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Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation

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Abstract

The generalized Korteweg–de Vries equation has the property that solutions with initial data that are analytic in a strip in the complex plane continue to be analytic in a strip as time progresses. Established here are algebraic lower bounds on the possible rate of decrease in time of the uniform radius of spatial analyticity for these equations. Previously known results featured exponentially decreasing bounds.

Résumé

Si la donnée initiale est analytique sur une bande dans le plan complexe, alors la solution de l'équation de Korteweg et de Vries généralisée le reste pour tout temps. Nous montrons que la largeur de cette bande décroit algébriquement en temps. Les résultats antérieurs ne donnaient qu'un taux de décroissance exponentiel.

1. Introduction

This paper deals with the initial-value problem for the generalized Korteweg–de Vries (gKdV) equation

$$u_t + u_{xxx} + u^p u_x = 0, \tag{1}$$

where $p \geqslant 1$ is a positive integer, and u is a function of the two real variables x and t. Eq. (1) with $p = 1$ or $p = 2$ arises in modeling wave phenomena in a variety of physical situations. For larger values of p, (1) has come to the

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fore in our collective efforts to understand fully the interaction between nonlinearity and dispersion in evolution equations.

In applications of (1) to physical problems, the dependent variable u is usually real-valued. However, for several reasons, complex-valued solutions have attracted interest lately. The present paper aims to add to the latter discussion. Interest will be focused upon solutions $u(x,t)$ of (1) which, while real-valued for real values x and t, admit an extension as an analytic function to a complex strip $S_\sigma = \{x + \mathrm{i}y\colon\ |y| < \sigma\}$, at least for small values of σ. In consequence, initial data $u_0(x) = u(x,0)$ will be drawn from a suitable class of analytic functions. It should be noted that there are situations where analytic solutions emanate from non-analytic initial data (see e.g. [7,16]). For example, it is proved in [16] that for the KdV equation itself, the case $p = 1$ in (1), a certain class of initial data with a single point singularity yields analytic solutions. However, these results do not produce explicit estimates on a radius σ of spatial analyticity of solutions. If, on the other hand, the initial datum u_0 is analytic in a symmetric strip around the real axis, it has recently been established that the solution will retain analyticity in the same strip at least for a small time [10] (see also [11,12]). The present work is focused on studying the asymptotics of the width σ of the strip of analyticity for large t, assuming that a certain Sobolev norm of the solution remains finite. The first result in this direction was proved by Kato and Masuda in [15] where the rate of decrease of σ in time was shown to be at most *super-exponential*. More recently, an *exponential* bound on the width of the strip was presented in [3] via a Gevrey-class technique. Our intuition, partly based on the existence of algebraic bounds on the rate of increase in time of Sobolev norms for (1), as shown by Staffilani [20], suggested that an *algebraic* lower bound for the width σ of the strip may hold. The goal of this paper is to provide an affirmative answer to this conjecture. The main ingredient in the proof is a new multilinear estimate in Bourgain–Gevrey spaces. This estimate effectively introduces a power of σ as a prefactor in the nonlinear term, and this induces the algebraic decrease of σ over time.

The appropriate notation and function spaces are introduced in the next section, while Section 3 contains some auxiliary linear estimates. Multilinear estimates are proved in Section 4, and the proof of the main theorem is given in Section 5.

2. Function space setting

The Fourier transform of a function v_0 belonging to the Schwartz class is defined by

$$\hat{v}_0(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v_0(x)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x.$$

For a function $v(x,t)$ of two variables, the spatial Fourier transform is denoted by

$$\mathcal{F}_x v(\xi,t)x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x,t)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x,$$

whereas the notation $\hat{v}(\xi,\tau)$ designates the space–time Fourier transform

$$\hat{v}(\xi,\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t)\, \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{e}^{-\mathrm{i}t\tau}\, \mathrm{d}x\, \mathrm{d}t.$$

Define Fourier multiplier operators A and Λ by

$$\widehat{Av}(\xi,\tau) = \big(1 + |\xi|\big)\hat{v}(\xi,\tau)$$

and

$$\widehat{\Lambda v}(\xi,\tau) = \big(1 + |\tau|\big)\hat{v}(\xi,\tau).$$

The following notation is used to signify the L^p-L^q space–time norms;

$$\|v\|_{L_pL_q} = \left\{ \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} |v(x,t)|^q \, dt \right|^{p/q} dx \right\}^{1/p}.$$

