^2(x)\,dx$ are bounded if $k_{ii}+m_{ii} \neq 0$ for $i = 1, 2$, and in this case, so is $\int_0^1 u(x)v(x)\,dx$. In the situation where one of $k_{11}+m_{11}$ and $k_{22}+m_{22}$ disappears, say, $k_{22}+m_{22} = 0$, then $k_{12}+m_{12}$ must not vanish by (S3), so at least $\int_0^1 u^2(x)\,dx$ and $\int_0^1 u(x)v(x)\,dx$ are guaranteed to be bounded, and hence a is bounded. In all events, the bounds are only dependent on the κ_{ij} and μ_{ij}, $1 \le i, j \le 2$. Referring this result to the formula (3.3), an L_∞-bound on u and v is deduced as follows. First, because of (3.3), we know that

$$\begin{aligned} 0 \le u(x) &= a + \int_{-\infty}^{\infty} k_{11}(x-y)u^2(y)\,dy \\ &\quad + \int_{-\infty}^{\infty} k_{12}(x-y)u(y)v(y)\,dy + \int_{-\infty}^{\infty} k_{22}(x-y)v^2(y)\,dy \\ &= a + \sum_{j=-\infty}^{\infty}\int_{2j-1}^{2j+1} k_{11}(x-y)u^2(y)\,dy + \sum_{j=-\infty}^{\infty}\int_{2j-1}^{2j+1} k_{12}(x-y)u(y)v(y)\,dy \\ &\quad + \sum_{j=-\infty}^{\infty}\int_{2j-1}^{2j+1} k_{22}(x-y)v^2(y)\,dy. \end{aligned}$$

On the other hand, we see that

$$\begin{aligned} &\sum_{j=-\infty}^{\infty}\int_{2j-1}^{2j+1} k_{11}(x-y)u^2(y)\,dy \\ &= \int_{-1}^{1}\Big[k_{11}(x-y)u^2(y) + \sum_{j=1}^{\infty}\big(k_{11}(x-y-2j) + k_{11}(x+y+2j)\big)u^2(y+2j)\Big]\,dy \\ &\le \int_{-1}^{1}\Big[k_{11}(x-y) + \sum_{j=1}^{\infty}\big(k_{11}(x-y-2j) + k_{11}(x+y+2j)\big)\Big]u^2(y)\,dy \\ &= \int_{-1}^{1}\sum_{j=-\infty}^{\infty} k_{11}(x-y+2j)u^2(y)\,dy \\ &\le \max_{-1\le z\le 1}\Big\{\sum_{j=-\infty}^{\infty} k_{11}(z+2j)\Big\}\int_{-1}^{1} u^2(y)\,dy \\ &= 2\max_{-1\le z\le 1}\Big\{\sum_{j=-\infty}^{\infty} k_{11}(z+2j)\Big\}\int_{0}^{1} u^2(y)\,dy. \end{aligned}$$

In a similar fashion, it follows that

$$\sum_{j=-\infty}^{\infty}\int_{2j-1}^{2j+1} k_{12}(x-y)u(y)v(y)\,dy$$
$$\le 2\max_{-1\le z\le 1}\Big\{\sum_{j=-\infty}^{\infty} k_{12}(z+2j)\Big\}\int_0^1 u(y)v(y)\,dy$$

and

$$\sum_{j=-\infty}^{\infty}\int_{2j-1}^{2j+1} k_{22}(x-y)v^2(y)\,dy$$
$$\le 2\max_{-1\le z\le 1}\Big\{\sum_{j=-\infty}^{\infty} k_{22}(z+2j)\Big\}\int_0^1 v^2(y)\,dy.$$

Since $k_{ij}\in L_1\cap L_\infty$ is monotone decreasing on $[0,\infty)$, the periodic function $\bar{k}_{ij}(x)=\sum_{j=-\infty}^{\infty}k_{12}(x+2j)$ is bounded, monotone decreasing on $[0,1]$, and lies in $L_1(0,1)$. Thus, $\max_{-1\le z\le 1}\bar{k}_{ij}(z)=\sum_{j=-\infty}^{\infty}k_{ij}(2j)$ for $i,j=1,2$, whence,

$$u(x)\le a+2\sum_{j=-\infty}^{\infty}k_{11}(2j)\int_0^1 u^2(y)\,dy$$
$$+2\sum_{j=-\infty}^{\infty}k_{12}(2j)\int_0^1 u(y)v(y)\,dy+2\sum_{j=-\infty}^{\infty}k_{22}(2j)\int_0^1 v^2(y)\,dy$$

and

$$v(x)\le a+2\sum_{j=-\infty}^{\infty}m_{11}(2j)\int_0^1 u^2(x)\,dx$$
$$+2\sum_{j=-\infty}^{\infty}m_{12}(2j)\int_0^1 u(y)v(y)\,dy+2\sum_{j=-\infty}^{\infty}m_{22}(2j)\int_0^1 v^2(x)\,dx.$$

Denote the upper bounds for u and v by κ and μ, respectively, and choose

$$R=\frac{2\max\{\kappa,\mu\}}{1+2\max\{\kappa,\mu\}}.$$

Then it follows that $d\big((u,v),0\big)<R$, which means $(u,v)\notin\mathbb{K}\cap\partial\mathcal{B}_R$; this contradiction proves (b). □

The last four Lemmas immediately imply the following potentially helpful result.

Proposition 3.6. *If r and R are as in (a) and (b) of Lemma 3.5, respectively, then $i(\mathbb{K},A,\mathbb{K}_r^R)=-1$, and therefore the operator A has a fixed point in the cone segment $\mathbb{K}_r^R$.*

Proof. First, remark that because the operator A is continuous and $\mathbb{K}$-compact, it follows that any relatively open set $U\subset\mathbb{K}\cap\mathcal{B}_r$ where $r<1$ with no fixed points on its relative boundary has a well-defined fixed-point index $i(\mathbb{K},A,U)$. Lemma 3.1 and part (a) of Lemma 3.5 imply $i(\mathbb{K},A,\mathbb{K}\cap\mathcal{B}_r)=1$ for any positive $r<\frac{1}{1+r_0}$ where r_0 is as specified in Lemma 3.5. On the other hand, Lemma 3.2 and part (b) of Lemma 3.5 imply $i(\mathbb{K},A,\mathbb{K}\cap\mathcal{B}_R)=0$ for any $R<1$ large enough. For such values of r and R, the additivity of the fixed-point index implies that

$$i(\mathbb{K},A,\mathbb{K}_r^R)=i(\mathbb{K},A,\mathbb{K}_R)-i(\mathbb{K},A,\mathbb{K}_r)=-1.$$

and consequently that A has a fixed point in $\mathbb{K}_r^R$. □

Remark: While suggestive, Proposition 3.6 does not settle the issue raised in Theorem 3.3 because the cone segment $\mathbb{K}_r^R$ always includes at least one trivial solution of (3.2). That is, there is a solution $(\phi, \psi) = (\phi_0, \psi_0) \in \mathbb{K}_r^R$ where ϕ_0 and ψ_0 are nonzero constant functions. The existence and number of such trivial solutions are an easily established algebraic fact. These constant solutions cannot be excluded from our considerations by appropriate choices of r and R. Indeed, consideration of the inequalities that are the basis for our choice of these parameters show that $(\phi_0, \psi_0) \in \mathbb{K}_r^R$.

Again following the approach in Benjamin *et al.* (1990), introduce two operators $r_l : \mathbb{C} \to \mathbb{C}_l$ and $s_l : \mathbb{C}_l \to \mathbb{C}$, where $\mathbb{C}_l$ is comprised of all continuous periodic functions with period $2l$. These operators are defined for $u \in \mathbb{C}$ by

$$(r_l\, u)(x) = \begin{cases} u(x) & \text{if } 0 \le |x| \le l, \\ u(2jl - |x|) & \text{if } (2j-1)l \le |x| \le (2j+1)l, \end{cases}$$

for $j = 1, 2, \cdots$, and

$$(s_l\, u)(x) = \begin{cases} u(x) & \text{if } 0 \le |x| \le l, \\ u(l) & \text{if } |x| \ge l. \end{cases}$$

The operators r_l and s_l are extended to Cartesian products componentwise; that is, if $(u,v) \in \mathbb{C}\times\mathbb{C}$, $r_l(u,v) = (r_l u, r_l v)$ and if $(u,v) \in C_l \times \mathbb{C}_l$ then $s_l(u,v) = (s_l u, s_l v)$.

Define a new cone

$$\mathbb{P}_l\mathbb{K} = \big\{(u,v) \in \mathbb{C}_l \times \mathbb{C}_l : u(x) = u(-x) \ge 0,\ v(x) = v(-x) \ge 0, \\ \text{and both } u \text{ and } v \text{ are non-increasing on } [0,l]\big\}.$$

One checks easily that $\mathbb{P}_l\mathbb{K}$ is a closed cone in the Fréchet space $X = \mathbb{C} \times \mathbb{C}$. The composite operator $r_l s_l$, when restricted to $\mathbb{P}_l\mathbb{K}$, is the identity map. Following Lemma 3.3 in Benjamin *et al.* (1990), it is determined that the operator A maps $\mathbb{P}_l\mathbb{K}$ into itself, is continuous, and for any $0 < \rho < 1$, the set $A(\mathbb{P}_l\mathbb{K} \cap \mathcal{B}_\rho)$ is a relatively compact subset of $\mathbb{P}_l\mathbb{K}$.

For $(u,v) \in \mathbb{K}$, the homotopy H_t defined for $0 \le t \le 1$ by

$$H_t \begin{pmatrix} u \\ v \end{pmatrix} = tA \begin{pmatrix} u \\ v \end{pmatrix} + (1-t) s_l\, A r_l \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.6}$$

will play an important role in our theory.

Lemma 3.7. *Suppose the only fixed points of the operator A in $\mathbb{K}_r^R$ are vectors of constant functions. By Hypothesis (S2), there are only a finite number of such solutions. Let $(u_0, v_0) \in \mathbb{K}_r^R$ be any one of them. Then it follows that $i(\mathbb{K}, A, \mathbb{K} \cap \mathcal{B}_\epsilon^0) = 0$ for $\epsilon > 0$ sufficiently small, where $\mathcal{B}_\epsilon^0 = \mathcal{B}_\epsilon(u_0, v_0)$ is the metric ball of radius ϵ about (u_0, v_0).*

The proof of the lemma consists of two steps. Step 1 is to show that for l sufficiently large, the operator A is homotopic to $s_l A r_l$ on a small neighborhood

$$\mathbb{K} \cap \mathcal{B}_\epsilon^0 = r_l^{-1}[\mathbb{P}_l\mathbb{K} \cap \mathcal{B}_\epsilon^0], \tag{3.7}$$

where $\mathcal{B}_\epsilon^0$ is as above and $\epsilon > 0$ is small enough that $\mathbb{K} \cap \mathcal{B}_\epsilon^0 \subset \mathbb{K}_r^R$ and that (u_0, v_0) is the unique fixed point of A in $\mathbb{K} \cap \mathcal{B}_{2\epsilon}^0$. Step 2 is to show $i(\mathbb{K}, s_l A r_l, \mathbb{K} \cap \mathcal{B}_\epsilon^0) = 0$.

Since $\mathbb{K}$ r-dominates $\mathbb{P}_l\mathbb{K}$ (see Benjamin *et al.* 1990), veracity of Lemma 3.7 is concluded since

$$\begin{aligned} i\big(A,(u_0,v_0)\big) &= i(\mathbb{K},A,\mathbb{K}\cap\mathcal{B}^0_\epsilon) = i(\mathbb{K},A,r_l^{-1}[\mathbb{P}_l\mathbb{K}\cap\mathcal{B}^0_\epsilon]) \\ &= i\big(\mathbb{P}_l\mathbb{K},A,\mathbb{P}_l\mathbb{K}\cap\mathcal{B}^0_\epsilon\big) = i\big(\mathbb{K},s_lAr_l,\mathbb{K}\cap\mathcal{B}^0_\epsilon\big) \\ &= 0. \end{aligned}$$

Remark: The equality (3.7) follows directly from the definitions.

Proof. (Step 1) We claim that A is homotopic to s_lAr_l on the set defined in (3.7). This will be true provided the homotopy H_t in (3.6) is admissible. The continuity and compactness aspects of H_t being clear, it remains only to check there is no element $(u,v)\in\mathbb{K}\cap\partial\mathcal{B}^0_\epsilon$ which satisfies the equation

$$\begin{pmatrix}u\\v\end{pmatrix} = tA\begin{pmatrix}u\\v\end{pmatrix} + (1-t)s_l\,Ar_l\begin{pmatrix}u\\v\end{pmatrix} \tag{3.8}$$

for some $t\in[0,1]$. By contradiction, suppose there is $(u,v)\in\mathbb{K}\cap\partial\mathcal{B}^0_\epsilon$ and $t\in[0,1]$ such that (3.8) is true. Rearranging the terms and only considering x in the range $0\le x\le l$, (3.8) becomes

$$Ar_l\begin{pmatrix}u\\v\end{pmatrix} - \begin{pmatrix}u\\v\end{pmatrix} = t\Big\{Ar_l\begin{pmatrix}u\\v\end{pmatrix} - A\begin{pmatrix}u\\v\end{pmatrix}\Big\}. \tag{3.9}$$

In detail, (3.9) has the form

$$\begin{cases} k_{11}*(r_lu)^2 + k_{12}*(r_lu)(r_lv) + k_{22}*(r_lv)^2 - u = t\psi_1(x), \\ m_{11}*(r_lu)^2 + m_{12}*(r_lu)(r_lv) + m_{22}*(r_lv)^2 - v = t\psi_2(x), \end{cases} \tag{3.10}$$

where

$$\begin{aligned} \psi_1(x) =& k_{11}*\{(r_lu)^2-u^2\} + k_{12}*\{(r_lu)(r_lv)-uv\} + k_{22}*\{(r_lv)^2-v^2\} \\ =& \int_{y>l}\{k_{11}(x-y)+k_{11}(x+y)\}\{(r_lu)^2(y)-u^2(y)\}\,dy \\ &+ \int_{y>l}\{k_{12}(x-y)+k_{12}(x+y)\}\{(r_lu)(r_lv)(y)-(uv)(y)\}\,dy \\ &+ \int_{y>l}\{k_{22}(x-y)+k_{22}(x+y)\}\{(r_lv)^2(y)-v^2(y)\}\,dy \end{aligned}$$

and

$$\begin{aligned} \psi_2(x) =& m_{11}*\{(r_lu)^2-u^2\} + m_{12}*\{(r_lu)(r_lv)-uv\} + m_{22}*\{(r_lv)^2-v^2\} \\ =& \int_{y>l}\{m_{11}(x-y)+m_{11}(x+y)\}\{(r_lu)^2(y)-u^2(y)\}\,dy \\ &+ \int_{y>l}\{m_{12}(x-y)+m_{12}(x+y)\}\{(r_lu)(r_lv)(y)-uv(y)\}\,dy \\ &+ \int_{y>l}\{m_{22}(x-y)+m_{22}(x+y)\}\{(r_lv)^2(y)-v^2(y)\}\,dy. \end{aligned}$$

Multiplying (3.10) by $\cos(\frac{\pi x}{l})$ and integrating the result over $(0,l)$ gives

$$\begin{cases} \int_0^l \Big\{k_{11} * (r_l u)^2 + k_{12} * (r_l u)(r_l v) + k_{22} * (r_l v)^2 - u\Big\} \cos\frac{\pi x}{l}\,dx, \\ \quad = t\int_0^l \psi_1(x)\cos\frac{\pi x}{l}\,dx, \\ \int_0^l \Big\{m_{11} * (r_l u)^2 + m_{12} * (r_l u)(r_l v) + m_{22} * (r_l v)^2 - u\Big\} \cos\frac{\pi x}{l}\,dx \\ \quad = t\int_0^l \psi_2(x)\cos\frac{\pi x}{l}\,dx. \end{cases} \tag{3.11}$$

The first component on the left-hand side of (3.11) is

$$\begin{aligned} &\int_0^l \Big\{k_{11} * (r_l u)^2(x) + k_{12} * \big(r_l(uv)\big)(x) + k_{22} * (r_l v)^2(x) - u(x)\Big\} \cos\frac{\pi x}{l}\,dx \\ =&\,\widehat{k}_{11}(\tfrac{\pi}{l})\int_0^l u^2(x)\cos\frac{\pi x}{l}\,dx + \widehat{k}_{12}(\tfrac{\pi}{l})\int_0^l u(x)v(x)\cos\frac{\pi x}{l}\,dx \\ &+ \widehat{k}_{22}(\tfrac{\pi}{l})\int_0^l v^2(x)\cos\frac{\pi x}{l}\,dx - \int_0^l u(x)\cos\frac{\pi x}{l}\,dx \\ =&\int_0^l \Big\{\widehat{k}_{11}(\tfrac{\pi}{l})u^2(x) + \widehat{k}_{12}(\tfrac{\pi}{l})uv(x) + \widehat{k}_{22}(\tfrac{\pi}{l})v^2(x) - u(x)\Big\}\cos\frac{\pi x}{l}\,dx, \end{aligned}$$

while the second is

$$\int_0^l \Big\{\widehat{m}_{11}(\tfrac{\pi}{l})u^2(x) + \widehat{m}_{12}(\tfrac{\pi}{l})uv(x) + \widehat{m}_{22}(\tfrac{\pi}{l})v^2(x) - v(x)\Big\}\cos\frac{\pi x}{l}\,dx.$$

Give the last two integrands a name, *viz.*

$$A[r_l(u,v) - (u,v)]_1 = \widehat{k}_{11}(\tfrac{\pi}{l})u^2 + \widehat{k}_{12}(\tfrac{\pi}{l})uv + \widehat{k}_{22}(\tfrac{\pi}{l})v^2 - u$$

and

$$A[r_l(u,v) - (u,v)]_2 = \widehat{m}_{11}(\tfrac{\pi}{l})u^2 + \widehat{m}_{12}(\tfrac{\pi}{l})uv + \widehat{m}_{22}(\tfrac{\pi}{l})v^2 - v.$$

Then (3.11) may be rewritten as

$$\begin{cases} \int_0^l [Ar_l(u,v) - (u,v)]_1 \cos\frac{\pi x}{l}\,dx = t\int_0^l \psi_1(x)\cos\frac{\pi x}{l}\,dx, \\ \int_0^l [Ar_l(u,v) - (u,v)]_2 \cos\frac{\pi x}{l}\,dx = t\int_0^l \psi_2(x)\cos\frac{\pi x}{l}\,dx. \end{cases} \tag{3.12}$$

We intend to show that (3.12) cannot hold, so that (3.8) is ruled out. If (3.12) holds, the following two points may be established. First, there are two linearly independent vectors (β_1, β_2) and (β_1', β_2') of non-negative numbers such that for $l > 0$ sufficiently large,

$$\int_0^l \Big\{\beta_1[Ar_l(u,v) - (u,v)]_1 + \beta_2[Ar_l(u,v) - (u,v)]_2\Big\}\cos\frac{\pi x}{l}\,dx \geq 0$$

and

$$\int_0^l \Big\{\beta_1'[Ar_l(u,v) - (u,v)]_1 + \beta_2'[Ar_l(u,v) - (u,v)]_2\Big\}\cos\frac{\pi x}{l}\,dx \geq 0,$$

and equality holds in both these relations if and only if the quantities in braces are constant for $0 \le x \le l$. The second point is that if $(u,v) \in \mathbb{K} \cap \partial\mathcal{B}^0_\epsilon$, then for sufficiently large values of l,

$$\int_0^l \psi_1(x) \cos\frac{\pi x}{l}\, dx \le 0 \qquad \text{and} \qquad \int_0^l \psi_2(x) \cos\frac{\pi x}{l}\, dx \le 0.$$

Moreover, if equality holds, so that

$$\int_0^l \left[\psi_1(x) + \psi_2(x)\right] \cos\frac{\pi x}{l}\, dx = 0,$$

then u and v are constants for $x \ge l$. Supposing for the moment these two points are valid, then for large l it is adduced that

$$\begin{cases} \beta_1[Ar_l(u,v) - (u,v)]_1 + \beta_2[Ar_l(u,v) - (u,v)]_2 = C_1, \\ \beta_1'[Ar_l(u,v) - (u,v)]_1 + \beta_2'[Ar_l(u,v) - (u,v)]_2 = C_2, \end{cases} \tag{3.13}$$

for $0 \le x \le l$, where C_1 and C_2 are constants, and thus

$$\begin{cases} t\displaystyle\int_0^l \left[\beta_1\psi_1(x) + \beta_2\psi_2(x)\right] \cos\frac{\pi x}{l}\, dx = 0, \\ t\displaystyle\int_0^l \left[\beta_1'\psi_1(x) + \beta_2'\psi_2(x)\right] \cos\frac{\pi x}{l}\, dx = 0. \end{cases} \tag{3.14}$$

Since (β_1, β_2) and (β_1', β_2') are linearly independent, (3.13) implies that

$$\begin{cases} [Ar_l(u,v) - (u,v)]_1 = C_1', \\ [Ar_l(u,v) - (u,v)]_2 = C_2', \end{cases}$$

where C_1' and C_2' are constants. It follows that the functions u and v are constant for $0 \le x \le l$. Appealing again to the linear independence of (β_1, β_2) and (β_1', β_2'), (3.14) entails that

$$t\int_0^l \left[\psi_1(x) + \psi_2(x)\right] \cos\frac{\pi x}{l}\, dx = 0.$$

On the other hand, t must not be zero, otherwise, (3.8) implies that u and v are constant for $x \ge l$, and it then follows that u and v are constant on the entire real line $\mathbb{R}$. Thus the pair (u,v) is a fixed point of $s_l A r_l$ in $\mathbb{K} \cap \mathcal{B}^0_{2\epsilon}$, and it is also a fixed point of A since u and v are constant functions and both r_l and s_l fix all constant functions. This is not possible since $\epsilon > 0$ was chosen so small that (u_0, v_0) is the only fixed point of A in $\mathbb{K} \cap \mathcal{B}^0_{2\epsilon}$. Hence, t is non-zero and therefore

$$\int_0^l \left[\psi_1(x) + \psi_2(x)\right] \cos\frac{\pi x}{l}\, dx = 0.$$

The validity of the second point indicates that u and v are constant for $x \ge l$, which leads to the same contradiction as discussed when $t = 0$. In summary, it is concluded that (3.12) and thus (3.8) does not hold.