A class of analytic functions suitable for our analysis is the analytic Gevrey class $G_{\sigma,s}$, introduced by Foias and Temam [8], which may be defined as the domain of the operator $A^s e^{\sigma A}$ in L_2. The Gevrey norm is defined to be

$$\|v_0\|^2_{G_{\sigma,s}} = \int_{-\infty}^{\infty} (1+|\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\widehat{v_0}(\xi)|^2 \, d\xi.$$

It is straightforward to check that a function in $G_{\sigma,s}$ is the restriction to the real axis of a function analytic on a symmetric strip of width 2σ. The strip $\{z = x + iy\colon |y| < \sigma\}$ will be denoted by S_σ.

To efficiently exploit the dispersive effects inherent in (1), we consider a space that is a hybrid between the analytic Gevrey space and a space of the Bourgain-type. More precisely, for $\sigma > 0$, $s \in \mathbb{R}$, and $b \in [-1, 1]$ define $X_{\sigma,s,b}$ to be the Banach space equipped with the norm

$$\|v\|^2_{\sigma,s,b} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+|\tau - \xi^3|)^{2b} (1+|\xi|)^{2s} e^{2\sigma(1+|\xi|)} |\hat{v}(\xi,\tau)|^2 \, d\xi \, d\tau.$$

For $\sigma = 0$, $X_{\sigma,s,b}$ coincides with the space $X_{s,b}$ introduced by Bourgain, and Kenig, Ponce and Vega. The norm of $X_{s,b}$ is denoted by $\|\cdot\|_{s,b}$ and is defined by the integral

$$\|v\|^2_{s,b} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+|\tau - \xi^3|)^{2b} (1+|\xi|)^{2s} |\hat{v}(\xi,\tau)|^2 \, d\xi \, d\tau.$$

It follows directly from the Sobolev embedding theorem that the inequality

$$\sup_{t\in[0,T]} \|v(\cdot,t)\|_{G_{\sigma,s}} \leqslant c\|v\|_{\sigma,s,b} \tag{2}$$

holds for $b > \frac{1}{2}$. In addition, if $v_0 \in G_\sigma$ and ϵ is such that $0 < \epsilon < \sigma$, then v_0 and all of its derivatives are bounded on the smaller strip $S_{\sigma-\epsilon}$.

Proposition 1. *Let $0 < \epsilon < \sigma$ and $n \in \mathbb{N}$ be given. Then there exists a constant c depending on ϵ and n, such that*

$$\sup_{x+iy\in S_{\sigma-\epsilon}} |\partial_x^n f(x+iy)| \leqslant c\|f\|_{G_\sigma}.$$

Proof. This is a direct consequence of the inequality

$$\|f\|_{G_{\sigma-\epsilon,n+1}} \leqslant c_{n,\epsilon}\|f\|_{G_\sigma} \tag{3}$$

which holds for $n \in \mathbb{N}$, and the Sobolev embedding theorem. The inequality (3) follows from the relation

$$\sup_{\xi\in\mathbb{R}} \{e^{-\epsilon(1+|\xi|)}(1+|\xi|)^{n+1}\} = c_{n,\epsilon},$$

where $c_{n,\epsilon} = ((n+1)/e)^{n+1}(1/\epsilon^{n+1})$. Note that $c_{n,\epsilon} \to \infty$ as $\epsilon \to 0$, as one would expect. □

The space $X_{\sigma,s,b}$ was introduced by two of the authors in [10], where it was useful to obtain local-in-time well-posedness of (1) in $G_{\sigma,s}$ for an appropriate range of parameters s and b. Here, interest is focused on the global behavior of solutions in $X_{\sigma,s,b}$, where σ will be allowed to vary in time.

3. Linear estimates

Since the analysis is based on boundedness in $X_{\sigma,s,b}$ of an integral operator given by a variation-of-constants formula, certain estimates of the solutions of the corresponding linear problem are needed. These estimates are addressed now. Denote by $\{W(t)\}_{t=-\infty}^{\infty}$ the solution group associated with the homogeneous linear problem

$$\left.\begin{aligned} &w_t + w_{xxx} = 0, \\ &w(x,0) = w_0(x). \end{aligned}\right\} \tag{4}$$

Let ψ be an infinitely differentiable cut-off function such that $0 \leqslant \psi \leqslant 1$ everywhere and

$$\psi(t) = \begin{cases} 0, & |t| \geqslant 2, \\ 1, & |t| \leqslant 1, \end{cases}$$

and, for $T > 0$, let $\psi_T(t) = \psi(t/T)$.

Lemma 1. *Let $\sigma \geqslant 0$, $b > \frac{1}{2}$, $b - 1 < b' < 0$, and $T \geqslant 1$. Then there is a constant c such that*

$$\left\|\psi_T(t) W(t) u_0(x)\right\|_{\sigma,s,b} \leqslant c T^{1/2} \|u_0\|_{G_{\sigma,s}}, \tag{5}$$

$$\left\|\psi_T(t) u(x,t)\right\|_{\sigma,s,b} \leqslant c \|u\|_{\sigma,s,b} \tag{6}$$

and

$$\left\|\psi_T(t) \int_0^t W(t-s) v(s)\, \mathrm{d}s\right\|_{\sigma,s,b} \leqslant c T \|v\|_{\sigma,s,b'}. \tag{7}$$

Proof. The proof of (5) is immediate from the definition of $X_{\sigma,s,b}$, the linearity of the operator $\mathrm{e}^{\sigma A}$ and Lemma 3.1 in [18]. In the same way, (6) follows from Lemma 3.2 in [18]. For the proof of (7), one follows the proof of Lemma 2.1 in [9] step by step, keeping in mind that $T \geqslant 1$. The actual bound that emerges from these ruminations is $c \max\{T, T^{1-b+b'}\}$, but since $1 - b + b' < \frac{1}{2}$ and $T \geqslant 1$, the first term is dominant. □

The second kind of linear estimates needed are Kato-type smoothing inequalities and maximal function-type inequalities. For a suitable function f, define F_ρ via its Fourier transform $\widehat{F}_\rho$, *viz.*

$$\widehat{F}_\rho(\xi,\tau) = \frac{f(\xi,\tau)}{(1 + |\tau - \xi^3|)^\rho}. \tag{8}$$

Lemma 2 (Bourgain). *Let $\rho > \frac{1}{4}$ be given. Then there is a constant c, depending on ρ, such that*

$$\|A^{1/2} F_\rho\|_{L_4 L_2} \leqslant c \|f\|_{L_2 L_2}. \tag{9}$$

For the proof of this lemma, the reader is referred to [6].

Lemma 3 (Kenig–Ponce–Vega). *Let s and ρ be given. There is a constant c, depending on s and ρ, such that*

(i) *If $\rho > \frac{1}{2}$, then*

$$\|A F_\rho\|_{L_\infty L_2} \leqslant c \|f\|_{L_2 L_2}; \tag{10}$$

(ii) *If $\rho > \frac{1}{2}$ and $s > 3\rho$, then*

$$\left\|A^{-s} F_\rho\right\|_{L_2 L_\infty} \leqslant c \|f\|_{L_2 L_2}; \tag{11}$$

(iii) *If* $\rho > \frac{1}{2}$ *and* $s > \frac{1}{4}$, *then*

$$\left\|A^{-s}F_\rho\right\|_{L_4L_\infty} \leqslant c\|f\|_{L_2L_2}; \tag{12}$$

(iv) *If* $\rho > \frac{1}{2}$ *and* $s > \frac{1}{2}$, *then*

$$\left\|A^{-s}F_\rho\right\|_{L_\infty L_\infty} \leqslant c\|f\|_{L_2L_2}. \tag{13}$$

The inequality (10) was proved in [18]. The estimates (11) and (13) were proved in [10], and (12) can be proved analogously using an estimate appearing in [17].

4. Multilinear estimates in Bourgain–Gevrey spaces

The goal of this section is to prove some multilinear estimates in analytic Bourgain–Gevrey spaces which feature explicit dependence on the radius of spatial analyticity σ. These inequalities will play a key role in obtaining the algebraically decreasing time-asymptotics for σ.