Attention is now turned to the two points which are outstanding in this discussion. We prove the second point first. Lemma 3.5 of Benjamin *et al.* (1990) will be repeatedly used, so it is briefly reviewed. This lemma states that if $k \in \mathbb{C}$ is even and convex for $x \ge \lambda$, where λ is a non-negative number, then $\int_0^l \big[k(x-y) + k(x+$

$y)] \cos \frac{\pi x}{l} \, dx \leq 0$ for any $l \geq 2\lambda$ and $y \geq l$. Moreover, if the convexity is strict, then the inequality is strict. Write the integrals $\int_0^l \psi_i(x) \cos \frac{\pi x}{l} \, dx$, $i = 1, 2$, as follows:

$$\begin{aligned}
&\int_0^l \psi_1(x) \cos \frac{\pi x}{l} \, dx \\
&= \int_{y>l} \Big\{(r_l u)^2(y) - u^2(y)\Big\} \int_0^l \Big\{k_{11}(x-y) + k_{11}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \, dy \\
&+ \int_{y>l} \Big\{(r_l u)(y)(r_l v)(y) - u(y)v(y)\Big\} \int_0^l \{k_{12}(x-y) \\
&\quad + k_{12}(x+y)\} \cos \frac{\pi x}{l} \, dx \, dy \\
&+ \int_{y>l} \Big\{(r_l v)^2(y) - v^2(y)\Big\} \int_0^l \Big\{k_{22}(x-y) + k_{22}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \, dy,
\end{aligned}$$

$$\begin{aligned}
&\int_0^l \psi_2(x) \cos \frac{\pi x}{l} \, dx \\
&= \int_{y>l} \Big\{(r_l u)^2(y) - u^2(y)\Big\} \int_0^l \Big\{m_{11}(x-y) + m_{11}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \, dy \\
&+ \int_{y>l} \Big\{(r_l u)(y)(r_l v)(y) - u(y)v(y)\Big\} \int_0^l \Big\{m_{12}(x-y) \\
&\quad + m_{12}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \, dy \\
&+ \int_{y>l} \Big\{(r_l v)^2(y) - v^2(y)\Big\} \int_0^l \Big\{m_{22}(x-y) + m_{22}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \, dy.
\end{aligned}$$

Since for $i, j = 1, 2$, the k_{ij} and m_{ij} are convex functions for $x \geq \lambda$, Lemma 3.5 of Benjamin *et al.* (1990) indicates that

$$\int_0^l \Big\{k_{ij}(x-y) + k_{ij}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \leq 0$$

and

$$\int_0^l \Big\{m_{ij}(x-y) + m_{ij}(x+y)\Big\} \cos \frac{\pi x}{l} \, dx \leq 0$$

when l is chosen greater than 2λ and for any $y > l$. In consequence, it is seen that

$$\int_0^l \psi_1(x) \cos \frac{\pi x}{l} \, dx \leq 0 \quad \text{and} \quad \int_0^l \psi_2(x) \cos \frac{\pi x}{l} \, dx \leq 0.$$

Moreover, if we have that

$$\int_0^l \{\psi_1(x) + \psi_2(x)\} \cos \frac{\pi x}{l} \, dx = 0$$

then

$$0 = \int_{y>l} \Big\{(r_l u)^2(y) - u^2(y)\Big\} \int_0^l \Big\{(k_{11}+m_{11})(x-y)$$
$$+ (k_{11}+m_{11})(x+y)\Big\} \cos\frac{\pi x}{l}\, dx\, dy$$
$$+ \int_{y>l} \Big\{(r_l u)(y)(r_l v)(y) - u(y)v(y)\Big\} \int_0^l \Big\{(k_{12}+m_{12})(x-y)$$
$$+ k_{12}+m_{12})(x+y)\Big\} \cos\frac{\pi x}{l}\, dx\, dy$$
$$+ \int_0^l \Big\{(r_l v)^2(y) - v^2(y)\Big\} \int_0^l \Big\{(k_{22}+m_{22})(x-y)$$
$$+ (k_{22}+m_{22})(x+y)\Big\} \cos\frac{\pi x}{l}\, dx\, dy.$$

Again, according to Lemma 3.5 of Benjamin *et al.* (1990), the strict convexity assumption in (S1) allows one to adduce that either

$$\int_0^l \{(k_{12}+m_{12})(x-y) + (k_{12}+m_{12})(x+y)\} \cos\frac{\pi x}{l}\, dx < 0,$$

or both

$$\int_0^l \{(k_{11}+m_{11})(x-y) + (k_{11}+m_{11})(x+y)\} \cos\frac{\pi x}{l}\, dx < 0$$

and

$$\int_0^l \{(k_{22}+m_{22})(x-y) + (k_{22}+m_{22})(x+y)\} \cos\frac{\pi x}{l}\, dx < 0$$

for $l > 2\lambda$ sufficiently large and any $y > l$. Hence, u and v must be constant for $x \geq l$. The proof of the second Ansatz is thus complete.

The proof of the other point is addressed now. According to Lemma 3.4 of Benjamin *et al.* (1990), if $f \in \mathbb{C}$ is decreasing on $[0,l]$, then the integral $\int_0^l f(x)\cos\frac{\pi x}{l}\,dx \geq 0$, and equality holds if and only if f is a constant on $[0,l]$. So it is sufficient to show that there are two independent pairs (β_1,β_2) and (β_1',β_2') of non-negative numbers, such that both

$$\beta_1[Ar_l(u,v)-(u,v)]_1 + \beta_2[Ar_l(u,v)-(u,v)]_2$$

and

$$\beta_1'[Ar_l(u,v)-(u,v)]_1 + \beta_2'[Ar_l(u,v)-(u,v)]_2$$

are monotone decreasing on $[0,l]$.

In establishing this, it is convenient to introduce some notation. For any $x \in \mathbb{R}$ and $\Delta x > 0$, denote the increment of a function $f \in \mathbb{C}$ at x by $\Delta f(x) = f(x+\Delta x) - f(x)$. Of course, f is monotone decreasing if and only if $\Delta f \leq 0$ for all the relevant values of x and Δx.

Return to considering $(u,v) \in \mathbb{K} \cap \partial\mathcal{B}_\epsilon^0$ and write $(u,v) = (u_0+\epsilon u_1, v_0+\epsilon v_1)$, where $u_1, v_1 \in \mathbb{C}$ are bounded in absolute value by $\frac{1}{1-\epsilon}$ and for $x \geq 0$ and $\Delta x \geq 0$,

$\Delta u_1(x) \le 0$, $\Delta v_1(x) \le 0$. A calculation reveals that for $x, \Delta x \ge 0$,

$$\begin{aligned}
&\Delta[Ar_l(u,v) - (u,v)]_1(x) \\
&= \epsilon\Big[2\widehat{k}_{11}(\frac{\pi}{l})u_0 + \widehat{k}_{12}(\frac{\pi}{l})v_0 - 1\Big]\Delta u_1(x) + \epsilon\Big[\widehat{k}_{12}(\frac{\pi}{l})u_0 + 2\widehat{k}_{22}(\frac{\pi}{l})v_0\Big]\Delta v_1(x) \\
&+\epsilon^2\Big[\widehat{k}_{11}(\frac{\pi}{l})\big(u_1(x) + u_1(x+\Delta x)\big) + \widehat{k}_{12}(\frac{\pi}{l})v_1(x+\Delta x)\Big]\Delta u_1(x) \\
&+\epsilon^2\Big[\widehat{k}_{12}(\frac{\pi}{l})u_1(x) + \widehat{k}_{22}(\frac{\pi}{l})\big(v_1(x) + v_1(x+\Delta x)\big)\Big]\Delta v_1(x).
\end{aligned}$$

Since $u, v \in \mathbb{K}$, $\Delta u_1(x) \le 0$ and $\Delta v_1(x) \le 0$ for $x, \Delta x \ge 0$. Note that u_1 or v_1 could be negative, but in any event that are both bounded below by $-\frac{1}{1-\epsilon}$. In consequence, the last expression may be bounded above as follows:

$$\begin{aligned}
&\Delta[Ar_l(u,v) - (u,v)]_1(x) \\
&\le \epsilon\Big[2\widehat{k}_{11}(\frac{\pi}{l})(u_0 - \frac{\epsilon}{1-\epsilon}) + \widehat{k}_{12}(\frac{\pi}{l})(v_0 - \frac{\epsilon}{1-\epsilon}) - 1\Big]\Delta u_1(x) \\
&+ \epsilon\Big[\widehat{k}_{12}(\frac{\pi}{l})(u_0 - \frac{\epsilon}{1-\epsilon}) + 2\widehat{k}_{22}(\frac{\pi}{l})(v_0 - \frac{\epsilon}{1-\epsilon})\Big]\Delta v_1(x).
\end{aligned}$$

Letting $\gamma = \max\{\frac{1}{u_0(1-\epsilon)}, \frac{1}{v_0(1-\epsilon)}\}$, it follows that

$$\begin{aligned}
&\Delta[Ar_l(u,v) - (u,v)]_1(x) \\
&\le \epsilon(1-\gamma\epsilon)\Big[2\widehat{k}_{11}(\frac{\pi}{l})u_0 + \widehat{k}_{12}(\frac{\pi}{l})v_0 - \frac{1}{1-\gamma\epsilon}\Big]\Delta u_1(x) \\
&+ \epsilon(1-\gamma\epsilon)\Big[\widehat{k}_{12}(\frac{\pi}{l})u_0 + 2\widehat{k}_{22}(\frac{\pi}{l})v_0\Big]\Delta v_1(x).
\end{aligned}$$

Hypothesis (S1) includes the assumptions $k_{ij} \in L_1 \cap \mathbb{C}$, so $\widehat{k_{ij}}$ is continuous and hence $K_{ij} = \widehat{k_{ij}}(0) = \lim_{l\to\infty} \widehat{k_{ij}}(\frac{\pi}{l})$. Therefore, for any δ with $0 < \delta < 1$, say, there is an $l_\delta > 0$ such that when $l > l_\delta$, then $\widehat{k_{ij}}(\frac{\pi}{l}) \ge (1-\delta)K_{ij}$ for $i, j = 1, 2$. For such values of l, the above estimates may be continued thusly:

$$\begin{aligned}
&\Delta[Ar_l(u,v) - (u,v)]_1(x) \\
&\le \epsilon(1-\gamma\epsilon)(1-\delta)\Big[2K_{11}u_0 + K_{12}v_0 - \frac{1}{(1-\gamma\epsilon)(1-\delta)}\Big]\Delta u_1(x) \\
&+ \epsilon(1-\gamma\epsilon)(1-\delta)\Big[K_{12}u_0 + 2K_{22}v_0\Big]\Delta v_1(x).
\end{aligned}$$

Choose $\delta = \gamma\epsilon$ and $\epsilon > 0$ sufficiently small that $(1-\gamma\epsilon)^{-2} \le 1 + 3\gamma\epsilon$. In these circumstances, the above inequality may be continued as

$$\begin{aligned}
&\Delta[Ar_l(u,v) - (u,v)]_1(x) \\
&\le \epsilon(1-\gamma\epsilon)^2\Big[2K_{11}u_0 + K_{12}v_0 - \big(1 + 3\gamma\epsilon\big)\Big]\Delta u_1(x) \\
&\quad + \epsilon(1-\gamma\epsilon)^2\Big[K_{12}u_0 + 2K_{22}v_0\Big]\Delta v_1(x) \\
&= \epsilon(1-\gamma\epsilon)^2\Big[K_{11}u_0^2 + K_{12}u_0v_0 - \big(1 + 3\gamma\epsilon\big)u_0\Big]\frac{\Delta u_1(x)}{u_0} \\
&\quad + \epsilon(1-\gamma\epsilon)^2\Big[K_{12}u_0v_0 + 2K_{22}v_0^2\Big]\frac{\Delta v_1(x)}{v_0}.
\end{aligned}$$

Similar considerations yield

$$\begin{aligned}&\Delta[Ar_l(u,v)-(u,v)]_2(x)\\&\quad\le\epsilon(1-\gamma\epsilon)^2\Big[2M_{11}u_0^2+M_{12}u_0v_0\Big]\frac{\Delta u_1(x)}{u_0}\\&\qquad+\epsilon(1-\gamma\epsilon)^2\Big[M_{12}u_0v_0+2M_{22}v_0^2-\big(1+3\gamma\epsilon\big)v_0\Big]\frac{\Delta v_1(x)}{v_0}.\end{aligned}$$

For any $\beta_1,\beta_2\ge 0$, the increment of the linear combination $\beta_1[Ar_l(u,v)-(u,v)]_1+\beta_2[Ar_l(u,v)-(u,v)]_2$ is bounded as follows:

$$\begin{aligned}&\beta_1\Delta[Ar_l(u,v)-(u,v)]_1(x)+\beta_2\Delta[Ar_l(u,v)-(u,v)]_2(x)\\&\quad\le\epsilon(1-\gamma\epsilon)^2\Big\{\beta_1\big\{2K_{11}u_0^2+K_{12}u_0v_0-\big(1+3\gamma\epsilon\big)u_0\big\}\\&\qquad+\beta_2\big\{2M_{11}u_0^2+M_{12}u_0v_0\big\}\Big\}\frac{\Delta u_1(x)}{u_0}\\&\qquad+\epsilon(1-\gamma\epsilon)^2\Big\{\beta_1\big\{K_{12}u_0v_0+2K_{22}v_0^2\big\}\\&\qquad+\beta_2\big\{M_{12}u_0v_0+2M_{22}v_0^2-\big(1+3\gamma\epsilon\big)v_0\big\}\Big\}\frac{\Delta v_1(x)}{v_0}.\end{aligned}$$

Hence, if the coefficients of $\frac{\Delta u_1}{u_0}$ and $\frac{\Delta v_1}{v_0}$ in the last relation are non-negative, then,

$$\beta_1\Delta[Ar_l(u,v)-(u,v)]_1+\beta_2\Delta[Ar_l(u,v)-(u,v)]_2\le 0,$$

which means that $\beta_1[Ar_l(u,v)-(u,v)]_1+\beta_2[Ar_l(u,v)-(u,v)]_2$ is decreasing. Therefore, to prove the second point, it suffices to show that for $\epsilon>0$ small enough there are independent pairs (β_1,β_2) and (β_1',β_2') of non-negative numbers such that

$$\begin{cases}\beta_1\Big\{2K_{11}u_0^2+K_{12}u_0v_0-(1+3\gamma\epsilon)u_0\Big\}+\beta_2\Big\{2M_{11}u_0^2+M_{12}u_0v_0\Big\}\ge 0,\\ \beta_1\Big\{K_{12}u_0v_0+2K_{22}v_0^2\Big\}+\beta_2\Big\{M_{12}u_0v_0+2M_{22}v_0^2-(1+3\gamma\epsilon)v_0\Big\}\ge 0,\end{cases}$$

and similarly for (β_1',β_2'). Rearranging these inequalities gives the relations

$$\begin{cases}(1+3\gamma\epsilon)(u_0\beta_1)\le\dfrac{1}{u_0}\Big\{2K_{11}u_0^2+K_{12}u_0v_0\Big\}(u_0\beta_1)\\ \qquad\qquad+\dfrac{1}{v_0}\Big\{2M_{11}u_0^2+M_{12}u_0v_0\Big\}(v_0\beta_2),\\ (1+3\gamma\epsilon)(v_0\beta_2)\le\dfrac{1}{u_0}\Big\{K_{12}u_0v_0+2K_{22}v_0^2\Big\}(u_0\beta_1)\\ \qquad\qquad+\dfrac{1}{v_0}\Big\{M_{12}u_0v_0+2M_{22}v_0^2\Big\}(v_0\beta_2),\end{cases}$$

where

$$\frac{1}{u_0}\Big\{2K_{11}u_0^2+K_{12}u_0v_0\Big\}+\frac{1}{u_0}\Big\{K_{12}u_0v_0+2K_{22}v_0^2\Big\}=2$$

and

$$\frac{1}{v_0}\Big\{2M_{11}u_0^2+M_{12}u_0v_0\Big\}+\frac{1}{v_0}\Big\{M_{12}u_0v_0+2M_{22}v_0^2\Big\}=2.$$

For $\epsilon>0$ sufficiently small, the assumption $K_{12}+K_{22}>0$ and $M_{11}+M_{12}>0$ in Hypothesis (S3), the positivity of u_0 and v_0 and an application of Corollary 6.4 in the Appendix gives the result that there are two independent pairs (β_1,β_2) and (β_1',β_2') of non-negative numbers which satisfy the last system of inequalities.

(Step 2) At this stage, it is asserted that

$$i(\mathbb{P}_l\mathbb{K}, A, \mathbb{P}_l\mathbb{K} \cap \mathcal{B}^0_\epsilon) = 0.$$

Let $(u^*, v^*) \in \mathbb{P}_l\mathbb{K}$ with both components strictly decreasing on $(0, l)$. If $I - A$ restricted to $\mathbb{P}_l\mathbb{K} \cap \partial\mathcal{B}^0_\epsilon$ excludes the entire half ray $\{a(u^*, v^*) : a \geq 0\}$, then the fixed-point index of (u_0, v_0) is 0 by Lemma 3.2. More precisely, we are trying to exclude the possibility that for fixed, but small values of $\epsilon > 0$, there is a pair $(u, v) \in \mathbb{P}_l\mathbb{K} \cap \partial\mathcal{B}^0_\epsilon$ such that

$$(u, v) - A(u, v) = a(u^*, v^*) \tag{3.15}$$

for some $a \geq 0$. Arguing by contradiction, if there is a (u^*, v^*) and an $a \geq 0$ such that (3.15) holds, then multiplying by $\cos(\frac{\pi x}{l})$, integrating both sides over $[0, l]$ and simplifying the results gives

$$\begin{cases} -\displaystyle\int_0^l [A(u,v) - (u,v)]_1 \cos\frac{\pi x}{l}\,dx = a\int_0^l u^*(x)\cos\frac{\pi x}{l}\,dx, \\ -\displaystyle\int_0^l [A(u,v) - (u,v)]_2 \cos\frac{\pi x}{l}\,dx = a\int_0^l v^*(x)\cos\frac{\pi x}{l}\,dx, \end{cases} \tag{3.16}$$

where, the notation is as in Step 1 (see (3.12)). It is known from Step 1 that there are two non-negative numbers β_1 and β_2, not both zero such that $\beta_1[A(u, v) - (u, v)]_1 + \beta_2[A(u, v) - (u, v)]_2$ is non-increasing on $[0, l]$, and consequently

$$-\int_0^l \left(\beta_1[A(u,v) - (u,v)]_1 + \beta_2[A(u,v) - (u,v)]_2\right)\cos\frac{\pi x}{l}\,dx \leq 0.$$

On the other hand, because both u^* and v^* are strictly decreasing and since both $\beta_1, \beta_2 \geq 0$ and at least one of β_1 and β_2 is positive, we know that

$$\int_0^l \left[\beta_1 u^*(x) + \beta_2 v^*(x)\right]\cos\frac{\pi x}{l}\,dx > 0.$$

These relations together with (3.16) imply $a = 0$, whence (3.15) is reduced to

$$(u, v) = A(u, v).$$

On the other hand, ϵ has been chosen so that (u_0, v_0) is the unique fixed point of A in $\mathbb{K} \cap \mathcal{B}^0_{2\epsilon}$. Thus, (3.15) is proved to be invalid, which is to say,

$$i(\mathbb{P}_l\mathbb{K}, A, \mathbb{P}_l\mathbb{K} \cap \mathcal{B}^0_\epsilon) = 0$$

for $\epsilon > 0$ small enough. □

The issue of existence of non-trivial solutions expounded in Theorem 3.3 has been settled by the preceding theory. Attention is next directed to the regularity of this non-trivial solution.

Proposition 3.8. *Suppose $(\phi, \psi) \in \mathbb{K}^R_r$ is a non-trivial solution of (3.2). Then it must be the case that*

$$\lim_{x\to\infty} \phi(x) = 0 \quad \textit{and} \quad \lim_{x\to\infty} \psi(x) = 0.$$

Proof. Since $(\phi, \psi) \in \mathbb{K}^R_r$, $\phi_0 = \lim_{x\to\infty} \phi(x)$ and $\psi_0 = \lim_{x\to\infty} \psi(x)$ exist and are both non-negative. Because $k_{ij}, m_{ij} \in L_1 \cap \mathbb{C}$, it is deduced that (ϕ_0, ψ_0) satisfies $A(\phi_0, \psi_0) = (\phi_0, \psi_0)$. Arguing by contradiction, suppose $(\phi_0, \psi_0) \neq (0, 0)$. Then both u_0 and v_0 must be strictly greater than zero from our earlier remark concerning fixed points with one component zero. If $\phi = u + \phi_0, \psi = v + \psi_0$, then $(u, v) \in \mathbb{K}$

and neither u nor v is a constant function. Substituting this form of ϕ and ψ into (3.2) and simplifying the result yields

$$\begin{cases} k_{11} * (u^2 + 2\phi_0 u) + k_{12} * (uv + \psi_0 u + \phi_0 v) + k_{22} * (v^2 + 2\psi_0 v) = u, \\ m_{11} * (u^2 + 2\phi_0 u) + m_{12} * (uv + \psi_0 u + \phi_0 v) + m_{22} * (v^2 + 2\psi_0 v) = v. \end{cases}$$

It follows readily that

$$\begin{cases} k_{11} * (2\phi_0 u) + k_{12} * (\psi_0 u + \phi_0 v) + k_{22} * (2\psi_0 v) \le u, \\ m_{11} * (2\phi_0 u) + m_{12} * (\psi_0 u + \phi_0 v) + m_{22} * (2\psi_0 v) \le v. \end{cases} \tag{3.17}$$

Since $k_{ij}, m_{ij} \in L_1 \cap \mathbb{C}$ for $i, j = 1, 2$, there is an $M > 0$ such that

$$\begin{aligned} \int_{-M}^{M} k_{ij}(x)\,dx &\ge \frac{3}{4}\int_{-\infty}^{\infty} k_{ij}(x)\,dx = \frac{3}{4}K_{ij}, \\ \int_{-M}^{M} m_{ij}(x)\,dx &\ge \frac{3}{4}\int_{-\infty}^{\infty} m_{ij}(x)\,dx = \frac{3}{4}M_{ij}, \end{aligned} \tag{3.18}$$

for $1 \le i, j \le 2$. Integrating (3.17) over $(-3M, 3M)$ leads to the system of inequalities

$$\int_{-3M}^{3M}\int_{-\infty}^{\infty} [2\phi_0 k_{11}(x-y)u(y) + \psi_0 k_{12}(x-y)u(y) + \phi_0 k_{12}(x-y)v(y)$$
$$+2\psi_0 k_{22}(x-y)v(y)]\,dy\,dx \le \int_{-3M}^{3M} u(x)\,dx,$$
$$\int_{-3M}^{3M}\int_{-\infty}^{\infty} [2\phi_0 m_{11}(x-y)u(y) + \psi_0 m_{12}(x-y)u(y) + \phi_0 m_{12}(x-y)v(y)$$
$$+2\psi_0 m_{22}(x-y)v(y)]\,dy\,dx \le \int_{-3M}^{3M} v(x)\,dx.$$

Because all the components of the integrand are non-negative, this system may be extended to

$$\left.\begin{aligned} &\int_{-3M}^{3M}\int_{-4M}^{4M} [2\phi_0 k_{11}(x-y)u(y) + \psi_0 k_{12}(x-y)u(y) + \\ &\phi_0 k_{12}(x-y)v(y) + 2\psi_0 k_{22}(x-y)v(y)]\,dy\,dx \le \int_{-3M}^{3M} u(x)\,dx, \\ &\int_{-3M}^{3M}\int_{-4M}^{4M} [2\phi_0 m_{11}(x-y)u(y) + \psi_0 m_{12}(x-y)u(y) + \\ &\phi_0 m_{12}(x-y)v(y) + 2\psi_0 m_{22}(x-y)v(y)]\,dy\,dx \le \int_{-3M}^{3M} v(x)\,dx. \end{aligned}\right\} \tag{3.19}$$

But, for any non-negative $f \in \mathbb{C}$,

$$\int_{-3M}^{3M}\int_{-4M}^{4M} k_{ij}(x-y)f(y)\,dy\,dx = \int_{-3M}^{3M}\int_{-4M-y}^{4M+y} k_{ij}(z)f(y)\,dz\,dy$$
$$\ge \int_{-3M}^{3M}\int_{-M}^{M} k_{ij}(z)f(y)\,dz\,dy \ge \frac{3}{4}K_{ij}\int_{-3M}^{3M} f(y)\,dy,$$

and similarly,

$$\int_{-3M}^{3M}\int_{-4M}^{4M} m_{ij}(x-y)f(y)\,dy\,dx \ge \frac{3}{4}M_{ij}\int_{-3M}^{3M} f(y)\,dy.$$

If $\bar{u} = \int_{-3M}^{3M} u(y)\,dy$ and $\bar{v} = \int_{-3M}^{3M} v(y)\,dy$, then $\bar{u} > 0$ and $\bar{v} > 0$ because $u, v \geq 0$ are not identically zero. The system (3.19) implies that

$$\begin{cases} \frac{3}{2}\Big(\phi_0 K_{11} + \frac{1}{2}\psi_0 K_{12}\Big)\bar{u} + \frac{3}{2}\Big(\frac{1}{2}\phi_0 K_{12} + \psi_0 K_{22}\Big)\bar{v} \leq \bar{u}, \\ \frac{3}{2}\Big(\phi_0 M_{11} + \frac{1}{2}\psi_0 M_{12}\Big)\bar{u} + \frac{3}{2}\Big(\frac{1}{2}\phi_0 M_{12} + \psi_0 M_{22}\Big)\bar{v} \leq \bar{v}, \end{cases}$$

or, what is the same,

$$\left.\begin{aligned} &\Big(\phi_0^2 K_{11} + \frac{1}{2}\phi_0\psi_0 K_{12} - \frac{2}{3}\phi_0\Big)\frac{\bar{u}}{\phi_0} + \Big(\frac{1}{2}\phi_0\psi_0 K_{12} + \psi^2 K_{22}\Big)\frac{\bar{v}}{\psi_0} \leq 0, \\ &\Big(\phi_0^2 M_{11} + \frac{1}{2}\phi_0\psi_0 M_{12}\Big)\frac{\bar{u}}{\phi_0} + \Big(\frac{1}{2}\phi_0\psi_0 M_{12} + \psi_0^2 M_{22} - \frac{2}{3}\psi_0\Big)\frac{\bar{v}}{\psi_0} \leq 0. \end{aligned}\right\} \tag{3.20}$$

Without loss of generality, suppose $\frac{\bar{u}}{\phi_0} = \max\{\frac{\bar{u}}{\phi_0}, \frac{\bar{v}}{\psi_0}\}$, so the left-hand side of the second inequality in (3.20) is greater than or equal to the quantity

$$\begin{aligned} &\Big(\phi_0^2 M_{11} + \frac{1}{2}\phi_0\psi_0 M_{12}\Big)\frac{\bar{v}}{\psi_0} + \Big(\frac{1}{2}\phi_0\psi_0 M_{12} + \psi_0^2 M_{22} - \frac{2}{3}\psi_0\Big)\frac{\bar{v}}{\psi_0} \\ &= \Big(\phi_0^2 M_{11} + \phi_0\psi_0 M_{12} + \psi_0^2 M_{22} - \frac{2}{3}\psi_0\Big)\frac{\bar{v}}{\psi_0} \\ &= (\psi_0 - \frac{2}{3}\psi_0)\frac{\bar{v}}{\psi_0} = \frac{1}{3}\bar{v} > 0. \end{aligned}$$

This contradiction completes the proof of the theorem. □

Proposition 3.9. *Let $(\phi, \psi) \in \mathbb{K}_r^R$ be any non-trivial solution of (3.2). Then $\phi, \psi \in L_1 \cap L_\infty$.*

In fact, it suffices to show that $n\phi(n)$ and $n\psi(n)$ are bounded for $n > 0$ sufficiently large. It then follows that $\phi, \psi \in L_p$ for any $p > 1$, and in particular, $\phi, \psi \in L_2$. As a consequence, $\phi^2, \psi^2, \phi\psi \in L_1$, and then (3.2) implies $\phi, \psi \in L_1 \cap L_\infty$.

Proof. Let $n > 0$ be given. Integrate the two equations in (3.2) over $(0, n)$ sum the outcome and reverse the order of integration to obtain

$$\begin{aligned} &\int_0^n [\phi(x) + \psi(x)]\,dx \\ &\qquad = \int_0^\infty \Big(\gamma_{11,n}(y)\phi^2(y) + \gamma_{12,n}(y)\phi(y)\psi(y) + \gamma_{22,n}(y)\psi^2(y)\Big)\,dy, \end{aligned}$$

where

$$\gamma_{ij,n}(y) = \int_0^n \big[k_{ij}(x-y) + k_{ij}(x+y) + m_{ij}(x+y) + m_{ij}(x-y)\big]\,dx$$

for $1 \leq i, j \leq 2$. By their definition, the $\gamma_{ij,n}$ are increasing with n and $\lim_{n\to\infty} \gamma_{ij,n}(y) = |k_{ij}|_1 + |m_{ij}|_1 = K_{ij} + M_{ij}$ uniformly on compact subsets of $[0, \infty)$.

Letting $n > \mu$ with μ to be determined and splitting the intervals $[0, n]$ and $[0, \infty)$ into $[0, \mu) \cup [\mu, n]$ and $[0, \mu) \cup [\mu, n) \cup [n, \infty)$ respectively, the last relation may be written in the revealing form

$$\Big(\int_0^\mu + \int_\mu^n\Big)[\phi(x)+\psi(x)]\,dx$$
$$=\Big(\int_0^\mu + \int_\mu^n + \int_n^\infty\Big)\Big(\gamma_{11,n}(y)\phi^2(y)+\gamma_{12,n}(y)\phi(y)\psi(y)+\gamma_{22,n}(y)\psi^2(y)\Big)\,dy.$$

Rearranging, this becomes

$$\int_0^\mu \big[\gamma_{11,n}(y)\phi^2(y)+\gamma_{12,n}(y)\phi(y)\psi(y)+\gamma_{22,n}(y)\psi^2(y)\big]-\big[\phi(y)+\psi(y)\big]dy$$
$$=\int_\mu^n \big[\phi(y)+\psi(y)\big]-\big[\gamma_{11,n}(y)\phi^2(y)+\gamma_{12,n}(y)\phi(y)\psi(y)+\gamma_{22,n}(y)\psi^2(y)\big]dy$$
$$-\int_n^\infty \big[\gamma_{11,n}(y)\phi^2(y)+\gamma_{12,n}(y)\phi(y)\psi(y)+\gamma_{22,n}(y)\psi^2(y)\big]\,dy.$$

Let I_n be the integral $\int_0^\mu[\cdots]$ in the last equation. Then, independently of n, I_n is bounded above by

$$I_\infty=\int_0^\mu \Big\{\big[\gamma_{11,\infty}\phi^2(y)+\gamma_{12,\infty}\phi(y)\psi(y)+\gamma_{22,\infty}\psi^2(y)\big]-\big[\phi(y)+\psi(y)\big]\Big\}\,dy,$$

where $\gamma_{11,\infty}=K_{11}+M_{11}$, $\gamma_{12,\infty}=K_{12}+M_{12}$ and $\gamma_{22,\infty}=K_{22}+M_{22}$.

From Proposition 3.8, it is known that $\lim_{x\to\infty}\phi(x)=0$ and $\lim_{x\to\infty}\psi(x)=0$. Choose $\mu>0$ large enough that when $y\ge\mu$,

$$\gamma_{11,\infty}\phi^2(y)+\gamma_{12,\infty}\phi(y)\psi(y)+\gamma_{22,\infty}\psi^2(y)\le\frac{1}{2}[\phi(y)+\psi(y)].$$

On the other hand, since ϕ and ψ are even, non-negative and non-increasing on $[0,\infty]$,

$$\int_n^\infty\Big(\gamma_{11,n}(y)\phi^2(y)+\gamma_{12,n}(y)\phi(y)\psi(y)+\gamma_{22,n}(y)\psi^2(y)\Big)\,dy$$
$$\le\phi^2(n)\int_n^\infty\gamma_{11,n}(y)\,dy+\phi(n)\psi(n)\int_n^\infty\gamma_{12,n}(y)\,dy+\psi^2(n)\int_n^\infty\gamma_{22,n}(y)\,dy$$
$$\le n\phi^2(n)\gamma_{11,\infty}+n\phi(n)\psi(n)\gamma_{12,\infty}+n\psi^2(n)\gamma_{22,\infty}$$
$$=n\tilde\gamma\Big(\phi(n)+\psi(n)\Big)^2$$

where $\tilde\gamma=\max\{\gamma_{11,\infty},\frac{1}{2}\gamma_{12,\infty},\gamma_{22,\infty}\}$. Therefore, for any $n>\mu$ with μ fixed as above,

$$I_\infty\ge I_n\ge\frac{1}{2}\int_\mu^n\Big(\phi(y)+\psi(y)\Big)\,dy-n\tilde\gamma\Big(\phi(n)+\psi(n)\Big)^2$$
$$\ge\Big(\frac{1}{2}(1-\frac{\mu}{n})-\tilde\gamma[\phi(n)+\psi(n)]\Big)n\Big(\phi(n)+\psi(n)\Big).$$

Since $\lim_{x\to\infty}\phi(x)=0$ and $\lim_{x\to\infty}\psi(x)=0$, $n>0$ can be chosen so large that $\frac{1}{2}(1-\frac{\mu}{n})-\tilde\gamma[\phi(n)+\psi(n)]\ge\frac{1}{4}$. A consequence of such a choice is that

$$I_\infty\ge\frac{1}{4}n[\phi(n)+\psi(n)],$$

whence,

$$n[\phi(n)+\psi(n)] \le 4I_\infty$$

is bounded. According to our opening remarks, this suffices to establish the result. □

Theorem 3.10. *If $k \in L_1 \cap \mathbb{C}$ is even, non-negative and decreasing on $[0,\infty)$, then $|\xi|\widehat{k}(\xi)$ is bounded. Therefore, any non-trivial solution (ϕ,ψ) of (3.2) in $\mathbb{K}$ lies in $H^\infty \times H^\infty$.*

Proof. For $\xi \neq 0$,

$$\widehat{k}(\xi) = \widehat{k}(|\xi|) = \int_{-\infty}^{\infty} k(x)e^{ix\xi}\,dx = 2\int_0^\infty k(x)\cos(x\xi)\,dx$$
$$= \frac{2}{|\xi|}\int_0^\infty k(\frac{x}{|\xi|})\cos x\,dx.$$

If $\xi > 0$, then

$$\int_0^\infty k(\frac{x}{\xi})\cos x\,dx = \int_0^{\frac{\pi}{2}} k(\frac{x}{\xi})\cos x\,dx + \sum_{n=0}^{\infty}\int_{\frac{2n+1}{2}\pi}^{\frac{2n+3}{2}\pi} k(\frac{x}{\xi})\cos x\,dx$$
$$= \int_0^{\frac{\pi}{2}} k(\frac{x}{\xi})\cos x\,dx + \sum_{n=0}^{\infty}(-1)^{n+1}\int_0^{\pi} k(\frac{x}{\xi}+\frac{2n+1}{2\xi}\pi)\sin x\,dx.$$

Since k is non-negative and decreasing on $(0,\infty)$, the above series is alternating and the absolute value of each term is decreasing with n, so $|\int_0^\infty k(\frac{x}{\xi})\cos x\,dx| \le k(0)$, whence $|\xi\widehat{k}(\xi)| \le 2k(0)$.

Since the kernels k_{ij} and m_{ij} in (3.2) for $i,j=1,2$ lie in $L_1\cap\mathbb{C}$, are even, non-negative and decreasing on $(0,\infty)$, it follows from the last argument that $|\xi\widehat{k}_{ij}(\xi)|$ and $|\xi\widehat{m}_{ij}(\xi)|$ are bounded. From Proposition 3.9, $\phi,\psi \in L_p$ for any $p \ge 1$. In particular, $\phi^2, \phi\psi, \psi^2 \in L_2$, and so

$$k_{11}*\phi^2, \quad k_{12}*\phi\psi, \quad k_{22}*\psi^2, \quad m_{11}*\phi^2, \quad m_{12}*\phi\psi, \quad m_{22}*\psi^2$$

all lie in H^1, whence

$$\phi = k_{11}*\phi^2 + k_{12}*\phi\psi + k_{22}*\psi^2 \in H^1,$$
$$\psi = m_{11}*\phi^2 + m_{12}*\phi\psi + m_{22}*\psi^2 \in H^1.$$

In consequence, we have

$$\phi^2,\ \phi\psi,\ \psi^2 \in H^1,$$

thus

$$k_{11}*\phi^2, \quad k_{12}*\phi\psi, \quad k_{22}*\psi^2, \quad m_{11}*\phi^2, \quad m_{12}*\phi\psi, \quad m_{22}*\psi^2$$

all lie in H^2 and therefore,

$$\phi,\psi \in H^2.$$

Continuation of this argument establishes the result in view. □

Corollary 3.11. *In the Gear-Grimshaw system (1.2), if the constants a_1, a_2, a_3 are non-negative numbers constrained by*

$$a_3 \le \min\left\{\frac{1}{\sqrt{b_2}}, \frac{1}{b_2 a_2}, \frac{a_2}{b_2 a_1}, a_1, a_2, \frac{a_1}{a_2}, \frac{1}{b_2 a_1}\right\},$$

then for any number $C > \max\{\frac{r_1}{b_1}, 0\}$, there is a solitary-wave solution propagating at speed C given by shape functions $(\phi,\psi) \in \mathbb{K}\cap(H^\infty \times H^\infty)$.

Proof. In the Gear-Grimshaw system (2.1), take the Fourier transform in the spatial variable and invert the resulting matrix of Fourier multipliers on the left. After then applying the inverse Fourier transform, there appears the equivalent system

$$\begin{cases}\phi = k_{11} * \phi^2 + k_{12} * (\phi\psi) + k_{22} * \psi^2,\\ \psi = m_{11} * \phi^2 + m_{12} * (\phi\psi) + m_{22} * \psi^2,\end{cases}$$

of integral equations where

$$k_{11}(x) = \frac{1}{4\sqrt{\Delta}}\Big\{\frac{\alpha-(1-b_2a_2a_3)r_-^2}{r_-}e^{-r_-|x|} - \frac{\alpha-(1-b_2a_2a_3)r_+^2}{r_+}e^{-r_+|x|}\Big\},$$
$$k_{12}(x) = \frac{1}{2\sqrt{\Delta}}\Big\{\frac{a_2\alpha-(a_2-b_2a_1a_3)r_-^2}{r_-}e^{-r_-|x|} - \frac{a_2\alpha-(a_2-b_2a_1a_3)r_+^2}{r_+}e^{-r_+|x|}\Big\},$$
$$k_{22}(x) = \frac{1}{4\sqrt{\Delta}}\Big\{\frac{a_1\alpha-(a_1-a_3)r_-^2}{r_-}e^{-r_-|x|} - \frac{a_1\alpha-(a_1-a_3)r_+^2}{r_+}e^{-r_+|x|}\Big\},$$
$$m_{11}(x) = \frac{b_2}{4\sqrt{\Delta}}\Big\{\frac{a_2C-(a_2-a_3)r_-^2}{r_-}e^{-r_-|x|} - \frac{a_2C-(a_2-a_3)r_+^2}{r_+}e^{-r_+|x|}\Big\},$$
$$m_{12}(x) = \frac{b_2}{2\sqrt{\Delta}}\Big\{\frac{a_1C-(a_1-a_2a_3)r_-^2}{r_-}e^{-r_-|x|} - \frac{a_1C-(a_1-a_2a_3)r_+^2}{r_+}e^{-r_+|x|}\Big\},$$
$$m_{22}(x) = \frac{1}{4\sqrt{\Delta}}\Big\{\frac{C-(1-b_2a_1a_3)r_-^2}{r_-}e^{-r_-|x|} - \frac{C-(1-b_2a_2a_3)r_+^2}{r_+}e^{-r_+|x|}\Big\},$$

$$\Delta = (C-\alpha)^2 + 4C\alpha\beta, \quad \alpha = b_1C - r, \quad \beta = b_2a_3^2,$$

and, $r_+ > 0$, $r_- > 0$ are defined by

$$r_+^2 = \frac{C+\alpha+\sqrt{\Delta}}{2(1-\beta)} \qquad \text{and} \qquad r_-^2 = \frac{C+\alpha-\sqrt{\Delta}}{2(1-\beta)}.$$

Under the stated conditions, the k_{ij} and m_{ij}, $1 \le i,j \le 2$, satisfy Hypotheses (S1)–(S3), so the corollary is a simple application of Theorem 3.3. □

Corollary 3.12. *For any speed C with $|C| > 1$, the regularized Boussinesq system of equations*

$$\begin{cases}\eta_t + u_x + (\eta u)_x - \eta_{xxt} = 0,\\ u_t + \eta_x + uu_x - u_{xxt} = 0,\end{cases} \tag{3.21}$$

possesses solitary-wave solutions of the form $\big(\eta(x-Ct), u(x-Ct)\big)$ if $C > 1$ and $\big(\eta(x-Ct), -u(x-Ct)\big)$ if $C < -1$ where $(\eta, u) \in \mathbb{K} \cap (H^\infty \times H^\infty)$.

Proof. It suffices to prove this result for $C > 1$ since the equations for the shape functions have the form

$$\begin{cases}C(1-D_x^2)\eta - u = \eta u,\\ -\eta + C(1-D_x^2)u = \frac{1}{2}u^2,\end{cases}$$

and these are invariant under the transformation

$$C \to -C, \quad \eta \to \eta, \quad u \to -u. \tag{3.22}$$

Inverting the linear operator on the left-hand side leads to the system of integral equations

$$\begin{cases} \eta(\xi) = \dfrac{1}{4C}\displaystyle\int \Big(b_- e^{-\frac{|\xi-y|}{b_-}} + b_+ e^{-\frac{|\xi-y|}{b_+}}\Big)\eta(y)u(y)\,dy \\ \qquad + \dfrac{1}{8C}\displaystyle\int \Big(b_- e^{-\frac{|\xi-y|}{b_-}} - b_+ e^{-\frac{|\xi-y|}{b_+}}\Big)u(y)^2\,dy, \\ u(\xi) = \dfrac{1}{4C}\displaystyle\int \Big(b_- e^{-\frac{|\xi-y|}{b_-}} - b_+ e^{-\frac{|\xi-y|}{b_+}}\Big)\eta(y)u(y)\,dy \\ \qquad + \dfrac{1}{8C}\displaystyle\int \Big(b_- e^{-\frac{|\xi-y|}{b_-}} + b_+ e^{-\frac{|\xi-y|}{b_+}}\Big)u(y)^2\,dy, \end{cases}$$

where

$$b_+ = \sqrt{\frac{C}{C+1}} \qquad \text{and} \qquad b_- = \sqrt{\frac{C}{C-1}}.$$

It is easily verified that these integral kernels satisfy the hypotheses (S1)-(S3), so an application of Theorem 3.3 concludes the result. □

Remark: In this example and for the next class of examples, the number of non-zero trivial solutions is exactly one.

As another application of the theory developed in this section, we inquire which other of the Bossinesq (1.4)-systems have solitary-wave solutions. For the general system, the equations for the shape functions η and u take the form

$$\begin{cases} C(1-bD_\xi^2)\eta - (1+aD_\xi^2)u = \eta u, \\ -(1+cD_\xi^2)\eta + C(1-dD_\xi^2)u = \dfrac{1}{2}u^2. \end{cases} \tag{3.23}$$

Note that like its specialization (3.21), (3.23) is invariant under the transformation (3.22). In consequence, we may restrict attention to the case where $C > 1$ and $u > 0$. Elementary considerations show that if $b, d > 0$ and $C > 1$ is sufficiently large that $C^2bd - ac > 0$ and $C^2(b+d) + a + c > 0$, then the operator

$$\begin{pmatrix} C(1-bD_\xi^2) & -(1+aD_\xi^2) \\ -(1+cD_\xi^2) & C(1-dD_\xi^2) \end{pmatrix}$$

is invertible, and hence (3.23) may be put in the form

$$\begin{cases} \eta = k_{12} * (\eta u) + \dfrac{1}{2}k_{22} * (u^2), \\ u = m_{12} * (\eta u) + \dfrac{1}{2}m_{22} * (u^2), \end{cases}$$

where the integral kernels k_{ij} and m_{ij} are

$$k_{12}(x) = \frac{C}{2(C^2bd-ac)}\Big(\frac{1-dr_-^2}{r_-(r_+^2-r_-^2)}e^{-r_-|x|} - \frac{1-dr_+^2}{r_+(r_+^2-r_-^2)}e^{-r_+|x|}\Big),$$

$$k_{22}(x) = \frac{1}{2(C^2bd-ac)}\Big(\frac{1+ar_-^2}{r_-(r_+^2-r_-^2)}e^{-r_-|x|} - \frac{1+ar_+^2}{r_+(r_+^2-r_-^2)}e^{-r_+|x|}\Big),$$

$$m_{12}(x) = \frac{1}{2(C^2bd-ac)}\Big(\frac{1+cr_-^2}{r_-(r_+^2-r_-^2)}e^{-r_-|x|} - \frac{1+cr_+^2}{r_+(r_+^2-r_-^2)}e^{-r_+|x|}\Big),$$

$$m_{22}(x) = \frac{C}{2(C^2bd-ac)}\Big(\frac{1-br_-^2}{r_-(r_+^2-r_-^2)}e^{-r_-|x|} - \frac{1-br_+^2}{r_+(r_+^2-r_-^2)}e^{-r_+|x|}\Big),$$

and $r_+, r_- > 0$ are defined by

$$r_+^2 = \frac{C^2(b+d)+(a+c)+\sqrt{\left(C^2(b+d)+(a+c)\right)^2-4(C^2-1)(C^2bd-ac)}}{2(C^2bd-ac)},$$

$$r_-^2 = \frac{C^2(b+d)+(a+c)-\sqrt{\left(C^2(b+d)+(a+c)\right)^2-4(C^2-1)(C^2bd-ac)}}{2(C^2bd-ac)}.$$

Detailed, but elementary calculation shows that, if $b, d > 0$, $C^2bd - ac > 0$, $C > 1$, and $a, c \leq 0, |a|, |c| \leq r_-^{-2}$, then k_{ij} and m_{ij} for $i, j = 1, 2$ satisfy Hypotheses (S1)-(S3).