Theorem 1. *Let* $\sigma > 0, s > \frac{3}{2}, b > \frac{1}{2}, b' < -\frac{1}{4}$ *and* $p \geqslant 2$. *Then there exists a constant* $c > 0$ *depending only on* s, b, *and* b' *such that*

$$\left\|\partial_x(u_1 \dots u_{p+1})\right\|_{\sigma,s,b'} \leqslant c\|u_1\|_{s,b}\cdots\|u_{p+1}\|_{s,b} + c\sigma^{1/2}\|u_1\|_{\sigma,s,b}\cdots\|u_{p+1}\|_{\sigma,s,b}. \tag{14}$$

Proof. We present the proof in the case $p = 2$ and comment on the case $p > 2$. First note that (14) can be written more explicitly as

$$\begin{aligned}&\left\|\left(1+|\tau-\xi^3|\right)^{b'}\left(1+|\xi|\right)^{s}\mathrm{e}^{\sigma(1+|\xi|)}|\xi|\widehat{u_1u_2u_3}(\xi,\tau)\right\|_{L^2_\xi L^2_\tau}\\ &\quad\leqslant c\|u_1\|_{s,b}\|u_2\|_{s,b}\|u_3\|_{s,b} + c\sigma^{1/2}\|u_1\|_{\sigma,s,b}\|u_2\|_{\sigma,s,b}.\|u_3\|_{\sigma,s,b}.\end{aligned}$$

Define v_i, $i = 1, 2, 3$, by

$$v_i(\xi,\tau) = \left(1+|\xi|\right)^{s}\left(1+|\tau-\xi^3|\right)^{b}\mathrm{e}^{\sigma(1+|\xi|)}\hat{u}_i(\xi,\tau).$$

Then, proving the inequality (14) is equivalent to establishing the estimate

$$\begin{aligned}&\left\|\frac{(1+|\xi|)^s|\xi|\,\mathrm{e}^{\sigma(1+|\xi|)}}{(1+|\tau-\xi^3|)^{-b'}}\int_{\mathbb{R}^4}\frac{v_1(\xi_1,\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1-\xi_1^3|)^{b}}\frac{v_2(\xi-\xi_2,\tau-\tau_2)\,\mathrm{e}^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^3|)^{b}}\right.\\ &\qquad\left.\times\frac{v_3(\xi_2-\xi_1,\tau_2-\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^3|)^{b}}\,\mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi_2\,\mathrm{d}\tau_2 g\right\|_{L^2_\xi L^2_\tau}\\ &\quad\leqslant c\left\|\mathrm{e}^{-\sigma(1+|\xi|)}v_1\right\|_{L^2_\xi L^2_\tau}\left\|\mathrm{e}^{-\sigma(1+|\xi|)}v_2\right\|_{L^2_\xi L^2_\tau}\left\|\mathrm{e}^{-\sigma(1+|\xi|)}v_3\right\|_{L^2_\xi L^2_\tau} + c\sigma\|v_1\|_{L^2_\xi L^2_\tau}\|v_2\|_{L^2_\xi L^2_\tau}\|v_3\|_{L^2_\xi L^2_\tau}.\end{aligned}$$

Using duality, it suffices to estimate a 6-fold integral of the form

$$\begin{aligned}&\int_{\mathbb{R}^6}\frac{h(\xi,\tau)(1+|\xi|)^{1+s}\,\mathrm{e}^{\sigma(1+|\xi|)}}{(1+|\tau-\xi^3|)^{-b'}}\frac{v_1(\xi_1,\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1-\xi_1^3|)^{b}}\\ &\quad\times\frac{v_2(\xi-\xi_2,\tau-\tau_2)\,\mathrm{e}^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^3|)^{b}}\\ &\quad\times\frac{v_3(\xi_2-\xi_1,\tau_2-\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^3|)^{b}}\,\mathrm{d}\mu\end{aligned}$$

where h is an arbitrary element of the unit ball B in $L^2(\mathbb{R}^2)$ and $d\mu = d\xi_2\, d\tau_2\, d\xi_1\, d\tau_1\, d\xi\, d\tau$. Using the simple inequality

$$e^{\sigma(1+|\xi|)} \leqslant e + \sigma^{1/2}\big(1+|\xi|\big)^{1/2} e^{\sigma(1+|\xi|)}, \tag{15}$$

it is plain that the latter integral is bounded by $I_1 + I_2$ where

$$\begin{aligned} I_1 = e \sup_{h\in B} \int_{\mathbb{R}^6} & \frac{|h(\xi,\tau)|(1+|\xi|)^{1+s}}{(1+|\tau-\xi^3|)^{-b'}} \frac{|v_1(\xi_1,\tau_1)|\, e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1-\xi_1^3|)^{b}} \\ & \times \frac{|v_2(\xi-\xi_2,\tau-\tau_2)|\, e^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^3|)^{b}} \\ & \times \frac{|v_3(\xi_2-\xi_1,\tau_2-\tau_1)|\, e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^3|)^{b}}\, d\mu \end{aligned}$$