Corollary 3.13. *If $b, d > 0$, $a, c \leq 0$ and $|a|, |c| \leq \sqrt{bd}$, then for any propagation velocity C with $|C| > 1$, the (1.4)-system has non-trivial solitary-wave solutions $\big(\eta(x-Ct), u(x-Ct)\big)$ if $C > 1$ and $\big(\eta(x-Ct), -u(x-Ct)\big)$ if $C < -1$, where $(\eta, u) \in \mathbb{K} \cap \big(H^\infty \times H^\infty\big)$.*

Remark: A special case where the forgoing hypotheses apply is the Bona-Smith example $a = 0$, $b = d = \frac{1}{3}$, $c = -\frac{1}{3}$. For this situation, our results reproduce aspects of the earlier theory of Toland (1981, 1984). Our theory applies also to the special case $a = \theta^2 - \frac{2}{3}$, $c = 0$, $b = d = \frac{1}{2}(1-\theta^2)$ where $0 \leq \theta^2 \leq \frac{2}{3}$ discussed by M. Chen (2000) using dynamical systems methods as in Toland (1986), Amick and Toland (1992), Champneys and Spence (1993), Champneys *et al.* (1996) and Champneys and Groves (1997).

4. More General Systems.

4.1. **Notation.** The Banach space $L_p = L_p(\mathbb{R})$ with $p \geq 1$ and the Sobolev space $H^s = H^s(\mathbb{R})$ with $s \geq 0$ are connoted as before. The class

$$L_p^m = L_p^m(\mathbb{R}) = L_p \times \cdots \times L_p = \{\mathbf{u} = (u_1, \cdots, u_m) : u_j \in L_p \text{ for } 1 \leq j \leq m\}$$

is a Banach space with the norm $|\mathbf{u}|_p = \left(\sum_{j=1}^m |u_j|_p^p\right)^{\frac{1}{p}}$. For $s_1, s_2, \cdots, s_m \geq 0$, the collection

$$H^{s_1 \times s_2 \times \cdots \times s_m} = \{\mathbf{u} = (u_1, \cdots, u_m) : u_j \in H^{s_j} \text{ for } 1 \leq j \leq m\}$$

is a Banach space with the norm $\|\mathbf{u}\|_{s_1 \times s_2 \times \cdots \times s_m} = \left(\sum_{j=1}^m \|u_j\|_{s_j}^2\right)^{\frac{1}{2}}$. The Fourier transform of $\mathbf{u} \in L_2^m$ is defined componentwise to be

$$\widehat{\mathbf{u}} = (\widehat{u}_1, \widehat{u}_2, \cdots, \widehat{u}_m).$$

The transpose of a vector of functions $\mathbf{u}$ is denoted by $\mathbf{u}^t$.

If $s \in \mathbb{R}$ is not an integer, the related integers

$$\lfloor s \rfloor = \max\{n \in \mathbb{Z} : n < s\}$$

and

$$\lceil s \rceil = \min\{n \in \mathbb{Z} : n > s\}$$

will appear in our analysis.

4.2. **Application of Concentration-Compactness Theory to (1.8).** This section comprises an extension of the theory developed in Subsection 2.3 from 2-equation systems to m-equation systems. Instead of introducing new notation, we continue to use the symbol $\mathcal{L} = \mathcal{L}_C = C - CN - L$, so the abstract m-equation system (1.8) may be rewritten in same form

$$\mathcal{L}\mathbf{u}^t = \mathbf{f}(\mathbf{u}), \tag{4.1}$$

as was the 2-equation system, where $\mathcal{L} = (A_{ij})_{m\times m}$ is an $m \times m$ matrix whose components A_{ij} are Fourier multiplier operators with symbols $a_{ij}(\xi)$, $1 \le i, j \le m$.

Hypotheses similar to those appearing in Subsection 2.3 will be in force.

(A1) The matrix $\big(a_{ij}(\xi)\big) = \big(a_{ij}(\xi)\big)_{m\times m}$ is symmetric, and each a_{ij} is an even function on $\mathbb{R}$, which is to say that for $i, j = 1, 2, \cdots, m$,

$$a_{ij}(\xi) = a_{ji}(\xi) = a_{ij}(|\xi|).$$

Moreover, each $a_{ij}(\xi)$ is a sum of finitely many terms of the form $\alpha_k|\xi|^{2r_k}$, where the α_k's and r_k's are real numbers, with $r_k > 0$ for all k.

(A2) The operator $\mathcal{L}$ is elliptic in the sense that there are two positive constants $\underline{\gamma}$, $\bar{\gamma}$, and positive numbers s_j for $j = 1, \cdots, m$ such that for any $\mathbf{x} = (x_1, x_2, \cdots, x_m)$, $\mathbf{y} = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m$,

$$\mathbf{x}\big(a_{ij}(\xi)\big)\mathbf{x}^t \ge \underline{\gamma}\sum_{j=1}^{m}(1+\xi^2)^{s_j}x_j^2,$$

and

$$\big|\mathbf{x}\big(a_{ij}(\xi)\big)\mathbf{y}^t\big| \le \bar{\gamma}\Big(\sum_{j=1}^{m}(1+\xi^2)^{s_j}x_j^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{m}(1+\xi^2)^{s_j}y_j^2\Big)^{\frac{1}{2}}.$$

(A3) There is a homogeneous polynomial F of degree $p \ge 3$ defined on $\mathbb{R}^m$ such that

$$\mathbf{f} = \operatorname{grad} F,$$

and moreover, there is a positive number $\gamma_0 > 0$ such that

$$|F(x_1, \cdots, x_m)| \le \gamma_0|(x_1, \cdots, x_m)|_p^p = \gamma_0\sum_{j=1}^{m}|x_j|^p.$$

Regarding the problem (4.1), we follow the lines laid out in Section 2.3 by defining a homogeneous functional Λ on $X = H^{s_1\times s_2\times\cdots\times s_m}$ as follows:

$$\Lambda(\mathbf{u}) = \frac{\int \mathbf{u}\mathcal{L}\mathbf{u}^t\,dx}{\big(\int F(\mathbf{u})\,dx\big)^{\frac{2}{p}}}. \tag{4.2}$$

Remark: For clarity, we omit the superscript t indicating the transpose $\mathbf{u}^t$ of $\mathbf{u}$ throughout this chapter. We believe no confusion results and some formulas are thereby made much easier to read.

Elementary calculations reveal that the Fréchet derivative Λ' evaluated at $\mathbf{u}$ and applied to $\mathbf{h} \in X$ is

$$\begin{aligned}\Lambda'(\mathbf{u})\mathbf{h} &= \frac{2\int \mathbf{u}\mathcal{L}\mathbf{h}\,dx}{\big(\int F(\mathbf{u})\,dx\big)^{\frac{2}{p}}} - \frac{2}{p}\frac{\int \operatorname{grad} F\cdot\mathbf{h}\,dx\int \mathbf{u}\mathcal{L}\mathbf{u}\,dx}{\big(\int F(\mathbf{u})\,dx\big)^{\frac{2}{p}+1}}\\ &= \frac{2p\big(\int F(\mathbf{u})\,dx\big)\int \mathbf{u}\mathcal{L}\mathbf{h}\,dx - 2\int \mathbf{f}(\mathbf{u})\cdot\mathbf{h}\,dx\int \mathbf{u}\mathcal{L}\mathbf{u}\,dx}{p\big(\int F(\mathbf{u})\,dx\big)^{\frac{p+2}{p}}}.\end{aligned} \tag{4.3}$$

If $\mathbf{u}$ is any critical point of Λ, in particular, if $\mathbf{u}$ is a minimizer of Λ, then $\Lambda'(\mathbf{u})\mathbf{h} = 0$ for all $\mathbf{h} \in X$, which is the same as

$$\int \mathbf{u}\mathcal{L}\mathbf{h}\,dx = \frac{\int \mathbf{u}\mathcal{L}\mathbf{u}\,dx}{p\int F(\mathbf{u})\,dx}\int \mathbf{f}(\mathbf{u})\cdot\mathbf{h}\,dx.$$

It follows that

$$\mathcal{L}\mathbf{u} = \frac{\int \mathbf{u}\mathcal{L}\mathbf{u}\,dx}{p\int F(\mathbf{u})\,dx}\mathbf{f}(\mathbf{u}).$$

If $\beta = \left\{\frac{\int \mathbf{u}\mathcal{L}\mathbf{u}\,dx}{p\int F(\mathbf{u})\,dx}\right\}^{\frac{1}{p-2}}$, then by homogeneity, $\beta\mathbf{u}$ satisfies the problem (4.1):

$$\mathcal{L}(\beta\mathbf{u}) = \mathbf{f}(\beta\mathbf{u}).$$

Therefore, the problem (4.1) can be solved by finding a minimizer Φ of the variational problem

$$\Lambda(\Phi) = \min\{\Lambda(\mathbf{u}) : \mathbf{u} \in X\}. \tag{4.4}$$

Since the functional Λ is homogeneous of degree zero, we may study instead the constrained minimization problem

$$\Theta(\lambda) = \inf\left\{J(\mathbf{u}) : \mathbf{u} \in X, \int F(\mathbf{u})\,dx = \lambda\right\}, \tag{4.5}$$

where $\lambda = 1$, say, and J is defined on X by

$$J(\mathbf{u}) = \int \mathbf{u}\mathcal{L}\mathbf{u}\,dx.$$

The main result in this section is stated in the next theorem.

Theorem 4.1. *Suppose the hypotheses (A1)-(A3) are valid and that $s = \min\{s_j : 1 \le j \le m\} \ge \frac{1}{2} - \frac{1}{p}$. Then every minimizing sequence $\{\mathbf{u}^{(n)}\}_{n=1}^{\infty}$ of the problem (4.5) is, up to translations in the underlying domain, relatively compact in $X = H^{s_1 \times \cdots \times s_m}$. In consequence, there exists at least one non-trivial solution $\Phi \in X$ to the problem (4.1). Moreover, if $s > \frac{1}{2} - \frac{1}{p}$, then $\Phi \in H^\infty \times H^\infty \times \cdots H^\infty$.*

The elliptic Hypothesis (A2) implies that if $\mathbf{u} \in X$, then J satisfies

$$\underline{\gamma}\|\mathbf{u}\|_X^2 \le J(\mathbf{u}) \le \bar{\gamma}\|\mathbf{u}\|_X^2,$$

and the Sobolev imbedding theorem indicates that there is a positive number $\gamma_{s,p}$ only dependent on s and p such that

$$\left(\gamma_0^{-1}\int F(\mathbf{u})\,dx\right)^{\frac{2}{p}} \le |\mathbf{u}|_p^2 \le \gamma_{s,p}\|\mathbf{u}\|_X^2. \tag{4.6}$$

Hence, for any $\lambda > 0$, it follows that $0 < \Theta(\lambda) < \infty$ and so any minimizing sequence

$$\{\mathbf{u}^{(n)} = (u_1^{(n)}, u_2^{(n)}, \cdots, u_m^{(n)})\}_{n=1}^{\infty} \subset X$$

of (4.5) is bounded. To apply the concentration-compactness principle, the sequence $\{\rho_n\}_{n=1}^{\infty}$ associated with $\{\mathbf{u}^{(n)}\}_{n=1}^{\infty}$ is constructed as follows:

$$\begin{aligned}\rho_n(x) = &\sum_{s_j\in\mathbb{N}} \left(\left|u_j^{(n)}(x)\right|^2 + \left|D_x^{s_j}u_j^{(n)}(x)\right|^2\right)\\ &+\sum_{s_j\notin\mathbb{N}} \left(\left|u_j^{(n)}(x)\right|^2 + \min\{1,\lfloor s_j\rfloor\}\left|D_x^{\lfloor s_j\rfloor}u_j^{(n)}(x)\right|^2\right.\\ &\left.+\int \frac{\left|D_x^{\lfloor s_j\rfloor}u_j^{(n)}(x) - D_y^{\lfloor s_j\rfloor}u_j^{(n)}(y)\right|^2}{|x-y|^{1+2(s_j-\lfloor s_j\rfloor)}}\,dy\right).\end{aligned}$$

Then, $\rho_n \geq 0$, $\{\rho_n\}_{n=1}^{\infty} \subset L_1$ is bounded and the L_1-norm of ρ_n is equivalent to the X-norm of $\mathbf{u}^{(n)}$. Let $\mu_n = \int \rho_n(x)\,dx$ be the L_1-norm of ρ_n, so that $\mu_n > 0$ is bounded and $\mu = \liminf_{n\to\infty}\mu_n > 0$. Without loss of generality, suppose $\mu_n \to \mu$ as $n \to \infty$. The proof of the theorem consists of ruling out the possibility of Vanishing or Dichotomy occurring within the sequence $\{\rho_n\}_{n=1}^{\infty}$.

Proof. If the conclusion of the theorem is not true, then there is a subsequence $\{\rho_{n_k}\}_{k=1}^{\infty}$ of $\{\rho_n\}_{n=1}^{\infty}$ which satisfies either Vanishing or Dichotomy.

If Vanishing occurs, then, for any $R > 0$,

$$\limsup_{k\to\infty, y\in\mathbb{R}} \int_{|x-y|\leq R} \rho_{n_k}\,dx = 0,$$

which implies that

$$\limsup_{k\to\infty, y\in\mathbb{R}} \int_{|x-y|\leq R} |\mathbf{u}^{(n_k)}|^2\,dx = 0.$$

Depending on whether or not the set $\{s_j : 1\leq j\leq m\}\subset\mathbb{N}$, Lemmas 2.4 or 6.3 in Chen and Bona (1998) following Lions (1984) imply that

$$\lim_{k\to\infty}\int |\mathbf{u}^{(n_k)}|^p\,dx = 0.$$

The inequality (4.6) then leads to the contradiction

$$1 = \int F(\mathbf{u}^{(n_k)})\,dx \leq \gamma_0|\mathbf{u}^{(n_k)}|_p^p = \gamma_0\int|\mathbf{u}^{(n_k)}|^p\,dx \to 0,$$

as $k\to\infty$.

If Dichotomy occurs, then there is a $\bar\mu\in(0,\mu)$ such that for any $\epsilon > 0$, there are two sequences $\{\rho_k^1\}_{k=1}^{\infty}, \{\rho_k^2\}_{k=1}^{\infty} \subset L_1$, $\rho_k^1, \rho_k^2 \geq 0$, and a positive number k_0 such that for $k \geq k_0$,

$$\begin{cases} \left|\rho_{n_k} - (\rho_k^1+\rho_k^2)\right|_1 \leq \epsilon, \\ \left|\int\rho_k^1\,dx - \bar\mu\right| \leq \epsilon, \qquad \left|\int\rho_k^2\,dx - (\mu-\bar\mu)\right| \leq \epsilon, \\ \text{supp }\rho_k^1 \subset (y_k - E_0, y_k+E_0) \qquad \text{and} \\ \text{supp }\rho_k^2 \subset (-\infty, y_k - 2R_k)\cup(y_k+2R_k,\infty), \end{cases} \tag{4.7}$$

where $E_0 > 0$ is fixed and $\{y_k\}_{k=1}^{\infty}, \{R_k\}_{k=1}^{\infty} \subset\mathbb{R}$ are two sequences with $R_k > E_0$ for all k and $\lim_{k\to\infty}R_k = \infty$. Let $\zeta, \psi \in C_b^{\infty}$ again denote the functions introduced in (2.6) in the proof of Theorem 2.2 of Section 2.3. For $E_1 > E_0$ and $x\in\mathbb{R}$, define

$$\zeta_k(x) = \zeta(\frac{x-y_k}{E_1}), \qquad \psi_k(x) = \psi(\frac{x-y_k}{R_k}),$$

and let

$$\eta_k^1(x) = \begin{cases} -\zeta(\frac{x-y_k}{E_1}) + \zeta(\frac{x-y_k}{R_k}), & \text{if } \; y_k + E_1 \le x \le y_k + 2R_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta_k^2(x) = \begin{cases} -\zeta(\frac{x-y_k}{E_1}) + \zeta(\frac{x-y_k}{R_k}), & \text{if } y_k - 2R_k \le x \le y_k - E_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $x \in \mathbb{R}$, $\zeta_k(x), \psi_k(x), \eta_k^1(x)$ and $\eta_k^2(x)$ are related by

$$\zeta_k(x) + \psi_k(x) + \eta_k^1(x) + \eta_k^2(x) = 1,$$

and

$$\operatorname{supp} \zeta_k \subset (y_k - 2E_1, y_k + 2E_1), \quad \operatorname{supp} \psi_k \subset (-\infty, y_k - R_k) \cup (y_k + R_k, \infty).$$
$$\operatorname{supp} \eta_k^1 \subset (y_k - 2R_k, y_k - E_1), \quad \operatorname{supp} \eta_k^2 \subset (y_k + E_1, y_k + 2R_k).$$

Hence,

$$\mathbf{u}^{(n_k)} = \zeta_k \mathbf{u}^{(n_k)} + \psi_k \mathbf{u}^{(n_k)} + \eta_k^1 \mathbf{u}^{(n_k)} + \eta_k^2 \mathbf{u}^{(n_k)},$$

and, for each $j \in [1, m]$, the supports of $\zeta_k u_j^{(n_k)}$, $\psi_k u_j^{(n_k)}$, $\eta_k^1 u_j^{(n_k)}$ and $\eta_k^2 u_j^{(n_k)}$ are contained in the supports of ζ_k, ψ_k, η_k^1 and η_k^2, respectively.

From Lemma 6.2 in the Appendix, if E_1 is sufficiently large, then

$$\|\eta_k^1 u_j^{n_k}\|_{s_j} \le C\epsilon \qquad \text{and} \qquad \|\eta_k^2 u_j^{n_k}\|_{s_j} \le C\epsilon$$

as $k \to \infty$, where C is a constant independent of k. And from Lemma 6.1 in the Appendix, for any $j = 1, 2, \cdots, m$,

$$\|\zeta_k u_j^{(n_k)} + \psi_k u_j^{(n_k)}\|_{s_j} = \|\zeta_k u_j^{(n_k)}\|_{s_j}^2 + \|\psi_k u_j^{(n_k)}\|_{s_j}^2 + \mathcal{R},$$

as $k \to \infty$, where the remainder $\mathcal{R}$ is bounded by a constant which is independent of k times $(R_k - E_1)^{-2s_j}$. In consequence, for all j, we have

$$\|u_j^{(n_k)}\|_{s_j}^2 = \|\zeta_k u_j^{(n_k)}\|_{s_j}^2 + \|\psi_k u_j^{(n_k)}\|_{s_j}^2 + \mathcal{R} + C\epsilon.$$

So, define

$$\mathbf{u}^{(k),1} = \zeta_k \mathbf{u}^{(n_k)}, \qquad \mathbf{u}^{(k),2} = \psi_k \mathbf{u}^{(n_k)} \quad \text{and} \quad \mathbf{w}^{(k)} = (\eta_k^1 + \eta_k^2)\mathbf{u}^{(n_k)},$$

or, in components,

$$u_j^{(k),1} = \zeta_k u_j^{n_k}, \qquad u_j^{(k),2} = \psi_k u_j^{n_k} \quad \text{and} \quad w_j^k = \eta_k^1 u_j^{(n_k)} + \eta_k^2 u_j^{n_k}$$

for $j = 1, 2, \cdots, m$. With this notation, it is seen that

$$\operatorname{supp} \mathbf{u}^{(k),1} = \cup_j \operatorname{supp} u_j^{(k),1} \subset (y_k - 2E_1, y_k + 2E_1),$$
$$\operatorname{supp} \mathbf{u}^{(k),2} = \cup_j \operatorname{supp} u_j^{(k),2} \subset (-\infty, y_k - R_k) \cup (y_k + R_k, \infty),$$
$$\operatorname{supp} \mathbf{w}^{(k)} = \cup_j \operatorname{supp} w_j^k \subset (y_k - 2R_k, y_k - E_1) \cup (y_k + E_1, y_k + 2R_k),$$

and, for k sufficiently large,

$$\|\mathbf{w}^{(k)}\|_X \le C\epsilon,$$
$$\|\mathbf{u}^{(n_k)}\|_X^2 = \|\mathbf{u}^{(k),1}\|_X^2 + \|\mathbf{u}^{(k),2}\|_X^2 + O(\epsilon)$$

as $\epsilon \to 0$. Hence, there exist subsequences of $\{\mathbf{u}^{(k),1}\}_{k=1}^\infty$ and $\{\mathbf{u}^{(k),2}\}_{k=1}^\infty$, still denoted by $\{\mathbf{u}^{(k),1}\}_{k=1}^\infty$, $\{\mathbf{u}^{(k),2}\}$, respectively, and a number λ such that, for any $\epsilon > 0$, there is a $k_1 > k_0$ so that $k > k_1$ implies

$$\left| \int F(\mathbf{u}^{(k),1})\, dx - \lambda \right| \le \epsilon \qquad \text{and} \qquad \left| \int F(\mathbf{u}^{(k),2})\, dx - (1 - \lambda) \right| \le \epsilon.$$

It follows that

$$\begin{aligned} J(\mathbf{u}^{(n_k)}) &= J\big(\mathbf{u}^{(k),1} + \mathbf{u}^{(k),2} + \mathbf{w}^{(k)}\big) \\ &= J\big(\mathbf{u}^{(k),1}\big) + J\big(\mathbf{u}^{(k),2}\big) + J\big(\mathbf{w}^{(k)}\big) + 2\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{u}^{(k),2}\,dx \\ &\quad + 2\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{w}^{(k)}\,dx + 2\int \mathbf{u}^{(k),2}\mathcal{L}\mathbf{w}^{(k)}\,dx. \end{aligned} \tag{4.8}$$

Notice that

$$\begin{aligned} &J(\mathbf{w}^{(k)}) \le \bar{\gamma}\|\mathbf{w}^{(k)}\|_X^2 \le C\epsilon, \\ &\Big|\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{w}^{(k)}\,dx\Big| \le \gamma\|\mathbf{u}^{(k),1}\|_X\|\mathbf{w}^{(k)}\|_X \le C\epsilon, \\ &\Big|\int \mathbf{u}^{(k),2}\mathcal{L}\mathbf{w}^{(k)}\,dx\Big| \le \gamma\|\mathbf{u}^{(k),2}\|_X\|\mathbf{w}^{(k)}\|_X \le C\epsilon. \end{aligned}$$

Also, it is asserted that

$$\Big|\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{u}^{(k),2}\,dx\Big| = \Big|\sum_{i,j=1}^{m}\int a_{ij}(\xi)\widehat{u_i^{(k),1}}\,\widehat{u_j^{(k),2}}\,d\xi\Big| \le C\epsilon.$$

If all the s_k are integers, there is no pseudo-differential operator involved, so $\mathcal{L}$ is a local operator and hence $\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{u}^{(k),2}\,dx = 0$ since the supports of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are disjoint. Otherwise, as the supports of $u_i^{(k),1}$ and $u_j^{(k),2}$ are $R_k - 2E_1$ apart and $\lim_{k\to\infty}(R_k - E_1) = \infty$, the desired estimate follows from the relation

$$\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{u}^{(k),2}\,dx = O\big((R_k - E_0)^{-2r_0}\big)$$

as $R_k - E_0 \to +\infty$, where $r_0 = \min\{\lfloor r_1\rfloor, \cdots, \lfloor r_n\rfloor\}$ and the r_j's run over the non-integer values of the $s_k's$ (see Lemma 6.1 in the Appendix). Therefore, letting $k \to \infty$ in (4.8) leads to

$$\begin{aligned} \Theta(1) &= \lim_n\{J(\mathbf{u}^{(n)})\} = \lim_k\{J(\mathbf{u}^{(n_k)})\} \\ &= \lim_k\Big\{J(\mathbf{u}^{(k),1}) + J(\mathbf{u}^{(k),2}) + J(\mathbf{w}_k) + 2\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{u}^{(k),2} \\ &\quad +2\int \mathbf{u}^{(k),1}\mathcal{L}\mathbf{w}_k + 2\int \mathbf{u}^{(k),2}\mathcal{L}\mathbf{w}_k\Big\} \\ &\ge \liminf_k J(\mathbf{u}^{(k),1}) + \liminf_k J(\mathbf{u}^{(k),2}) + O(\epsilon) \end{aligned}$$

as $\epsilon \to 0$. If $\int F\big(\mathbf{u}^{(k),1}(x)\big)\,dx \to \lambda = 0$, then by (4.6),

$$\liminf_k J(\mathbf{u}^{(k),1}) \ge \liminf_k \underline{\gamma}\|\mathbf{u}^{(k),1}\|_X^2 \ge \liminf_k \underline{\gamma}|\rho_k^1|_1 \ge \underline{\gamma}\bar{\mu},$$

whence,

$$\Theta(1) > \underline{\gamma}\bar{\mu} + \liminf_k J(\mathbf{u}^{(k),2}),$$

or

$$\Theta(1) \ge \underline{\gamma}\bar{\mu} + \Theta(1) > \Theta(1),$$

a contradiction. If, on the other hand, $\int F\big(\mathbf{u}^{(k),1}(x)\big)\,dx \to \lambda \ne 0$, then

$$\Theta(1) \ge \Theta(\lambda) + \Theta(1-\lambda).$$

However, by homogeneity and positivity, $\Theta(\lambda) = |\lambda|^{\frac{2}{p}}\Theta(1)$ and $\Theta(1) > 0$, whence

$$\Theta(1) \ge \Theta(\lambda) + \Theta(1-\lambda) = \{|\lambda|^{\frac{2}{p}} + |1-\lambda|^{\frac{2}{p}}\}\Theta(1) > \Theta(1),$$

another contradiction. Thus Dichotomy is seen to be impossible.

Since Vanishing and Dichotomy have been ruled out, it is concluded that there is a sequence $\{y_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that for any $\epsilon > 0$, there is an $R < \infty$ and an integer $n_0 > 0$ such that for $n > n_0$,

$$\int_{|x-y_n|\le R} \rho_n(x)\,dx \ge \mu - \epsilon, \quad \int_{|x-y_n|\ge R} \rho_n(x)\,dx \le \epsilon,$$

$$\begin{aligned} \Big|\int_{|x-y_n|\ge R} F\big(\mathbf{u}^{(n)}(x)\big)\,dx\Big| \le & \int_{|x-y_n|\ge R} \Big|F\big(\mathbf{u}^{(n)}(x)\big)\Big|\,dx \\ \le & \|\mathbf{u}^{(n)}\|_X^{p-2} \int_{|x-y_n|\ge R} \rho_n(x)\,dx \\ = & O(\epsilon), \end{aligned}$$

as $\epsilon \to 0$. It follows that

$$\Big|\int_{|x-y_n|\le R} F\big(\mathbf{u}^{(n)}(x)\big)\,dx - 1\Big| \le \epsilon.$$

Letting $\tilde{\mathbf{u}}^{(n)}(x) = \mathbf{u}^{(n)}(x - y_n)$ for $x \in \mathbb{R}$, the above property means that $\tilde{\mathbf{u}}^{(n)}$ (or a subsequence) converges weakly in X, almost everywhere on $\mathbb{R}$, and strongly in L_p to some function $\tilde{\mathbf{u}} \in X$, say, and, moreover,

$$\int F\big(\tilde{\mathbf{u}}(x)\big)\,dx = \lim \int F\big(\mathbf{u}^{(n)}(x)\big)\,dx = 1.$$

Furthermore,

$$\Theta(1) = \lim_n \int \tilde{\mathbf{u}}^{(n)} \mathcal{L} \tilde{\mathbf{u}}^{(n)}\,dx \ge \int \tilde{\mathbf{u}} \mathcal{L} \tilde{\mathbf{u}}\,dx.$$

Thus the vector function $\tilde{\mathbf{u}}$ solves the variational problem (4.5), and therefore, $\Phi = (\frac{\Theta(1)}{p})^{\frac{1}{p-2}} \tilde{\mathbf{u}}$ is a solution of (4.1).

For the regularity result, if $s = \min\{s_j : 1 \le j \le m\} > \frac{1}{2}$, then X is a Banach-algebra, and since $\Phi \in X$, then $\mathbf{f} \in X$ and $\Phi = \mathcal{L}^{-1}\mathbf{f} \in H^{3s\times\cdots\times 3s}$. Inductively, it follows that $\Phi \in H^\infty \times \cdots \times H^\infty$. If $\frac{1}{2} - \frac{1}{p} < s \le \frac{1}{2}$, then $\mathbf{f} \in H^{q\times\cdots\times q}$ where $q = s - (\frac{1}{2} - \frac{1}{p})$, so $\Phi = \mathcal{L}^{-1}\mathbf{f} \in H^{r\times\cdots\times r}$ where $r = 3s - (\frac{1}{2} - \frac{1}{p})$. Since $3s - (\frac{1}{2} - \frac{1}{p}) > 2s$, another inductive argument will establish the advertised regularity result, which is that if $s > \frac{1}{2} - \frac{1}{p}$, the solution Φ of (4.2) lies in $H^\infty \times \cdots \times H^\infty$.

Theorem 4.1 is proved.

□

5. Further Results for General Systems.

5.1. **Notation.** The symbol $\mathbb{Z}_+$ stands for the set of all non-negative integers. For any integer $m > 1$, $\mathbb{Z}_+^m = \{\alpha = (\alpha_1, \cdots, \alpha_m) : \alpha_k \in \mathbb{Z}_+ \text{ for } 1 \le k \le m\}$. The bold-face letter $\mathbf{x} = (x_1, \cdots, x_m)$ connotes an m-dimensional vector; for any $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}_+^m$, define $|\alpha| = |\alpha_1| + \cdots + |\alpha_m|$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. In this situation, $|\alpha|$ is called the degree of $\mathbf{x}^\alpha$. For a multi-variable polynomial $f(\mathbf{x}) = \sum_{j\in\mathbf{Z}_+, \alpha\in\mathbf{Z}_+^m} c_{j,\alpha}\mathbf{x}^\alpha$, the degree of f is the largest value of $|\alpha|$ whose corresponding coefficient $c_{j,\alpha}$ is non-zero.

5.2. **Generalizations of the Results in Section 3.** In Section 4, we studied a certain type of system of m-equations (4.1) assuming that the operator $\mathcal{L}$ is elliptic in the sense defined by (A1)-(A2), and that the vector of nonlinear terms $\mathbf{f}$ is the gradient of a homogeneous polynomial on $\mathbb{R}^m$. The problem was replaced by a variational problem, and Lions' theory came to fore. In this form, the theory has an interesting range of applications. However, the family of Boussinesq systems (1.4) does not belong to the preceding category of models. Bearing this in mind, interest is now turned to the situation where the operator $\mathcal{L}$ in (4.1) lacks symmetry and the nonlinear vector $\mathbf{f}$ is not necessarily homogeneous, nor is it the gradient of some function. The extended degree theory of positive operators will be brought to bear on the problem in this more general form.

Assuming that the matrix $\big(a_{ij}(\xi)\big)$ associated with the symbol of the operator $\mathcal{L}$ in (4.1) is invertible, take the spatial Fourier transform termwise in (4.1), invert $\big(a_{ij}(\xi)\big)$ and then take the inverse Fourier transform to reach a system of integral equations

$$\begin{cases} u_1 = N_{11} * f_1(\mathbf{u}) + \cdots + N_{1q} * f_m(\mathbf{u}), \\ \qquad\vdots \\ u_m = N_{m1} * f_1(\mathbf{u}) + \cdots + N_{mm} * f_m(\mathbf{u}), \end{cases}$$

as before, where $\mathbf{u} = (u_1, \cdots, u_m)$. Suppose that each component f_k of $\mathbf{f} = (f_1, f_2, \cdots, f_m)$ is a polynomial of degree greater than or equal to 2, the highest degree being q, say, and that $f_k(0,0,\cdots,0) = \partial_j f_k(0,0,\cdots,0) = 0$ for $k,j = 1,2,\cdots,m$. Rearranging these formulas according to the degree of the nonlinearity, the above system may be rewritten in the form,

$$\begin{cases} u_1 = A_{12}F_{12}(\mathbf{u}) + \cdots + A_{1q}F_{1q}(\mathbf{u}), \\ \qquad\vdots \\ u_m = A_{m2}F_{m2}(\mathbf{u}) + \cdots + A_{mq}F_{mq}(\mathbf{u}), \end{cases} \tag{5.1}$$

where, for $1 \le k \le m$ and $2 \le j \le q$, F_{kj} is a sum of homogeneous monomials of degree j defined on $\mathbb{R}^m$ and A_{kj} is an integral operator defined by

$$\begin{aligned} A_{kj}F_{kj}(\mathbf{u}) &= \sum_{\alpha \in \mathbb{Z}_+^m, |\alpha| = j} a_{kj}^{\alpha} * \mathbf{u}^{\alpha} \\ &= \sum_{\alpha \in \mathbb{Z}_+^m, |\alpha| = j} \int a_{kj}^{\alpha}(\cdot - y) u_1^{\alpha_1}(y) \cdots u_m^{\alpha_m}(y)\, dy. \end{aligned} \tag{5.2}$$

The first step is to choose an appropriate function-space setting. Guided by the theory for one and two equations, let

$$X = \underbrace{\mathbb{C}(\mathbb{R}) \times \cdots \times \mathbb{C}(\mathbb{R})}_{m \text{ factors}} = \{\mathbf{f} = (f_1, \cdots, f_m) : f_k \in \mathbb{C}, 1 \le k \le m\},$$

where $\mathbb{C} = \mathbb{C}(\mathbb{R})$ is as before the class of all continuous real-valued functions of a real variable. The generating family of semi-norms for the topology on X is

$$p_j(\mathbf{f}) = \max_{-jl \le x \le jl} \Big\{ |f_k(x)| : 1 \le k \le m \Big\}, \quad j = 1, 2, \cdots,$$

$\mathbf{f} = (f_1, \cdots, f_m) \in X$, where $l > 0$ is to be determined later. The cone

$$\begin{aligned}\mathbb{K} =&\{\mathbf{f} = (f_1, \cdots, f_m) \in X : f_k(x) = f_k(-x) \geq 0,\\ &f_k \text{ is non-increasing on } (0, \infty), 1 \leq k \leq m\}\end{aligned}$$

is the m-dimensional analog of the cone used in Section 3 for a system of 2-equations. Note that, if $\mathbf{u} = (u_1, \cdots, u_m) \in \mathbb{K}$, then the distance of $\mathbf{u}$ to the origin is

$$d(\mathbf{u}, 0) = \frac{\max\{u_j(0) : 1 \leq j \leq m\}}{1 + \max\{u_j(0) : 1 \leq j \leq m\}},$$

or, what is the same, if $d(\mathbf{u}, 0) = r$, then $\max\{u_k(0) : 1 \leq k \leq m\} = \frac{r}{1-r}$.

Define an operator A on $\mathbb{K}$ by asking that $A\mathbf{u}$ is the result of applying the integral operators on the right-hand side of (5.1) to $\mathbf{u}$. Then equation (5.1) may be written briefly as

$$\mathbf{u} = A\mathbf{u}. \tag{5.3}$$

The study of non-trivial fixed points of A is now undertaken.

As in Section 3, the following hypotheses about the kernels a_{kj}^{α} are made.

(B1) $a_{kj}^{\alpha}(x) = a_{kj}^{\alpha}(-x)$, $a_{kj}^{\alpha} \in L_1 \cap \mathbb{C}$, $a_{kj}^{\alpha} \geq 0$ is non-increasing on $(0, \infty)$, there is a number $\lambda \geq 0$ such that when $x \geq \lambda$, all the a_{kj}^{α} are convex. Moreover, for any integer k in $[1, m]$, there is at least one element α in the set

$$\mathcal{A} = \{\alpha : \sum_{k=1}^{m} a_{kj}^{\alpha} \text{ is strictly convex for } |x| \geq \lambda\},$$

whose kth component is positive.

(B2) If $(u_1, \cdots, u_m)$ is a fixed point of A in $\mathbb{K}$ and one of its components is a constant function, then all m-components are constants. Such solutions will be called *trivial* solutions. It is assumed moreover that the number of trivial solutions in $\mathbb{K}$ is finite and that a trivial solution is either identically zero or has all m-components positive.

(B3) Not too many of the A_{kj} and a_{kj}^{α} vanish. Precisely, let, $\mu_{kj}^{\alpha} = \int_0^2 a_{kj}^{\alpha}(x)\, dx$. Assume that there is a number $a \geq 0$ and a vector $\mathbf{u} \in \mathbb{K}$ such that the system of inequalities

$$\begin{cases} a + \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} \mu_{1j}^{\alpha} \int_0^1 \mathbf{u}^{\alpha}(x)\, dx \leq \int_0^1 u_1(x)\, dx, \\ \qquad\vdots \\ a + \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} \mu_{mj}^{\alpha} \int_0^1 \mathbf{u}^{\alpha}(x)\, dx \leq \int_0^1 u_m(x)\, dx \end{cases}$$

holds. Then there is a constant c depending only on the $\{\mu_{kj}^{\alpha}\}$ such that $a \leq c$ and, for any α such that there is a k and j with $\mu_{kj}^{\alpha} > 0$, the quantity $\int_0^1 \mathbf{u}^{\alpha}(x)\, dx$ is bounded by a constant μ^{α} which depends only on the non-zero elements of $\{\mu_{kj}^{\alpha}\}$.

(B4) The non-negative square matrix

$$\begin{pmatrix} \sum_{j=2}^{q} \sum_{|\alpha|=j} \alpha_1 |a_{1j}^{\alpha}|_1 & \cdots & \sum_{j=2}^{q} \sum_{|\alpha|=j} \alpha_1 |a_{mj}^{\alpha}|_1 \\ & \vdots & \\ \sum_{j=2}^{q} \sum_{|\alpha|=j} \alpha_m |a_{1j}^{\alpha}|_1 & \cdots & \sum_{j=2}^{q} \sum_{|\alpha|=j} \alpha_m |a_{mj}^{\alpha}|_1 \end{pmatrix}$$

is irreducible.

Remark: A square matrix $A = (a_{ij}) = (a_{ij})_{n\times n}$ is called *reducible* (see Gantmacher 1960) if the index set $1, 2, \cdots, n$ can be split into two complimentary sets (without common indices) $i_1, i_2, \cdots, i_\mu$; $k_1, k_2, \cdots, k_{n-\mu}$ such that

$$a_{i_\alpha k_\beta} = 0, \quad \text{for } \alpha = 1, 2, \cdots, \mu; \quad \beta = 1, 2, \cdots, n-\mu.$$

In other words, A is reducible if there is a permutation of $\{1, \cdots, n\}$, $\{\sigma_1, \cdots, \sigma_n\}$ say, such that

$$\tilde{A} = (a_{\sigma_i\sigma_j})_{i,j=1,\cdots,n} = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where $\mu \in [1, n)$, B is a $\mu \times \mu$ square matrix and D is an $(n-\mu)\times(n-\mu)$ square matrix. If A is not reducible, it is called *irreducible.* In the special case when $m = 2$ and $q = 2$, the irreducibility assumption is equivalent to the first part of Hypothesis (S3) in Section 3.2, which requires both off-diagonal elements to be strictly positive.

The major result of this section is the following theorem.

Theorem 5.1. *Under Hypotheses (B1)-(B4), the system (5.1) has a nontrivial solution* $\mathbf{u} = (u_1, \cdots, u_m) \in \mathbb{K}$.

The proof consists of the following three lemmas and a proposition.

Lemma 5.2. *The operator A defined in (5.3) maps* $\mathbb{K}$ *to* $\mathbb{K}$ *and is continuous and* $\mathbb{K}$*-compact.*

Proof. This result follows as in Lemma 3.1 of Benjamin *et al.* (1990). □

Lemma 5.3. *If* $r_0 = \max\{\sum_{|\alpha|=j} |a^\alpha_{kj}|_1 : 1 \le k \le m,\ 2 \le j \le q\}$, *where* $|a^\alpha_{kj}|_1$ *is the* L_1*-norm of the kernel* a^α_{kj}, *that is* $|a^\alpha_{kj}|_1 = \int |a^\alpha_{kj}(x)|\,dx = \int a^\alpha_{kj}(x)\,dx = \widehat{\alpha^\alpha_{kj}}(0)$, *then*

(a) for $0 < r < \frac{1}{2+r_0}$, $\mathbf{u} \ne tA\mathbf{u}$ *for any* $\mathbf{u} \in \mathbb{K}\cap\partial\mathcal{B}_r$ *and* $t \in [0,1]$,

(b) for $\frac{1}{2+r_0} < R < 1$ *with* R *sufficiently close to* 1, $\mathbf{u} - A\mathbf{u} \ne a\mathbf{1}$ *for any* $\mathbf{u} \in \mathbb{K}\cap\partial\mathcal{B}_R$ *and all* $a \ge 0$, *where* $\mathbf{1}$ *is the constant function which takes the value* $(1, 1, \cdots, 1)$ *at every point in* $\mathbb{R}$.

Proof. Part (a). Suppose the statement is not true. Then there must be some $\mathbf{u} \in \mathbb{K}\cap\partial\mathcal{B}_r$ and $t \in [0,1]$ such that $\mathbf{u} = tA\mathbf{u}$. Since each component u_k and each kernel a^α_{kj} is non-negative, even and decreasing on $[0,\infty)$, for any $x \in \mathbb{R}$, $\int a^\alpha_{kj}(x-y)\mathbf{u}^\alpha(y)\,dy \le \int a^\alpha_{kj}(x-y)\mathbf{u}^\alpha(0)\,dy = |a^\alpha_{kj}|_1\mathbf{u}^\alpha(0)$. In particular, at $x = 0$, $\mathbf{u}(0) = t(A\mathbf{u})(0)$ leads to the system of inequalities

$$\begin{cases} \dfrac{1}{t}u_1(0) \le \displaystyle\sum_{|\alpha|=2} |\alpha^\alpha_{12}|_1\mathbf{u}^\alpha(0) + \cdots + \sum_{|\alpha|=q} |\alpha^\alpha_{1q}|_1\mathbf{u}^\alpha(0), \\ \qquad\vdots \\ \dfrac{1}{t}u_m(0) \le \displaystyle\sum_{|\alpha|=2} |\alpha^\alpha_{m2}|_1\mathbf{u}^\alpha(0) + \cdots + \sum_{|\alpha|=q} |\alpha^\alpha_{mq}|_1\mathbf{u}^\alpha(0). \end{cases}$$

Since $\mathbf{u} \in \partial\mathcal{B}_r$, it must be the case that $\frac{r}{1-r} = \max\{u_k(0) : 1 \le k \le m\}$, so $\mathbf{u}^\alpha(0) \le \left(\frac{r}{1-r}\right)^{|\alpha|}$ and the above inequalities therefore imply

$$\begin{cases} \dfrac{1}{t}u_1(0) \le \displaystyle\sum_{j=2}^{q} r_0\left(\frac{r}{1-r}\right)^j, \\ \quad\vdots \\ \dfrac{1}{t}u_m(0) \le \displaystyle\sum_{j=2}^{q} r_0\left(\frac{r}{1-r}\right)^j. \end{cases}$$

It follows that

$$\begin{aligned} \frac{1}{t}\frac{r}{1-r} &= \frac{1}{t}\max_{1\le k\le m}\{u_k(0)\} \le r_0\sum_{j=2}^{q}\left(\frac{r}{1-r}\right)^j \\ &= r_0\left(\frac{r}{1-r}\right)^2\frac{1-\left(\frac{r}{1-r}\right)^{q-1}}{1-\frac{r}{1-r}} \le r_0\left(\frac{r}{1-r}\right)\left(\frac{r}{1-2r}\right), \end{aligned}$$

from which one deduces that

$$r \ge \frac{1}{2+tr_0} \ge \frac{1}{2+r_0}.$$

This contradicts the assumption $r < \frac{1}{2+r_0}$ and Part (a) is proved.

Part (b). Supposing (b) is invalid, there must be a $\mathbf{u} \in \mathbb{K} \cap \partial\mathcal{B}_R$ and an $a \ge 0$ such that

$$\mathbf{u} - A\mathbf{u} = a\mathbf{1}$$

or, in concrete form,

$$\begin{cases} u_1(x) = a + \displaystyle\sum_{|\alpha|=2}\int a_{12}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy + \cdots \\ \qquad + \displaystyle\sum_{|\alpha|=j}\int a_{1j}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy + \cdots + \sum_{|\alpha|=q}\int a_{1q}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy, \\ \quad\vdots \\ u_m(x) = a + \displaystyle\sum_{|\alpha|=2}\int a_{m2}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy + \cdots \\ \qquad + \displaystyle\sum_{|\alpha|=j}\int a_{mj}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy + \cdots + \sum_{|\alpha|=q}\int a_{mq}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy. \end{cases} \tag{5.4}$$

Notice that

$$\begin{aligned} &\frac{1}{2}\int_{-1}^{1}\int a_{kj}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy\,dx \ge \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} a_{kj}^\alpha(x-y)\mathbf{u}^\alpha(y)\,dy\,dx \\ &= \frac{1}{2}\int_{-1}^{1}\int_{-1-y}^{1-y} a_{kj}^\alpha(z)\mathbf{u}^\alpha(y)\,dz\,dy \ge \frac{1}{2}\int_0^2 a_{kj}^\alpha(z)\,dz\int_{-1}^{1}\mathbf{u}^\alpha(y)\,dy \\ &= \mu_{kj}^\alpha\int_0^1 \mathbf{u}^\alpha(y)\,dy, \end{aligned}$$

so, integrating (5.4) over $[0,1]$ leads to the system of inequalities

$$\begin{cases} a+\sum_{|\alpha|=2}\mu_{12}^{\alpha}\int_0^1 \mathbf{u}^{\alpha}(x)\,dx+\cdots+\sum_{|\alpha|=j}\mu_{1j}^{\alpha}\int_0^1 \mathbf{u}^{\alpha}(x)\,dx \\ \qquad +\cdots\sum_{|\alpha|=q}\mu_{1q}^{\alpha}\int_0^1 \mathbf{u}^{\alpha}(x)\,dx\le\int_0^1 u_1(x)\,dx, \\ \qquad\vdots \\ a+\sum_{|\alpha|=2}\mu_{m2}^{\alpha}\int_0^1 \mathbf{u}^{\alpha}(x)\,dx+\cdots+\sum_{|\alpha|=j}\mu_{mj}^{\alpha}\int_0^1 \mathbf{u}^{\alpha}(x)\,dx \\ \qquad +\cdots\sum_{|\alpha|=q}\mu_{mq}^{\alpha}\int_0^1 \mathbf{u}^{\alpha}(x)\,dx\le\int_0^1 u_m(x)\,dx. \end{cases}$$

From Hypothesis (B3), a is bounded and each integral $\int_0^1 \mathbf{u}^{\alpha}(x)\,dx$ with at least one positive coefficient μ_{kj}^{α} is bounded by a constant μ^{α}. So returning to (5.4) and following the arguments used in proving part (b) of Lemma 3.5 leads to

$$\begin{aligned} 0\le u_k(x)\le a+2&\sum_{|\alpha|=2,a_{k2}^{\alpha}\neq 0}\sum_{j=-\infty}^{\infty}a_{k2}^{\alpha}(2j)\mu^{\alpha}+\cdots \\ &+2\sum_{|\alpha|=j,a_{kj}^{\alpha}\neq 0}\sum_{j=-\infty}^{\infty}a_{kj}^{\alpha}(2j)\mu^{\alpha}+\cdots+2\sum_{|\alpha|=q,a_{kq}^{\alpha}\neq 0}\sum_{j=-\infty}^{\infty}a_{kq}^{\alpha}(2j)\mu^{\alpha} \end{aligned}$$

for $k=1,2,\cdots,m$. This means that each component u_k is bounded above by some number μ_k, say. It follows that $d(\mathbf{u},0)\le R_0$, where

$$R_0=\frac{2\max\{\mu_k:1\le k\le m\}}{1+2\max\{\mu_k:1\le k\le m\}},$$

and thus if we choose $R\in(R_0,1)$, then $\mathbf{u}\notin\mathbb{K}\cap\partial\mathcal{B}_R$ and thus (b) is established. □

The argument appearing in the proof of Proposition 3.6 leads to the following result.