and

$$\begin{aligned} I_2 = \sigma^{1/2} \sup_{h\in B} \int_{\mathbb{R}^6} & \frac{|h(\xi,\tau)|(1+|\xi|)^{1+s}(1+|\xi|)^{1/2} e^{\sigma(1+|\xi|)}}{(1+|\tau-\xi^3|)^{-b'}} \frac{|v_1(\xi_1,\tau_1)|\, e^{-\sigma(1+|\xi_1|)}(1+|\xi_1|)^{-s}}{(1+|\tau_1-\xi_1^3|)^{b}} \\ & \times \frac{|v_2(\xi-\xi_2,\tau-\tau_2)|\, e^{-\sigma(1+|\xi-\xi_2|)}(1+|\xi-\xi_2|)^{-s}}{(1+|\tau-\tau_2-(\xi-\xi_2)^3|)^{b}} \\ & \times \frac{|v_3(\xi_2-\xi_1,\tau_2-\tau_1)|\, e^{-\sigma(1+|\xi_2-\xi_1|)}(1+|\xi_2-\xi_1|)^{-s}}{(1+|\tau_2-\tau_1-(\xi_2-\xi_1)^3|)^{b}}\, d\mu. \end{aligned}$$

To analyze I_1, split the integration with respect to ξ, ξ_1, ξ_2 into six regions corresponding to combinations of inequalities such as $|\xi_2 - \xi_1| \leqslant |\xi - \xi_2| \leqslant |\xi_1|$, and estimate the integral on each region separately. The portion of I_1 corresponding to the particular region just delineated can be dominated by the supremum over all h in B of the duality relation

$$\big\langle A^{1/2} H^+_{-b'},\, e^{-\sigma A} A^{1/2} (V_1)^+_b\, e^{-\sigma A} A^{-s} (V_2)^+_b\, e^{-\sigma A} A^{-s} (V_3)^+_b \big\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^2)$ and $H^+_{-b'}$ and $(V_i)^+_b$ are related to $|h|$ and $|v_i|$, respectively, as in (8). This inner product can be bounded by the quantity

$$\big\|A^{1/2} H_{-b'}\big\|_{L_4L_2} \big\|e^{-\sigma A} A^{1/2} (V_1)_b\big\|_{L_4L_2} \big\|e^{-\sigma A} A^{-s} (V_2)_b\big\|_{L_2L_\infty} \big\|e^{-\sigma A} A^{-s} (V_3)_b\big\|_{L_\infty L_\infty}.$$

Using the estimates (9), (11), and (13), it is deduced that

$$I_1 \leqslant c \big\|e^{-\sigma A} v_1\big\|_{L_2L_2} \big\|e^{-\sigma A} v_2\big\|_{L_2L_2} \big\|e^{-\sigma A} v_3\big\|_{L_2L_2}.$$

The other five cases (e.g. $|\xi_2 - \xi_1| \leqslant |\xi_1| \leqslant |\xi - \xi_2|$, $|\xi - \xi_2| \leqslant |\xi_2 - \xi_1| \leqslant |\xi_1|$, etc.) follow by symmetry.

To effect a similar analysis of I_2 first note that

$$e^{\sigma(1+|\xi|)} \leqslant e^{\sigma(1+|\xi_1|)}\, e^{\sigma(1+|\xi-\xi_2|)}\, e^{\sigma(1+|\xi_2-\xi_1|)}$$

and then split the ξ-integrations exactly as in the treatment of I_1. This strategy yields the inequality

$$\begin{aligned} I_2 &\leqslant \sigma^{1/2} \sup_{h\in B} \big\|A^{1/2} H_{-b'}\big\|_{L_4L_2} \big\|A(V_1)_b\big\|_{L_\infty L_2} \big\|A^{-s}(V_2)_b\big\|_{L_2L_\infty} \big\|A^{-s}(V_3)_b\big\|_{L_4L_\infty} \\ &\leqslant c\sigma^{1/2} \|v_1\|_{L_2L_2} \|v_2\|_{L_2L_2} \|v_3\|_{L_2L_2} \end{aligned}$$

where the estimates (9)–(11) and (12) were used. Notice that in estimating I_2, both types of smoothing results were needed to compensate for the extra half-derivative coming from the inequality (15). This concludes the proof in the case $p = 2$.