Proposition 5.4. *Let r and R be defined as in (a) and (b) of Lemma 5.3, respectively. Then $i(\mathbb{K},A,\mathbb{K}_r^R)=-1$, and therefore, there is at least one fixed point of A in the cone segment $\mathbb{K}_r^R$.*

Remark: As in Section 3, this result does not settle the issue of existence of solitary travelling waves. The reason is that there always exists at least one non-zero trivial fixed point in the cone segment $\mathbb{K}_r^R$, that has all of its components positive constant functions. To prove this claim, consider a much narrower cone $\mathbb{K}=\mathbb{R}_+^m=\{(x_1,\cdots,x_m)\in\mathbb{R}^m:\quad x_1,\cdots,x_n\ge 0\}$. Notice that the operator A maps $\mathbb{K}$ to $\mathbb{K}$ continuously. As the cone is finite-dimensional, this means A is continuous and compact on $\mathbb{K}$. Positive operator theory can be applied just as in the infinite-dimensional setting to prove the existence of a fixed point of A in this new cone, all of whose components are non-zero. In the situation where $m=2$ and the nonlinearity is homogenous, the existence of such a fixed point follows from algebraic considerations. This point is not pursued here, but a separate note about this issue will appear in due course.

As in Section 3, the existence of a non-trivial solution may be inferred if the fixed-point index of the operator A at any of the trivial solutions in the cone segment $\mathbb{K}_r^R$ is shown to be zero. This is our next goal, which is the most challenging aspect of the theory.

The two operators $r_l : \mathbb{C} \to \mathbb{C}_l$ and $s_l : \mathbb{C}_l \to \mathbb{C}$ defined in Section 3.2 are extended componentwise to the m-variable case. Thus if $\mathbf{u} = (u_1, \cdots, u_m) \in \mathbb{C}^m$, $\mathbf{r}_l\mathbf{u} = (r_l u_1, \cdots, r_l u_m)$, and for any $\mathbf{v} = (v_1, \cdots, v_m) \in \mathbb{C}_l^m$, $\mathbf{s}_l\mathbf{v} = (s_l v_1, \cdots, s_l v_m)$. Following the development in Section 3, define

$$\mathbb{P}_l\mathbb{K} = \Big\{ \mathbf{u} = (u_1, u_2, \cdots, u_m) \in \mathbb{C}_l^m : u_j \text{ is even, non-negative and non-increasing on } [0, l], 1 \le j \le m \Big\}.$$

The homotopy $\{H_t\}_{t\in[0,1]}$ given by

$$H_t(\mathbf{u}) = tA\mathbf{u} + (1-t)\mathbf{s}_l A \mathbf{r}_l \mathbf{u} \tag{5.5}$$

is defined for $t \in [0, 1]$ and $\mathbf{u} \in \mathbb{K}$. As in Section 3, it plays an important role.

Lemma 5.5. *Suppose the fixed points of A in the cone segment $\mathbb{K}_r^R$ are all trivial, that is, if $\mathbf{u}_0 = (u_1^0, \cdots, u_m^0) \in \mathbb{K}_r^R$ satisfies $A\mathbf{u}_0 = \mathbf{u}_0$, then each u_k^0 is a constant function. In that case, each such fixed point is isolated and has fixed-point index equal to zero.*

Following the strategy displayed in Lemma 3.7, this result is established in two steps. The first is to show that the operator A is homotopic to $\mathbf{s}_l A \mathbf{r}_l$ on a small neighborhood

$$\mathbb{K} \cap \mathcal{B}_\epsilon(\mathbf{u}_0) = \mathbf{r}_l^{-1}[\mathbb{P}_l\mathbb{K} \cap \mathcal{B}_\epsilon(\mathbf{u}_0)], \tag{5.6}$$

where $\mathcal{B}_\epsilon(\mathbf{u}_0)$ is the ball in X centered at $\mathbf{u}_0$, $\epsilon > 0$ is small enough that $\mathbb{K}\cap\mathcal{B}_\epsilon(\mathbf{u}_0) \subset \mathbb{K}_r^R$, $\mathbf{u}_0$ is the unique fixed point of A in $\mathbb{K}\cap\mathcal{B}_{2\epsilon}^0$, and $l > 0$ is sufficiently large. The second step is to show that $i\big(\mathbb{K}, \mathbf{s}_l A \mathbf{r}_l, \mathbb{K}\cap\mathcal{B}_\epsilon(\mathbf{u}_0)\big) = 0$. As $\mathbb{K}$ $\mathbf{r}$-dominates $\mathbb{P}_l\mathbb{K}$, the lemma is concluded via the calculation

$$\begin{aligned} i(A, \mathbf{u}_0) &= i\big(\mathbb{K}, A, \mathbb{K}\cap\mathcal{B}_\epsilon(\mathbf{u}_0)\big) = i\big(\mathbb{K}, A, \mathbf{r}_l^{-1}[\mathbb{P}_l\mathbb{K}\cap\mathcal{B}_\epsilon(\mathbf{u}_0)]\big) \\ &= i\big(\mathbb{P}_l\mathbb{K}, A, \mathbb{P}_l\mathbb{K}\cap\mathcal{B}_\epsilon(\mathbf{u}_0)\big) = i\big(\mathbb{K}, \mathbf{s}_l A \mathbf{r}_l, \mathbb{K}\cap\mathcal{B}_\epsilon(\mathbf{u}_0)\big) \\ &= 0. \end{aligned}$$

Proof. (Step 1) By way of obtaining a contradiction, suppose A is not homotopic to $\mathbf{s}_l A \mathbf{r}_l$ on the set defined in (5.6). Then there must be an element $\mathbf{u} \in \mathbb{K}\cap\partial\mathcal{B}_\epsilon(\mathbf{u}_0)$ and a $t \in [0, 1]$ such that

$$\mathbf{u} = tA\mathbf{u} + (1-t)\mathbf{s}_l A r_l \mathbf{u}, \tag{5.7}$$

or,

$$\mathbf{s}_l A r_l \mathbf{u} - \mathbf{u} = t(\mathbf{s}_l A \mathbf{r}_l \mathbf{u} - A\mathbf{u}). \tag{5.8}$$

On the interval $[-l, l]$, (5.8) reduces to

$$A\mathbf{r}_l\mathbf{u} - \mathbf{u} = t(A\mathbf{r}_l\mathbf{u} - A\mathbf{u}),$$

or, componentwise,

$$\sum_{j=2}^{q} \sum_{|\alpha|=j} a_{kj}^\alpha * (\mathbf{r}_l \mathbf{u}^\alpha) - u_k = t \sum_{j=2}^{q} \sum_{|\alpha|=j} a_{kj}^\alpha * (\mathbf{r}_l \mathbf{u}^\alpha - \mathbf{u}^\alpha) \tag{5.9}$$

for $k = 1, 2, \cdots, m$. Multiplying (5.9) by $\cos\frac{\pi x}{l}$ and integrating the result over $[0, l]$ gives

$$\begin{aligned}&\int_0^l \Big\{\sum_{j=2}^q \sum_{|\alpha|=j} a_{kj}^\alpha * (\mathbf{r}_l\mathbf{u}^\alpha)(x) - u_k(x)\Big\}\cos\frac{\pi x}{l}\,dx\\ &= t\int_0^l \sum_{j=2}^q \sum_{|\alpha|=j} a_{kj}^\alpha * (\mathbf{r}_l\mathbf{u}^\alpha - \mathbf{u}^\alpha)\cos\frac{\pi x}{l}\,dx,\end{aligned} \tag{5.10}$$

for $1 \le k \le m$. Notice that for all k, j and α,

$$\int_0^l a_{kj}^\alpha * (\mathbf{r}_l\mathbf{u}^\alpha)(x)\cos\frac{\pi x}{l}\,dx = \int_0^l \widehat{a_{kj}^\alpha}(\frac{\pi}{l})\mathbf{u}^\alpha(x)\cos\frac{\pi x}{l}\,dx,$$

so (5.10) may be written in an expanded form as

$$\left.\begin{aligned}&\int_0^l \Big\{\sum_{j=2}^q \sum_{|\alpha|=j} \widehat{a_{1j}^\alpha}(\frac{\pi}{l})\mathbf{u}^\alpha(x) - u_1(x)\Big\}\cos\frac{\pi x}{l}\,dx = t\int_0^l \psi_1(x)\cos\frac{\pi x}{l}\,dx,\\ &\qquad\vdots\\ &\int_0^l \Big\{\sum_{j=2}^q \sum_{|\alpha|=j} \widehat{a_{mj}^\alpha}(\frac{\pi}{l})\mathbf{u}^\alpha(x) - u_m(x)\Big\}\cos\frac{\pi x}{l}\,dx = t\int_0^l \psi_m(x)\cos\frac{\pi x}{l}\,dx,\end{aligned}\right\}$$

where, for $k = 1, \cdots, m$,

$$\begin{aligned}\psi_k(x) &= \sum_{j=2}^q \sum_{|\alpha|=j} a_{kj}^\alpha * (\mathbf{r}_l\mathbf{u}^\alpha - \mathbf{u}^\alpha)\\ &= \sum_{j=2}^q \sum_{|\alpha|=j} \int_{y>l} \big(a_{kj}^\alpha(x-y) + a_{kj}^\alpha(x+y)\big)\big(\mathbf{r}_l\mathbf{u}^\alpha(y) - \mathbf{u}^\alpha(y)\big)\,dy.\end{aligned}$$

Denote by Ψ the vector $(\psi_1, \cdots, \psi_m)$, and write $A\mathbf{r}_l\mathbf{u}-\mathbf{u}$ in components as $\big(\{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}_1, \cdots, \{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}_m\big)$, where, for $k = 1, \cdots, m$,

$$\{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}_k(x) = \sum_{j=2}^q \sum_{|\alpha|=j} \widehat{a_{kj}^\alpha}(\frac{\pi}{l})\mathbf{u}^\alpha(x) - u_k(x).$$

Then (5.10) may be written in the compact form

$$\int_0^l \{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}\cos\frac{\pi x}{l}\,dx = t\int_0^l \Psi\cos\frac{\pi x}{l}\,dx. \tag{5.11}$$

Step 1 is complete if (5.10) or (5.11) can be proved invalid. For this purpose, we make a pair of Ansatz's. The first one is that there are m independent vectors $\{\Lambda^{(k)} = (\lambda_1^{(k)}, \cdots, \lambda_m^{(k)})\}_{k=1}^m \subset \mathbb{R}_+^m = \{(x_1, \cdots, x_m) : x_1, \cdots, x_m \ge 0\}$ such that

$$\int_0^l \Lambda^{(k)} \cdot \{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}\cos\frac{\pi x}{l}\,dx \ge 0,$$

where $\Lambda^{(k)} \cdot \{A\mathbf{u}-\mathbf{u}\} = \sum_{j=1}^m \lambda_j^{(k)}\{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}_j$, $k = 1, 2, \cdots, m$, and that equality holds if and only if $\Lambda_k \cdot \{A\mathbf{r}_l\mathbf{u}-\mathbf{u}\}$ is identical to some constant for $0 \le x \le l$. The second conjecture is, for any k with $1 \le k \le m$, when l is chosen large enough,

$$\int_0^l \psi_k \cos\frac{\pi x}{l}\, dx \le 0,$$

and equality holds for all k if and only if each component u_j of $\mathbf{u}$, $1 \le j \le m$, is identical to some constant for $x \ge l$. Supposing for the moment the validity of these two propositions, then (5.10) or (5.11) leads to

$$0 \le \int_0^l \Lambda^{(k)} \cdot \{A\mathbf{r}_l\mathbf{u} - \mathbf{u}\} \cos\frac{\pi x}{l}\, dx = t\int_0^l \Lambda^{(k)} \cdot \Psi \cos\frac{\pi x}{l}\, dx \le 0,$$

so that for $1 \le k \le m$,

$$\Lambda^{(k)} \cdot \{A\mathbf{r}_l\mathbf{u} - \mathbf{u}\} = C_k, \quad \text{for} \quad 0 \le x \le l \tag{5.12}$$

and

$$\begin{aligned} t\int_0^l \Lambda^{(k)} \cdot \Psi \cos\frac{\pi x}{l}\, dx &= t\int_0^l \left(\lambda_1^{(k)}\psi_1 + \cdots + \lambda_m^{(k)}\psi_m\right) \cos\frac{\pi x}{l}\, dx \\ &= 0, \end{aligned} \tag{5.13}$$

where the C_k are constants. Since the vectors $\{\Lambda^{(k)}\}_{k=1}^m \subset \mathbb{R}_+^m$ are independent, (5.12) implies each component $\{A\mathbf{u} - \mathbf{u}\}_k$ is equal to some constant for $0 \le x \le l$, so that each component u_k of $\mathbf{u}$ is identical to some constant for $0 \le x \le l$. Again applying the independence of the $\{\Lambda^{(k)}\}_{k=1}^m$, (5.13) yields in turn that

$$t\int_0^l \psi_k(x) \cos\frac{\pi x}{l}\, dx = 0,$$

for $1 \le k \le m$. In fact, t must not be zero, otherwise, (5.7) becomes $\mathbf{u} = \mathbf{s}_l A\mathbf{r}_l\mathbf{u}$. In this case, each component u_k of $\mathbf{u}$ is constant for $x \ge l$; the evenness and continuity of $\mathbf{u}$ then implies that each u_k is a constant function on all of $\mathbb{R}$. In consequence, $\mathbf{u} \in \mathbb{K} \cap \partial\mathcal{B}_\epsilon(\mathbf{u}_0)$ is also a fixed point of A, and this contradicts the presumption that $\mathbf{u}_0$ is the unique fixed point of A in the small set $\mathbb{K} \cap \mathcal{B}_{2\epsilon}(\mathbf{u}_0)$. As $t > 0$, we must have

$$\int_0^l \psi_k(x) \cos\frac{\pi x}{l}\, dx = 0,$$

for $1 \le k \le m$. The second conjecture indicates that each component u_k of $\mathbf{u}$ is constant for $x \ge l$, and repeating the above argument made to discard the prospect $t = 0$ then rules out the possibility of (5.10) or (5.11).

Now, attention is turned to proving the two conjectures whose validity was just used to deduce the desired overall conclusion. We start with the second conjecture. From the representation of the ψ_k, it follows that

$$\begin{aligned} &\int_0^l \psi_k(x) \cos\frac{\pi x}{l}\, dx \\ &= \int_0^l \sum_{j=2}^q \sum_{|\alpha|=j} \int_{y>l} \Big(a_{kj}^\alpha(x+y) + a_{kj}^\alpha(x-y)\Big)\Big(\mathbf{r}_l\mathbf{u}^\alpha(y) - \mathbf{u}^\alpha(y)\Big) \cos\frac{\pi x}{l}\, dy\, dx \\ &= \sum_{j=2}^q \sum_{|\alpha|=j} \int_{y>l} \Big(\mathbf{r}_l\mathbf{u}^\alpha(y) - \mathbf{u}^\alpha(y)\Big)\, dy \int_0^l \Big(a_{kj}^\alpha(x+y) + a_{kj}^\alpha(x-y)\Big) \cos\frac{\pi x}{l}\, dx, \end{aligned}$$

for $1 \le k \le m$. From Hypothesis (B1), all the a^α_{kj} are convex for $|x| \ge \lambda$, so by Lemma 3.5 of Benjamin *et al.* (1990), when $l > 2\lambda$ is chosen sufficiently large, it follows that

$$\int_0^l \left(a^\alpha_{kj}(x+y) + a^\alpha_{kj}(x-y)\right)\cos\frac{\pi x}{l}\,dx \le 0$$

for $y \ge l$, and if the convexity is strict, then the integral is strictly less than zero. Therefore, for $1 \le k \le m$, $\int_0^l \psi_k(x)\cos\frac{\pi x}{l}\,dx \le 0$. In these circumstances, $\int_0^l \psi_k(x)\cos\frac{\pi x}{l}\,dx = 0$ for all k with $1 \le k \le m$ if and only if

$$\begin{aligned}
0 &= \int_0^l (\psi_1 + \cdots + \psi_m)\cos\frac{\pi x}{l}\,dx \\
&= \sum_{j=1}^q \sum_{|\alpha|=j} \int_{y>l} \Big(r_l\mathbf{u}^\alpha(y) - \mathbf{u}^\alpha(y)\Big)\,dy \times \\
&\qquad \int_0^l \sum_{k=1}^m \Big(a^\alpha_{kj}(x+y) + a^\alpha_{kj}(x-y)\Big)\cos\frac{\pi x}{l}\,dx.
\end{aligned}$$

For any $\alpha \in \mathcal{A}$, Hypothesis (B1) and Lemma 3.5 of Benjamin *et al.* (1990) imply that when $l > 2\lambda$ is large enough, then

$$\int_0^l \sum_{k=1}^m \Big(a^\alpha_{kj}(x+y) + a^\alpha_{kj}(x-y)\Big)\cos\frac{\pi x}{l}\,dx < 0$$

for any $y \ge l$. Hence, for such an α, $\mathbf{r}_l\mathbf{u}^\alpha(y) - \mathbf{u}^\alpha(y)$ must vanish identically for $y \ge l$. Since for each $k \in [1,m]$, there is at least one $\alpha \in \mathcal{A}$ whose kth component is non-zero, it is concluded that for $1 \le k \le m$, u_k is constant for $y \ge l$. The second conjecture is proved.

According to Lemma 3.4 of Benjamin *et al.* (1990), the condition

$$\int_0^l \Lambda^{(k)} \cdot \{A\mathbf{r}_l\mathbf{u} - \mathbf{u}\}\cos\frac{\pi x}{l}\,dx \ge 0$$

is implied if $\Lambda^{(k)} \cdot \{A\mathbf{r}_l\mathbf{u} - \mathbf{u}\}$ is monotone decreasing on $[0,l]$. Hence it suffices for establishing the first Ansatz to show there are linearly independent vectors $\{\Lambda^{(k)}\}_{k=1}^m$ such that $\Lambda^{(k)} \cdot \{A\mathbf{r}_l\mathbf{u} - \mathbf{u}\}$ is monotone decreasing.

For $f \in \mathbb{C}$, $x \in \mathbb{R}$, fix $\Delta x > 0$ and let $\Delta f(x) = f(x+\Delta x) - f(x)$ as in Section 3; then f is decreasing on $\mathbb{R}_+$, say, if and only if $\Delta f(x) \le 0$ for all positive values of x and Δx. In particular, for any $\mathbf{u} = (u_1, \cdots, u_m) \in \mathbb{K}$, if $x \ge 0$, then $\Delta u_k(x) \le 0$, $1 \le k \le m$. Notice that, for any functions f and g,

$$\begin{aligned}
\Delta(fg)(x) &= f(x+\Delta x)g(x+\Delta x) - f(x)g(x) \\
&= \big(f(x+\Delta x) - f(x)\big)g(x+\Delta x) + f(x)\big(g(x+\Delta x) - g(x)\big) \\
&= \Delta f(x)g(x+\Delta x) + f(x)\Delta g(x),
\end{aligned}$$

so, inductively, it is adduced that

$$\Delta(f_1\cdots f_m)(x) = \sum_{k=1}^m \Delta f_k(x)\prod_{j=1}^{k-1} f_j(x)\prod_{j=k+1}^m f_j(x+\Delta x). \qquad (*)$$

For $k = 1, \cdots, m$, $\Delta\{Ar_l\mathbf{u} - \mathbf{u}\}_k$ is given in detail as follows:

$$\begin{aligned}
&\Delta\{Ar_l\mathbf{u} - \mathbf{u}\}_k(x) \\
&=\Delta\Big(\sum_{j=2}^{q}\sum_{|\alpha|=j}\widehat{a_{kj}^{\alpha}}(\frac{\pi}{l})(\mathbf{u}_0+\epsilon\mathbf{v})^{\alpha}\Big)(x) - \Delta\big(u_k^0+\epsilon v_k\big)(x) \\
&=\sum_{j=2}^{q}\sum_{|\alpha|=j}\widehat{a_{kj}^{\alpha}}(\frac{\pi}{l})\Delta(\mathbf{u}_0+\epsilon\mathbf{v})^{\alpha}(x) - \epsilon\Delta v_k(x) \\
&=u_k^0\Big\{\sum_{j=2}^{q}\sum_{|\alpha|=j}\widehat{a_{kj}^{\alpha}}(\frac{\pi}{l})\frac{\mathbf{u}_0^{\alpha}}{u_k^0}\Delta(\mathbf{1}+\epsilon\mathbf{u}_0^{-\alpha}\mathbf{v})^{\alpha}(x) - \frac{\epsilon\Delta v_k(x)}{u_k^0}\Big\},
\end{aligned}$$

where $\mathbf{1}$ represents the constant function $(1, 1, \cdots, 1)$, and $\mathbf{u}_0^{-\alpha} = (\frac{1}{u_1^0}, \cdots, \frac{1}{u_m^0})^{\alpha}$. Note that if the kernels a_{kj}^{α} in (5.2) are renormalized by

$$a_{kj}^{\alpha}\frac{(\mathbf{u}_0)^{\alpha}}{u_k^0},$$

then $(1, 1, \cdots, 1)$ becomes the trivial solution. Under this renormalization, the quantities $|a_{kj}^{\alpha}|_1$ in the matrix defined in (B4) are then replaced by

$$|a_{kj}^{\alpha}|_1\frac{(\mathbf{u}_0)^{\alpha}}{u_k^0}.$$

It is easy to see that this renormalization does not change the irreducibility of the corresponding matrix. So without loss of generality, it may be supposed that $\mathbf{u}_0 = (1, 1, \cdots, 1)$ and $\mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{v} = (1+\epsilon v_1, \cdots, 1+\epsilon v_m) \in \mathbb{K} \cap \partial\mathcal{B}_{\epsilon}(\mathbf{u}_0)$, where each $v_k \in \mathbb{C}$ and $|v_k(x)| \leq \frac{1}{1-\epsilon}$ for all $x \in \mathbb{R}$, by the definition of the metric in the Fréchet space X. Consequently, we have $1 + \epsilon v_k(x) \geq 1 - \frac{\epsilon}{1-\epsilon}$ for all $x \in \mathbb{R}$. The relation $(*)$ may be used to determine that

$$\begin{aligned}
&\Delta(\mathbf{u}_0+\epsilon\mathbf{v})^{\alpha}(x) = \Delta\big((1+\epsilon v_1)^{\alpha_1}\cdots(1+\epsilon v_1)^{\alpha_m}\big)(x) \\
&=\sum_{k=1}^{m}\Delta\big(1+\epsilon v_k\big)^{\alpha_k}(x)\prod_{j=1}^{k-1}\big(1+\epsilon v_j(x)\big)^{\alpha_j}\prod_{j=k+1}^{m}\big(1+\epsilon v_j(x+\Delta x)\big).
\end{aligned}$$

Because $\Delta v_j(x) \leq 0$ and $v_j(x) \geq -\frac{1}{1-\epsilon}$, it transpires that

$$\begin{aligned}
\Delta(1+\epsilon v_k)^{\alpha_k}(x) &= \sum_{j=1}^{\alpha_k}\big(1+\epsilon v_k(x)\big)^{j-1}\epsilon\Delta v_k(x)\big(1+\epsilon v_k(x+\Delta x)\big)^{\alpha_k-j} \\
&\leq \epsilon\sum_{j=1}^{\alpha_k}(1-\frac{\epsilon}{1-\epsilon})^{\alpha_k-1}\Delta v_k(x) \\
&= \epsilon\alpha_k(1-\frac{\epsilon}{1-\epsilon})^{\alpha_k-1}\Delta v_k(x),
\end{aligned}$$

so long as $0 < \epsilon < 1$. Then, the preceding relations imply

$$\begin{aligned}\Delta(\mathbf{u_0}+\epsilon\mathbf{v})^{\alpha}(\mathbf{x}) \leq & \sum_{k=1}^{m}\Delta(1+\epsilon v_k)^{\alpha_k}(x)\prod_{j=1}^{k-1}(1-\frac{\epsilon}{1-\epsilon})^{\alpha_j}\prod_{j=k+1}^{m}(1-\frac{\epsilon}{1-\epsilon})^{\alpha_j} \\ \leq & \sum_{k=1}^{m}\epsilon\alpha_k(1-\frac{\epsilon}{1-\epsilon})^{|\alpha|-1}\Delta v_k(x) \\ = & \epsilon(1-\frac{\epsilon}{1-\epsilon})^{|\alpha|-1}\sum_{k=1}^{m}\alpha_k\Delta v_k(x)\end{aligned}$$

and so it follows that

$$\begin{aligned}\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_k(x) \leq & \epsilon\sum_{j=2}^{q}\sum_{|\alpha|=j}\widehat{a^{\alpha}_{kj}}(\frac{\pi}{l})(1-\frac{\epsilon}{1-\epsilon})^{|\alpha|-1}\sum_{n=1}^{m}\alpha_n\Delta v_n(x)-\epsilon\Delta v_k(x) \\ \leq & \epsilon(1-\frac{\epsilon}{1-\epsilon})^{q-1}\sum_{j=2}^{q}\sum_{|\alpha|=j}\widehat{a^{\alpha}_{kj}}(\frac{\pi}{l})\sum_{n=1}^{m}\alpha_n\Delta v_n(x)-\epsilon\Delta v_k(x).\end{aligned}$$

Since $a^{\alpha}_{kj} \in L_1 \cap \mathbb{C}$ for all the relevant α, k and j, $\widehat{a^{\alpha}_{kj}} \in \mathbb{C}$ are bounded and $\lim_{l\to\infty}\widehat{a^{\alpha}_{kj}}(\frac{\pi}{l}) = \widehat{a^{\alpha}_{kj}}(0) = |a^{\alpha}_{kj}|_1 = B^{\alpha}_{kj}$, say. Hence, for any δ with $0 < \delta < \frac{1}{2}$, there is an l_δ such that when $l \geq l_\delta$, $\widehat{a^{\alpha}_{kj}}(\frac{\pi}{l}) \geq (1-\delta)B^{\alpha}_{kj}$. For such values of l,

$$\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_k(x) \leq \epsilon(1-\frac{\epsilon}{1-\epsilon})^{q-1}(1-\delta)\sum_{j=2}^{q}\sum_{|\alpha|=j}\sum_{n=1}^{m}\alpha_n B^{\alpha}_{kj}\Delta v_n(x)-\epsilon\Delta v_k(x).$$

In particular, if $\delta \leq \frac{\epsilon}{1-\epsilon}$, then the last inequality becomes

$$\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_k(x) \leq \epsilon(1-\frac{\epsilon}{1-\epsilon})^{q}\sum_{j=2}^{q}\sum_{|\alpha|=j}\sum_{n=1}^{m}\alpha_n B^{\alpha}_{kj}\Delta v_n(x)-\epsilon\Delta v_k(x).$$

Choosing $\epsilon > 0$ small enough so that $(1-\frac{\epsilon}{1-\epsilon})^q > 1-2q\epsilon$, it follows that

$$\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_k(x) \leq \epsilon(1-2q\epsilon)\sum_{j=2}^{q}\sum_{|\alpha|=j}\sum_{n=1}^{m}\alpha_n B^{\alpha}_{kj}\Delta v_n(x)-\epsilon\Delta v_k(x).$$

So, for any $\lambda_1, \cdots, \lambda_m \geq 0$, we have

$$\begin{aligned}&\lambda_1\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_1(x)+\lambda_2\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_2(x)+\cdots+\lambda_m\Delta\{Ar_l\mathbf{u}-\mathbf{u}\}_m(x) \\ \leq & \epsilon(1-2q\epsilon)\sum_{k=1}^{m}\lambda_k\sum_{j=2}^{q}\sum_{|\alpha|=j}\sum_{n=1}^{m}\alpha_n B^{\alpha}_{kj}\Delta v_n(x)-\epsilon\sum_{k=1}^{m}\lambda_k\Delta v_k(x) \\ = & \epsilon(1-2q\epsilon)\Big\{\sum_{n=1}^{m}\Delta v_n(x)\sum_{k=1}^{m}\sum_{j=2}^{q}\sum_{|\alpha|=j}\lambda_k\alpha_n B^{\alpha}_{kj}-\frac{1}{1-2q\epsilon}\sum_{n=1}^{m}\lambda_n\Delta v_n(x)\Big\} \\ \leq & \epsilon(1-2q\epsilon)\Big\{\sum_{n=1}^{m}\Delta v_n(x)\sum_{k=1}^{m}\sum_{j=2}^{q}\sum_{|\alpha|=j}\lambda_k\alpha_n B^{\alpha}_{kj}-(1+3q\epsilon)\sum_{n=1}^{m}\lambda_n\Delta v_n(x)\Big\}.\end{aligned}$$

For $1 \le n \le m$, $\Delta v_n(x) \le 0$ for $x \ge 0$, and hence if $\lambda_1, \cdots, \lambda_m \ge 0$ exist such that

$$(1+3q\epsilon)\lambda_n \le \sum_{k=1}^{m}\sum_{j=2}^{q}\sum_{|\alpha|=j} \lambda_k \alpha_n B_{kj}^{\alpha}, \qquad (**)$$

for $1 \le n \le m$, then for $x \ge 0$,

$$\Delta\Big(\lambda_1\{Ar_l\mathbf{u}-\mathbf{u}\}_1 + \lambda_2\{Ar_l\mathbf{u}-\mathbf{u}\}_2 + \cdots + \lambda_m\{Ar_l\mathbf{u}-\mathbf{u}\}_m\Big)(x) \le 0,$$

which is to say

$$\lambda_1\{Ar_l\mathbf{u}-\mathbf{u}\}_1 + \lambda_2\{Ar_l\mathbf{u}-\mathbf{u}\}_2 + \cdots + \lambda_m\{Ar_l\mathbf{u}-\mathbf{u}\}_m$$

is a decreasing function on $\mathbb{R}_+$. In the present normalized variables, the fact that $\mathbf{u}_0 = (1, \cdots, 1)$ is a solution of (5.1) means

$$\sum_{j=2}^{q}\sum_{|\alpha|=j} B_{kj}^{\alpha} = 1.$$

In consequence, we have

$$\sum_{j=2}^{q}\sum_{|\alpha|=j}\sum_{n=1}^{m} \alpha_n B_{kj}^{\alpha} = \sum_{j=2}^{q}\sum_{|\alpha|=j} |\alpha| B_{kj}^{\alpha} \ge 2$$

for $k = 1, \cdots, m$. In fact, the irreducibility assumption (B4), the above inequality and Corollary 6.4 in the Appendix guarantee that the matrix

$$\begin{pmatrix} \sum_{j=2}^{q}\sum_{|\alpha|=j} \alpha_1 B_{1j}^{\alpha} & \cdots & \sum_{j=2}^{q}\sum_{|\alpha|=j} \alpha_1 B_{mj}^{\alpha} \\ & \vdots & \\ \sum_{j=2}^{q}\sum_{|\alpha|=j} \alpha_m B_{1j}^{\alpha} & \cdots & \sum_{j=2}^{q}\sum_{|\alpha|=j} \alpha_m B_{mj}^{\alpha} \end{pmatrix}$$

has a dominant eigenvalue $r \ge 2$ corresponding to an eigenvector $(\beta_1, \cdots, \beta_m)$ all of whose components $\beta_1, \cdots, \beta_m$ are strictly positive. Therefore there must exist m linearly independent vectors

$$\Big\{\Lambda^{(k)} = \big(\lambda_1^{(k)}, \cdots, \lambda_m^{(k)}\big)\Big\}_{k=1}^{m} \subset \mathbb{R}_+^m$$

which satisfy (**) if $\epsilon > 0$ and likewise $3q\epsilon$ is sufficiently small. This says exactly that for $1 \le k \le m$, $\Lambda^{(k)} \cdot \{A\mathbf{u} - \mathbf{u}\}$ is a decreasing function of $x \ge 0$.

(Step 2) It is asserted that

$$i\big(\mathbb{P}_l\mathbb{K}, A, \mathbb{P}_l\mathbb{K} \cap \mathcal{B}_\epsilon(\mathbf{u}_0)\big) = 0.$$

Let $\mathbf{u}^* \in \mathbb{P}_l\mathbb{K}$ with at least one component strictly decreasing on $(0, l)$. If we can rule out the possibility that

$$\mathbf{u} - A\mathbf{u} = a\mathbf{u}^* \qquad (5.14)$$

for any $a \ge 0$ and $\mathbf{u} \in \mathbb{P}_l\mathbb{K} \cap \partial\mathcal{B}_\epsilon(\mathbf{u}_0)$ for $\epsilon > 0$ small, then Proposition 2.2 of Benjamin *et al.* (1990) shows the validity of the claim. Arguing by contradiction, suppose there is such a $\mathbf{u}^*$ and an $a \ge 0$ satisfying (5.14). Multiplying (5.14) by

$\cos \frac{\pi x}{l}$ and integrating over $(0, l)$ gives

$$-\int_0^l \Big(\sum_{j=2}^{q} \sum_{|\alpha|=j} \widehat{a^{\alpha}_{1j}}(\frac{\pi}{l}) \mathbf{u}^{\alpha}(x) - u_1(x) \Big) \cos \frac{\pi x}{l}\, dx = \int_0^l a u_1^*(x) \cos \frac{\pi x}{l}\, dx$$

$$\vdots$$

$$-\int_0^l \Big(\sum_{j=2}^{q} \sum_{|\alpha|=j} \widehat{a^{\alpha}_{mj}}(\frac{\pi}{l}) \mathbf{u}^{\alpha}(x) - u_m(x) \Big) \cos \frac{\pi x}{l}\, dx = \int_0^l a u_m^*(x) \cos \frac{\pi x}{l}\, dx,$$

or, in short form,

$$-\int_0^l \{A\mathbf{u} - \mathbf{u}\} \cos \frac{\pi x}{l}\, dx = a \int_0^l \mathbf{u}^*(x) \cos \frac{\pi x}{l}\, dx. \tag{5.15}$$

From the previous step, it is known that there are m independent vectors $\{\Lambda^{(k)}\}_{k=1}^m \subset \mathbb{R}_+^m$ such that $\Lambda^{(k)} \cdot \{A\mathbf{u} - \mathbf{u}\}$ is decreasing on $[0, l]$, $1 \le k \le m$. Fix such a collection $\{\Lambda^{(k)}\}_{k=1}^m$. Then, as in Section 3,

$$-\int_0^l \Lambda^{(k)} \cdot \{A\mathbf{u} - \mathbf{u}\} \cos \frac{\pi x}{l}\, dx \le 0$$

and

$$a \int_0^l \Lambda^{(k)} \cdot \mathbf{u}^* \cos \frac{\pi x}{l}\, dx \ge 0.$$

Because of (5.15), it must be the case that for $1 \le k \le m$,

$$-\int_0^l \Lambda^{(k)} \cdot \{A\mathbf{u} - \mathbf{u}\} \cos \frac{\pi x}{l}\, dx = 0 \tag{5.16}$$

and

$$a \int_0^l \Lambda^{(k)} \cdot \mathbf{u}^* \cos \frac{\pi x}{l}\, dx = 0. \tag{5.17}$$

The strict monotonicity of at least one component of $\mathbf{u}^*$ and the independence of $\{\Lambda^{(k)}\}_{k=1}^m$ plus (5.17) implies $a = 0$. On the other hand, (5.16) implies that $\Lambda^{(k)} \cdot \{A\mathbf{u} - \mathbf{u}\}$ must be a constant for $0 \le x \le l$ and $k = 1, 2, \cdots, m$. Because of the independence of the $\{\Lambda^{(k)}\}_{k=1}^m$, each component function of $\mathbf{u}$ must be constant for $0 \le x \le l$. The periodicity of $\mathbf{u}$ then implies that each component of $\mathbf{u}$ is a constant function on $\mathbb{R}$, so $\mathbf{u}$ is seen to be a (trivial) fixed point of A in $\mathbb{K} \cap \mathcal{B}_{2\epsilon}(\mathbf{u}_0)$ other than $\mathbf{u}_0$. This contradiction proves (5.14) to be impossible for any $a \ge 0$. The proposition is established. □

The proof of Theorem 5.1 is complete. We now initiate a study of the properties of non-trivial solutions of the system (5.1).

Theorem 5.6. *If $\mathbf{u} = (u_1, \cdots, u_m)$ is any non-trivial solution of the system (5.1), then*

$$\lim_{x \to \pm\infty} u_k(x) = 0, k = 1, \cdots, m.$$

Proof. Since each component u_k is even, continuous, non-negative and non-increasing on $(0, \infty)$, it follows that $\lim_{x \to \infty} u_k(x)$ exists, for $1 \le k \le m$. If $u_k^0 = \lim_{x \to \infty} u_k(x)$, then $u_k^0 \ge 0$. Since $a^{\alpha}_{kj} \in L_1 \cap \mathbb{C}$, $\mathbf{u}_0 = (u_1^0, u_2^0, \cdots, u_m^0)$ is easily inferred to be a fixed point of the operator A. Suppose the statement isn't true, that is to say, there is some k such that $u_k^0 > 0$. Then, Hypothesis (B2) implies all components

$u_1^0, \cdots, u_m^0$ are strictly positive. Let $u_k = \phi_k + u_k^0$, so that $\Phi = (\phi_1, \phi_2, \cdots, \phi_m) \in \mathbb{K}$ is non-trivial and satisfies the system of equations

$$\begin{cases} \phi_1(x) + u_1^0 = \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} a_{1j}^{\alpha} * \left(\phi_1 + u_1^0\right)^{\alpha_1} \cdots \left(\phi_m + u_m^0\right)^{\alpha_m}, \\ \qquad \vdots \\ \phi_m(x) + u_m^0 = \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} a_{mj}^{\alpha} * \left(\phi_1 + u_1^0\right)^{\alpha_1} \cdots \left(\phi_m + u_m^0\right)^{\alpha_m}. \end{cases}$$

Because $A\mathbf{u}_0 = \mathbf{u}_0$, there obtains the obvious estimate

$$\begin{cases} \phi_1(x) \geq \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} a_{1j}^{\alpha} * \left(\alpha_1 \mathbf{u}_0^{\alpha} \frac{\phi_1}{u_1^0} + \cdots + \alpha_k \mathbf{u}_0^{\alpha} \frac{\phi_k}{u_k^0} \cdots + \alpha_m \mathbf{u}_0^{\alpha} \frac{\phi_m}{u_m^0}\right), \\ \qquad \vdots \\ \phi_m(x) \geq \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} a_{mj}^{\alpha} * \left(\alpha_1 \mathbf{u}_0^{\alpha} \frac{\phi_1}{u_1^0} + \cdots + \alpha_k \mathbf{u}_0^{\alpha} \frac{\phi_k}{u_k^0} \cdots + \alpha_m \mathbf{u}_0^{\alpha} \frac{\phi_m}{u_m^0}\right). \end{cases} \tag{5.18}$$

Since $a_{kj}^{\alpha} \in L_1 \cap C(\mathbb{R})$ for all k, j and α, there is an $M > 0$ such that

$$\int_{-M}^{M} a_{kj}^{\alpha}(x)\, dx \geq \frac{3}{4} \int_{-\infty}^{\infty} a_{kj}^{\alpha}(x)\, dx = \frac{3}{4} B_{kj}^{\alpha}. \tag{5.19}$$

Since $u_k^0 > 0$ for $k = 1, \cdots, m$, by normalization of the kernels a_{kj}^{α} to

$$\widetilde{a_{kj}^{\alpha}} = a_{kj}^{\alpha} \frac{(\mathbf{u}_0)^{\alpha}}{u_k^0},$$

it may be supposed again without loss of generality that $u_1^0 = \cdots = u_m^0 = 1$ and that

$$\sum_{j=2}^{q} \sum_{|\alpha|=j} B_{kj}^{\alpha} = 1, \tag{5.20}$$

for $1 \leq k \leq m$. Integrating (5.18) with respect to x over the interval $(-3M, 3M)$ gives

$$\int_{-3M}^{3M} \phi_k(x)\, dx \geq$$
$$\sum_{j=2}^{q} \sum_{|\alpha|=j} \int_{-3M}^{3M} \int_{-4M}^{4M} a_{kj}^{\alpha}(x-y) \Big(\alpha_1 \phi_1(y) + \cdots + \alpha_m \phi_m(y)\Big)\, dy\, dx,$$

for $k = 1, 2 \cdots, m$, where the tilde over a_{kj}^{α} has been omitted. If $\bar{\phi}_k = \int_{-3M}^{3M} \phi_k(x)\, dx$ for $k = 1, 2, \cdots, m$, then the above inequalities imply

$$\begin{cases} \bar{\phi}_1 \geq \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} \frac{3}{4} B_{1j}^{\alpha} \Big(\alpha_1 \bar{\phi}_1 + \cdots + \alpha_m \bar{\phi}_m \Big), \\ \qquad \vdots \\ \bar{\phi}_m \geq \displaystyle\sum_{j=2}^{q} \sum_{|\alpha|=j} \frac{3}{4} B_{mj}^{\alpha} \Big(\alpha_1 \bar{\phi}_1 + \cdots + \alpha_m \bar{\phi}_m \Big). \end{cases}$$

Renumber the indices so that $\bar{\phi}_1 = \min \{ \bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_m \}$. Then the first inequality above becomes

$$\begin{aligned} \bar{\phi}_1 &\geq \sum_{j=2}^{q} \sum_{|\alpha|=j} \frac{3}{4} B_{1j}^{\alpha} \Big(\alpha_1 \bar{\phi}_1 + \cdots + \alpha_m \bar{\phi}_1 \Big) \\ &= \sum_{j=2}^{q} \sum_{|\alpha|=j} \frac{3}{4} B_{1j}^{\alpha} |\alpha| \bar{\phi}_1 \geq \frac{3}{2} \bar{\phi}_1, \end{aligned}$$

which contradicts the fact that $\bar{\phi}_1 > 0$. The theorem is proved. □

Theorem 5.7. *If* $\mathbf{u} = (u_1, \cdots, u_m)$ *is a non-trivial solution of (5.1), then for any* $p \geq 1$ *and* k *with* $1 \leq k \leq m$*, the components* u_k *of* $\mathbf{u}$ *lie in* L_p*.*

As in Proposition 3.9, it suffices to prove that for each k, $nu_k(n)$ is bounded for $n > 0$ large. In particular, it will then follow that $\mathbf{u}^{\alpha} \in L_1$ for $\alpha \in \mathbb{Z}_+^m$, $2 \leq |\alpha| \leq q$, whence $u_k = \sum_{j=2}^{q} \sum_{|\alpha|=j} a_{kj}^{\alpha} * \mathbf{u}^{\alpha} \in L_1$.