In case $p > 2$, the same scheme of estimation will yield $p-2$ additional factors of the form

$$\left\|A^{-s}(V_i)_b\right\|_{L_\infty L_\infty}.$$

These remaining factors can be handled using (13). □

For $p = 1$, there are too few factors to absorb an extra power of the spatial derivative in an analogous way. However, by carefully splitting the (ξ, τ)-Fourier space into a number of regions and then applying the smoothing and the maximal function-type estimates, it is possible to overcome this difficulty, as is now demonstrated.

Theorem 2. *Let $\sigma > 0$, $s \geqslant 0$, $b > \frac{1}{2}$ and $b' \leqslant -\frac{3}{8}$. Then there exists a constant c depending only on s, b, and b' such that*

$$\left\|\partial_x(uv)\right\|_{\sigma,s,b'} \leqslant c\|u\|_{s,b}\|v\|_{s,b} + c\sigma^{1/4}\|u\|_{\sigma,s,b}\|v\|_{\sigma,s,b}.$$

Proof. Only the case $s = 0$ is treated; the case $s > 0$ is straightforwardly reduced to the case $s = 0$. The inequality

$$\mathrm{e}^{\sigma(1+|\xi|)} \leqslant \mathrm{e} + \sigma^{1/4}\big(1+|\xi|\big)^{1/4}\mathrm{e}^{\sigma(1+|\xi|)} \tag{16}$$

will play the role of (15) in the proof of the previous theorem. Setting

$$f(\xi,\tau) = \big(1+|\tau-\xi^3|\big)^b\,\mathrm{e}^{\sigma(1+|\xi|)}\hat{u}(\xi,\tau)$$

and

$$g(\xi,\tau) = \big(1+|\tau-\xi^3|\big)^b\,\mathrm{e}^{\sigma(1+|\xi|)}\hat{v}(\xi,\tau),$$

it is required to bound appropriately the quantity

$$\int_{\mathbb{R}^4} \frac{h(\xi,\tau)|\xi|\,\mathrm{e}^{\sigma(1+|\xi|)}}{(1+|\tau-\xi^3|)^{-b'}}\,\frac{f(\xi_1,\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi_1|)}}{(1+|\tau_1-\xi_1^3|)^b}\,\frac{g(\xi-\xi_1,\tau-\tau_1)\,\mathrm{e}^{-\sigma(1+|\xi-\xi_1|)}}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu \tag{17}$$

uniformly in h belonging to the unit ball B in $L^2(\mathbb{R}^2)$ where $\mathrm{d}\mu = \mathrm{d}\xi_1\,\mathrm{d}\tau_1\,\mathrm{d}\xi\,\mathrm{d}\tau$. Using the inequality (16), the integral (17) is bounded by the sum of the two terms

$$I_1 = \mathrm{e}\sup_{h\in B}\int_{\mathbb{R}^4} \frac{|h(\xi,\tau)||\xi|}{(1+|\tau-\xi^3|)^{-b'}}\,\frac{|f(\xi_1,\tau_1)|\,\mathrm{e}^{-\sigma(1+|\xi_1|)}}{(1+|\tau_1-\xi_1^3|)^b}\,\frac{|g(\xi-\xi_1,\tau-\tau_1)|\,\mathrm{e}^{-\sigma(1+|\xi-\xi_1|)}}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu$$

and

$$I_2 = \sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4} \frac{|h(\xi,\tau)||\xi|(1+|\xi|)^{1/4}\,\mathrm{e}^{\sigma(1+|\xi|)}}{(1+|\tau-\xi^3|)^{-b'}}\,\frac{|f(\xi_1,\tau_1)|\,\mathrm{e}^{-\sigma(1+|\xi_1|)}}{(1+|\tau_1-\xi_1^3|)^b}\,\frac{|g(\xi-\xi_1,\tau-\tau_1)|\,\mathrm{e}^{-\sigma(1+|\xi-\xi_1|)}}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu.$$