Proof. Fix $n > 0$ and integrate each equation in (5.1) over $(0, n)$. Rearranging the order of integration, there appears

$$\int_0^n u_1(x)\, dx = \sum_{j=2}^{q} \sum_{|\alpha|=j} \int_0^{\infty} \int_0^n \Big(a_{1j}^{\alpha}(x-y) + a_{1j}^{\alpha}(x+y) \Big)\, dx\, \mathbf{u}^{\alpha}(y)\, dy,$$

$$\vdots$$

$$\int_0^n u_m(x)\, dx = \sum_{j=2}^{q} \sum_{|\alpha|=j} \int_0^{\infty} \int_0^n \Big(a_{mj}^{\alpha}(x-y) + a_{mj}^{\alpha}(x+y) \Big)\, dx\, \mathbf{u}^{\alpha}(y)\, dy.$$

Summing these equations leads to

$$\int_0^n \Big(u_1(x) + \cdots + u_m(x) \Big)\, dx = \int_0^{\infty} \sum_{j=2}^{q} \sum_{|\alpha|=j} \gamma_{j,n}(y) \mathbf{u}^{\alpha}(y)\, dy, \tag{5.21}$$

where

$$\gamma_{j,n}(y) = \sum_{k=1}^{m} \int_0^n \Big(a_{kj}^{\alpha}(x-y) + a_{kj}^{\alpha}(x+y) \Big)\, dx.$$

Let $n > \mu$ with $\mu > 0$ to be determined later, and rewrite (5.21) in the form,

$$\Big(\int_0^\mu + \int_\mu^n\Big)\Big(u_1(x) + \cdots + u_m(x)\Big)\,dx$$
$$=\Big(\int_0^\mu + \int_\mu^n + \int_n^\infty\Big)\sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,n}(y)\mathbf{u}^\alpha(y)\,dy.$$

Define the quantity I_n by

$$I_n = \int_0^\mu \Big\{\sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,n}(y)\mathbf{u}^\alpha(y) - \Big(u_1(y) + \cdots + u_m(y)\Big)\Big\}\,dy,$$

and write it as

$$I_n = \int_\mu^n \Big\{\Big(u_1(y) + \cdots + u_m(y)\Big) - \sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,n}(y)\mathbf{u}^\alpha(y)\Big\}\,dy$$
$$- \int_n^\infty \Big\{\sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,n}(y)\mathbf{u}^\alpha(y)\Big\}\,dy.$$

From its definition, I_n is bounded above by

$$I_\infty = \int_0^\mu \Big\{\sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,\infty}\mathbf{u}^\alpha(y) - \Big(u_1(y) + \cdots + u_m(y)\Big)\Big\}\,dy,$$

where

$$\gamma_{j,\infty} = \sum_{k=1}^{m}\int_0^\infty \Big(a_{kj}^\alpha(x-y) + a_{kj}^\alpha(x+y)\Big)\,dx = \sum_{k=1}^{m} B_{kj}^\alpha.$$

On the other hand, since $\lim_{y\to\infty} u_k(y) = 0$, there is a $\mu > 0$ such that when $y \geq \mu$,

$$\sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,\infty}\mathbf{u}^\alpha(y) \leq \frac{1}{2}\Big(u_1(y) + \cdots + u_m(y)\Big).$$

Moreover, since all the components u_k are non-negative and non-increasing on $[0,\infty)$, it follows that

$$\int_n^\infty \Big\{\sum_{j=2}^{q}\sum_{|\alpha|=j}\gamma_{j,n}(y)\mathbf{u}^\alpha(y)\Big\}\,dy \leq \sum_{j=2}^{q}\sum_{|\alpha|=j}\mathbf{u}^\alpha(n)\int_n^\infty \gamma_{j,n}(y)\,dy$$
$$\leq n\sum_{j=2}^{q}\sum_{|\alpha|=j}\mathbf{u}^\alpha(n)\gamma_{j,\infty}$$
$$\leq n\tilde{\gamma}\sum_{j=2}^{q}\Big(u_1(n) + \cdots + u_m(n)\Big)^j,$$

where $\tilde{\gamma} = \max_{1\le j\le m}\{\gamma_{j,\infty}\}$. Thus, for a fixed μ and any $n > \mu$,

$$\begin{aligned}I_\infty \ge I_n &\ge \frac{1}{2}\int_\mu^n \Big(u_1(y) + \cdots + u_m(y)\Big)\,dy - n\tilde{\gamma}\sum_{j=2}^{q}\Big(u_1(n) + \cdots + u_m(n)\Big)^j \\ &\ge \frac{1}{2}(n-\mu)\Big(u_1(n) + \cdots + u_m(n)\Big) - n\tilde{\gamma}\sum_{j=2}^{q}\Big(u_1(n) + \cdots + u_m(n)\Big)^j \\ &= n\Big(u_1(n) + \cdots + u_m(n)\Big)\Big\{\frac{1}{2}\Big(1 - \frac{\mu}{n}\Big) - \tilde{\gamma}\sum_{j=1}^{q-1}\Big(u_1(n) + \cdots + u_m(n)\Big)^j\Big\}.\end{aligned}$$

Choose n large enough that

$$\frac{1}{2}\Big(1 - \frac{\mu}{n}\Big) - \tilde{\gamma}\sum_{j=1}^{q-1}\Big(u_1(n) + \cdots + u_m(n)\Big)^j \le \frac{1}{4}.$$

For such values of n, one has

$$I_\infty \ge \frac{1}{4}n\Big(u_1(n) + \cdots + u_m(n)\Big),$$

and the theorem is thereby proved. □

Theorem 5.8. *Let $\mathbf{u} = (u_1, \cdots, u_m)$ be a non-trivial solution of (5.1) in the space $(L_1 \cap L_\infty)^m$. For $1 \le k \le m$ and $\alpha \in \mathbb{Z}_+^m$ with $2 \le |\alpha| \le q$, if all the integral kernels $a_{kj}^\alpha \in L_1 \cap L_\infty$ are even functions, non-negative and non-increasing on $[0, \infty)$, then $\mathbf{u} \in (H^\infty)^m$.*

Proof. Since $\mathbf{u} \in (L_1 \cap L_\infty)^m$, $\mathbf{u}^\alpha \in L_2$ for any $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| \ge 1$. On the other hand, the proof of Theorem 3.10 shows that the quantities $|\xi|\widehat{a_{kj}^\alpha}(\xi)$ are bounded, so $a_{kj}^\alpha * \mathbf{u}^\alpha \in H^1$, whence $\mathbf{u} \in (H^1)^m$. Inductively, it is adduced that $\mathbf{u} \in (H^\infty)^m$. The proof is complete.

□

6. Appendix.

Lemma 6.1. *Let $r, s \in (0,1)$ and suppose $f \in H^r$ and $g \in H^s$. If there are two positive numbers $R_2 > R_1 > 0$ such that*

$$\operatorname{supp} f \subset (-R_1, R_1) \quad \text{and} \quad \operatorname{supp} g \subset (-\infty, -R_2) \cup (R_2, \infty), \tag{6.1}$$

then the inequality

$$\Big|\int_{-\infty}^{\infty} |\xi|^{r+s}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi\Big| \le \frac{2\|f\|^2 + \|g\|^2}{c_{r,s}\,(s+r)(R_2 - R_1)^{(s+r)}} \tag{6.2}$$

is valid, where $c_{r,s} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|e^{ix}-1|^2}{|x|^{1+r+s}}\,dx$.

Proof. Note that

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[f(x) - f(y)]\,[g(x) - g(y)]}{|x-y|^{1+r+s}}\,dy\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[f(z+y) - f(y)]\,[g(z+y) - g(y)]}{|z|^{1+r+s}}\,dy\,dz \\ &= \int_{-\infty}^{\infty}\frac{1}{|z|^{1+r+s}}\,dz\int_{-\infty}^{\infty}[f(z+y) - f(y)]\,[g(z+y) - g(y)]\,dy,\end{aligned}$$

and, by Plancherel's formula,

$$\begin{aligned}&\int_{-\infty}^{\infty}[f(z+y)-f(y)]\,[g(z+y)-g(y)]\,dy\\ =&\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)[e^{i\xi z}-1]\overline{\widehat{g}(\xi)}[e^{-i\xi z}-1]\,d\xi\\ =&\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}|e^{i\xi z}-1|^2\,d\xi.\end{aligned}$$

Parseval's formula $\int_{-\infty}^{\infty}f(x)g(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi$ implies that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[f(x)-f(y)]\,[g(x)-g(y)]}{|x-y|^{1+r+s}}\,dy\,dx=c_{r,s}\int_{-\infty}^{\infty}|\xi|^{r+s}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi \tag{6.3}$$

where $c_{r,s}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|e^{ix}-1|^2}{|x|^{1+r+s}}\,dx$. Because of (6.1), the left-hand side of (6.3) is equal to

$$\begin{aligned}&\left\{\int_{\operatorname{supp}f}\int_{\operatorname{supp}g}+\int_{\operatorname{supp}g}\int_{\operatorname{supp}f}\right\}\frac{[f(x)-f(y)]\,[g(x)-g(y)]}{|x-y|^{1+r+s}}\,dy\,dx\\ =&-2\int_{-\infty}^{-R_2}\int_{-R_1}^{R_1}\frac{f(y)g(x)}{|x-y|^{1+r+s}}\,dy\,dx-2\int_{R_2}^{\infty}\int_{-R_1}^{R_1}\frac{f(y)g(x)}{|x-y|^{1+r+s}}\,dy\,dx,\end{aligned}$$

whence

$$\begin{aligned}\left|c_{r,s}\int_{-\infty}^{\infty}|\xi|^{r+s}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\,d\xi\right|&\le\left\{\int_{-\infty}^{-R_2}+\int_{R_2}^{\infty}\right\}\int_{-R_1}^{R_1}\frac{f^2(y)+g^2(x)}{|x-y|^{1+r+s}}\,dy\,dx\\ &\le\frac{2\|f\|^2+\|g\|^2}{(s+r)(R_2-R_1)^{s+r}}.\end{aligned}$$

The lemma is proved. □

Let $\zeta\in\mathbb{C}_b^\infty$ be the function defined in (2.6).

Lemma 6.2. *Let m be a non-negative integer, $s\in(0,1)$ and $\{f_n\}_{n=1}^\infty$ a bounded sequence in H^{m+s}. Define $\{\rho_n\}_{n=1}^\infty$ by*

$$\begin{aligned}\rho_n(x)=\rho(f_n)(x)=f_n^2(x)&+\min\{1,m\}|D_x^m f_n(x)|^2\\ &+\int_{-\infty}^{\infty}\frac{|D_x^m f_n(x)-D_y^m f_n(y)|^2}{|x-y|^{1+2s}}\,dy.\end{aligned}$$

Suppose that for any $\epsilon\in(0,1)$, there is an $E_0>0$ and a sequence $\{R_n\}_{n=1}^\infty\subset\mathbb{R}$ with $R_n>E_0$ for all n and $\lim_{n\to\infty}R_n=\infty$ such that for large n

$$\int_{E_0}^{2R_n}\rho_n(x)\,dx+\int_{-2R_n}^{-E_0}\rho_n(x)\,dx\le\epsilon.$$

If $E_1>E_0$ is chosen so large that $(E_1-E_0)^{-2s}\le\epsilon$ and n is large enough that $R_n\ge 2E_1$, then there exists a constant C_0 independent of n for which

$$\|\eta_n f_n\|_{m+s}^2\le C_0\epsilon\quad\text{and}\quad\|\zeta_n f_n\|_{m+s}^2\le C_0\epsilon,$$

where

$$\eta_n(x)=\begin{cases}-\zeta(\frac{x}{E_1})+\zeta(\frac{x}{R_n}) & \text{if } E_1\le x\le 2R_n,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\zeta_n(x)=\begin{cases}-\zeta(\frac{x}{E_1})+\zeta(\frac{x}{R_n}) & \text{if } -2R_n\le x\le -E_1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Since the H^{m+s}-norm of a function $h : \mathbb{R} \to \mathbb{R}$ is equivalent to

$$\left\{ \|h\|^2 + \min\{1, m\} \|D_x^m h\|^2 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|D_x^m h(x) - D_y^m h(y)|^2}{|x-y|^{1+2s}} \, dy \, dx \right\}^{\frac{1}{2}},$$

the L_1-norm of $\rho(f)$ is equivalent to $\|f\|_s^2$. Hence, for $f \in H^{m+s}$, there are two constants $\underline{\gamma} > 0$ and $\overline{\gamma} > 0$ such that

$$\underline{\gamma} \|f\|_{m+s}^2 \le |\rho(f)|_1 \le \overline{\gamma} \|f\|_{m+s}^2. \tag{6.4}$$

In particular, the sequence $\{\rho_n = \rho(f_n)\}_n^\infty \subset L_1$ is bounded.

In the simplest case $m = 0$, the equivalence (6.4) implies that

$$\begin{aligned}
\underline{\gamma} \|\eta_n f_n\|_s^2 &\le \int_{-\infty}^{\infty} \rho(\eta_n f_n) \, dx \\
&= \|\eta_n f_n\|^2 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\eta_n(x) f_n(x) - \eta_n(y) f_n(y)|^2}{|x-y|^{1+2s}} \, dy \, dx \\
&\le \|\eta_n f_n\|^2 \\
&\quad + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\eta_n(x)|^2 |f_n(x) - f_n(y)|^2 + |\eta_n(x) - \eta_n(y)|^2 f_n^2(y)}{|x-y|^{1+2s}} \, dy \, dx \\
&\le \|\eta_n f_n\|^2 + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|\eta_n^2(x)| |f_n(x) - f_n(y)|^2}{|x-y|^{1+2s}} \, dy \, dx \\
&\quad + 2 \int_{-\infty}^{\infty} f_n^2(x) \int_{-\infty}^{\infty} \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}} \, dy \, dx \\
&\le 2 \int_{E_1}^{2R_n} \rho_n(x) \, dx + 2 \int_{E_0}^{2R_n} f_n^2(x) \int_{-\infty}^{\infty} \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}} \, dy \, dx \\
&\quad + 2 \int_{-\infty}^{E_0} f_n^2(x) \int_{-\infty}^{\infty} \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}} \, dy \, dx \\
&\quad + 2 \int_{2R_n}^{\infty} f_n^2(x) \int_{-\infty}^{\infty} \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}} \, dy \, dx.
\end{aligned}$$

The first term on the right-hand side of the last inequality is bounded by 2ϵ. Regarding the second term, for any $E_0 \le x \le 2R_n$, the fraction $\frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}}$ is bounded by

$$\frac{4 \max_{x \in \mathbb{R}} \{|\zeta'(x)|^2\}}{E_1^2} |x-y|^{1-2s} \qquad \text{as } |x-y| \to 0$$

and by

$$4|x-y|^{-(1+2s)} \qquad \text{as } |x-y| \to \infty,$$

so $\int \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}} \, dy$ is bounded, and the bound is independent of n, call it C_1, say. The second term is thus bounded by

$$2C_1 \int_{E_0}^{2R_n} f_n^2(x) \, dx \le 2C_1 \int_{E_0}^{2R_n} \rho_n^2(x) \, dx \le 2C_1 \epsilon.$$

In the third term, $x \in (-\infty, E_0)$, so $\eta_n(x) = 0$ by its definition. It follows that

$$\int_{-\infty}^{\infty} \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}} \, dy = \int_{E_1}^{2R_n} \frac{|\eta_n(y)|^2}{|x-y|^{1+2s}} \, dy \le \frac{1}{2s(E_1 - E_0)^{2s}} \le \frac{\epsilon}{2s}.$$

With respect to the fourth term, $x \in (2R_n, \infty)$, so $\eta_n(x) = 0$. Hence when n is large enough that $R_n > 2E_1$, we have the estimate

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|\eta_n(x) - \eta_n(y)|^2}{|x-y|^{1+2s}}\, dy = \int_{E_1}^{2R_n} \frac{|\eta_n(y)|^2}{|x-y|^{1+2s}}\, dy \\
&\le \int_{E_1}^{R_n} \frac{1}{|x-y|^{1+2s}}\, dy + \int_{R_n}^{2R_n} \frac{|\zeta(\frac{y}{E_1}) - \zeta(\frac{y}{R_n})|^2}{|x-y|^{1+2s}}\, dy \\
&\le \frac{1}{2sR_n^{2s}} + \int_{R_n}^{2R_n} \frac{|\zeta(\frac{y}{R_n})|^2}{|x-y|^{1+2s}}\, dy \\
&\le \frac{1}{2sR_n^{2s}} + \int_1^2 \frac{|\zeta(y)|^2}{R_n^{2s}\left(\frac{x}{R_n} - y\right)^{1+2s}}\, dy \\
&\le \frac{1}{2sR_n^{2s}} + \frac{\max_{x\in\mathbb{R}} |\zeta'(x)|^2}{(2-2s)R_n^{2s}} \\
&\le \left(\frac{1}{2s} + \frac{\max_{x\in\mathbb{R}} |\zeta'(x)|^2}{(2-2s)}\right)\epsilon.
\end{aligned}$$

Combining these four inequalities gives

$$\underline{\gamma}\big\|\eta_n f_n\big\|_s^2 \le 2\epsilon + 2C_1\epsilon + \frac{\epsilon}{s}\big\|f_n\big\|^2 + \left(\frac{1}{s} + \frac{\max_{x\in\mathbb{R}} |\zeta'(x)|^2}{(1-s)}\right)\big\|f_n\big\|^2\epsilon$$

for n sufficiently large. It may be deduced in a similar manner that

$$\underline{\gamma}\big\|\zeta_n f_n\big\|_s^2 \le 2\epsilon + 2C_1\epsilon + \frac{\epsilon}{s}\big\|f_n\big\|^2 + \left(\frac{1}{s} + \frac{\max_{x\in\mathbb{R}} |\zeta'(x)|^2}{(1-s)}\right)\big\|f_n\big\|^2\epsilon$$

for n sufficiently large. The overall result is thus seen to be valid in case $m = 0$ because $\|f_n\|_s$ is bounded.

In case $m > 0$, (6.4) implies that

$$\begin{aligned}
\underline{\gamma}\big\|\eta_n f_n\big\|_{m+s}^2 \le \big\|\eta_n f_n\big\|^2 + \big\|D_x^m(\eta_n f_n)\big\|^2 \\
+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|D_x^m(\eta_n f_n)(x) - D_y^m(\eta_n f_n)(y)|^2}{|x-y|^{1+2s}}\, dy\, dx.
\end{aligned}$$

Note that

$$D_x^m(\eta_n f_n) = \eta_n D_x^m f_n + \sum_{j=1}^m \binom{m}{j} D_x^j \eta_n D_x^{m-j} f_n,$$

where the notation $D_x^0 f$ represents f. In consequence of the last two formulas, it appears that

$$\begin{aligned}
\underline{\gamma}\|\eta_n f_n\|_{m+s}^2 &\le \|\eta_n f_n\|^2 + 2\|\eta_n D_x^m f_n\|^2 + 2\Big\|\sum_{j=1}^{m}\binom{m}{j} D_x^j \eta_n d_x^{m-j} f_n\Big\|^2 \\
&+ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\sum_{j=1}^{m}\binom{m}{j} D_x^j\eta_n(x) D_x^{m-j} f_n(x) - \sum_{j=1}^{m}\binom{m}{j} D_y^j \eta_n(y) D_y^{m-j} f(y)|^2}{|x-y|^{1+2s}}\, dy\, dx \\
&+ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\eta_n(x) D_x^m f^n(x) - \eta_n(y) D_y^m f_n(y)|^2}{|x-y|^{1+2s}}\, dy\, dx \\
&\le \|\eta_n f_n\|^2 + 2\overline{\gamma}\Big\|\sum_{j=1}^{m}\binom{m}{j} D_x^j \eta_n D_x^{m-j} f_n\Big\|_s^2 + 2\overline{\gamma}\|\eta_n D_x^m f_n\|_s^2 \\
&\le \|\eta_n f_n\|^2 + 2\overline{\gamma}\Big\|\sum_{j=1}^{m}\binom{m}{j} D_x^j \eta_n D_x^{m-j} f_n\Big\|_1^2 + 2\overline{\gamma}\|\eta_n D_x^m f_n\|_s^2.
\end{aligned}$$

Using the result for the case $m = 0$, we derive that

$$\|\eta_n f_n\|^2 + \|\eta_n D_x^m f_n\|_s^2 \le C_0\epsilon,$$

where C_0 is a constant independent of n. With regard to the second term, denote by ζ_b the quantity $\max\{|D_x^j\zeta(x)| : x \in \mathbb{R}, 1 \le j \le m\}$ and then note that

$$\Big\|\sum_{j=1}^{m}\binom{m}{j} D_x^j \eta_n D_x^{m-j} f_n\Big\|_1^2 \le 2^m \sum_{j=1}^{m}\binom{m}{j}\Big(\frac{4\zeta_b^2}{E_1^{2j}} + \frac{4\zeta_b^2}{E_1^{2(j+1)}}\Big)\|f_n\|_m^2.$$

Since $\|f_n\|_m$ is bounded and $E_1^{-2j} \le (E_1 - E_0)^{-2s} \le \epsilon$ for $j \ge 1$, it transpires that $\|\eta_n f_n\|_{m+s}^2$ is bounded by ϵ times a constant which is independent of n for large values of n. The same is true of $\|\zeta_n f_n\|_{m+s}^2$, a fact that may be established by the same line of argument. □

For the readers' convenience, the following lemma is quoted from Gantmacher (1960) Chapter XIII.

Lemma 6.3. *Let $A = (a_{ij})_{n\times n}$ be a square matrix with non-negative components. If A is irreducible, then A has a positive eigenvalue r, called the dominant eigenvalue, that is a simple root of the characteristic equation. The absolute value of any other eigenvalue of A does not exceed r. Furthermore, to the dominant eigenvalue r, there corresponds an eigenvector all of whose components are strictly positive.*

Corollary 6.4. *In Lemma 6.3, suppose $\mathbf{X} = (x_1, \cdots, x_n)$ is an eigenvector corresponding to the dominant eigenvalue r with $x_i > 0$ for $1 \le i \le n$. Then*

$$\min\Big\{\sum_{i=1}^{n} a_{ij} : 1 \le j \le n\Big\} \le r \le \max\Big\{\sum_{i=1}^{n} a_{ij} : 1 \le j \le n\Big\}. \tag{6.5}$$

Moreover, for ϵ in the interval $(0, \frac{1}{2}r)$ small enough, there are n linearly independent vectors $\{\mathbf{X}^j\}_{j=1}^n$ with strictly positive components such that

$$A\mathbf{X}^j > (r-\epsilon)\mathbf{X}^j$$

for $1 \le j \le n$.

Proof. Since $\mathbf{X}$ is an eigenvector of A corresponding to the eigenvalue r,

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = rx_1, \\ \quad\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = rx_n. \end{cases} \tag{6.6}$$

Summing these n equations yields

$$x_1 \sum_{i=1}^{n} a_{i1} + \cdots + x_n \sum_{i=1}^{n} a_{in} = r \sum_{i=1}^{n} x_i. \tag{6.7}$$

Since $x_i > 0$ for all i, (6.5) follows by dividing both sides of (6.7) by $\sum_{i=1}^{n} x_i$.

To prove the second part of the corollary, notice that since A is irreducible and non-negative, $\sum_{j \neq i} a_{ij} > 0$ for any fixed i in the range of $[1, n]$. The equations (6.6) and the strict positivity of all the components of $\mathbf{X}$ imply

$$a_{ii}x_i < rx_i$$

for $1 \leq i \leq n$, and hence that

$$r > a_{ii}$$

for all i. If ϵ is chosen so that $0 < \epsilon \leq \frac{1}{2}\big(r - \max\{a_{ii} : 1 \leq i \leq n\}\big)$, then for any i and any δ such that $0 < \delta x_i < \frac{\epsilon x_i}{r - a_{ii} - \epsilon}$, it is easy to verify that

$$\begin{cases} a_{11}x_1 + \cdots + a_{ii}(x_i + \delta x_i) + \cdots + a_{in}x_n > (r - \epsilon)x_1, \\ \quad\vdots \\ a_{i1}x_1 + \cdots + a_{ii}(x_i + \delta x_i) + \cdots + a_{in}x_n > (r - \epsilon)(x_i + \delta x_i), \\ \quad\vdots \\ a_{n1}x_1 + \cdots + a_{ii}(x_i + \delta x_i) + \cdots + a_{in}x_n > (r - \epsilon)x_n. \end{cases}$$

For $1 \leq i \leq n$, let $\mathbf{X}^{\mathbf{i}} = (x_1, \cdots, x_{i-1}, x_i + \delta x_i, x_{i+1}, \cdots, x_n)$. The last inequalities are the same as

$$A\mathbf{X}^{i} > (r - \epsilon)\mathbf{X}^{i}.$$

The linear independence of $\{\mathbf{X}^{i}\}_{i=1}^{n}$ follows directly, and the corollary is proved.

□

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