The first term can be dominated by

$$c\left\|\mathrm{e}^{-\sigma A}f\right\|_{L_2L_2}\left\|\mathrm{e}^{-\sigma A}g\right\|_{L_2L_2}$$

in a way completely analogous to the estimate in the case $p \geqslant 2$. Estimating I_2 in the case $p = 1$ turns out to be a bit more challenging. First, observe that I_2 can be dominated by

$$\sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4} \frac{|h(\xi,\tau)|(1+|\xi|)^{5/4}}{(1+|\tau-\xi^3|)^{-b'}}\,\frac{|f(\xi_1,\tau_1)|}{(1+|\tau_1-\xi_1^3|)^b}\,\frac{|g(\xi-\xi_1,\tau-\tau_1)|}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu \tag{18}$$

because $\mathrm{e}^{(1+|\xi|)} \leqslant \mathrm{e}^{(1+|\xi_1|)}\,\mathrm{e}^{(1+|\xi-\xi_1|)}$. Proceeding as in [6] and [18], the relation

$$\tau - \xi^3 - \big[(\tau_1-\xi_1^3) + (\tau-\tau_1) - (\xi-\xi_1)^3\big] = 3\xi_1(\xi-\xi_1)\xi$$

implies that one of the cases

$$\begin{aligned}&\text{(a) } |\tau-\xi^3|\geqslant|\xi_1||\xi-\xi_1||\xi|,\\ &\text{(b) } |\tau_1-\xi_1^3|\geqslant|\xi_1||\xi-\xi_1||\xi| \text{ or}\\ &\text{(c) } \bigl|\tau-\tau_1-(\xi-\xi_1)^3\bigr|\geqslant|\xi_1||\xi-\xi_1||\xi|\end{aligned}\tag{19}$$

always occurs. In case (a), the quantity in (18) is bounded by

$$\sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4}\bigl|h(\xi,\tau)\bigr|\bigl(1+|\xi|\bigr)^{5/4+b'}\frac{(1+|\xi_1|)^{b'}|f(\xi_1,\tau_1)|}{(1+|\tau_1-\xi_1^3|)^b}\frac{(1+|\xi-\xi_1|)^{b'}|g(\xi-\xi_1,\tau-\tau_1)|}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu.$$

We now split the domain of integration into two further subregions, $|\xi_1|>|\xi-\xi_1|$ and $|\xi_1|\leqslant|\xi-\xi_1|$. In the region where $|\xi_1|>|\xi-\xi_1|$, the quantity in the just displayed integral is dominated by

$$\sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4}\bigl|h(\xi,\tau)\bigr|\bigl(1+|\xi|\bigr)^{b'+3/8}\frac{(1+|\xi_1|)^{7/8+b'}|f(\xi_1,\tau_1)|}{(1+|\tau_1-\xi_1^3|)^b}\frac{(1+|\xi-\xi_1|)^{b'}|g(\xi-\xi_1,\tau-\tau_1)|}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu.$$

The latter integral can be further bounded by

$$\begin{aligned}\sup_{h\in B}\bigl\langle A^{b'+3/8}H_0^+,\ A^{7/8+b'}F_b^+A^{b'}G_b^+\bigr\rangle&\leqslant c\sup_{h\in B}\bigl\|A^{b'+3/8}H_0\bigr\|_{L_2L_2}\bigl\|A^{7/8+b'}F_b\bigr\|_{L_4L_2}\bigl\|A^{b'}G_b\bigr\|_{L_4L_\infty}\\&\leqslant c\|f\|_{L_2L_2}\|g\|_{L_2L_2}\end{aligned}$$

where H_0^+, F_b^+ and G_b^+ are related to $|h|$, $|f|$ and $|g|$, respectively, as in (8). Since the last two factors in the integral have identical structure, the analysis in the region $|\xi_1|\leqslant|\xi-\xi_1|$ is the same.

In case (b), the quantity in (18) is dominated by

$$\sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4}\frac{|h(\xi,\tau)|(1+|\xi|)^{5/4-b}}{(1+|\tau-\xi^3|)^{-b'}}\frac{|f(\xi_1,\tau_1)|}{(1+|\xi_1|)^b}\frac{|g(\xi-\xi_1,\tau-\tau_1)|}{(1+|\xi-\xi_1|)^b(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu.$$

We split the domain of integration into the same two subregions as before. In the region where $|\xi_1|>|\xi-\xi_1|$, the quantity is dominated by

$$\sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4}\frac{|h(\xi,\tau)|(1+|\xi|)^{5/4-2b}}{(1+|\tau-\xi^3|)^{-b'}}\bigl|f(\xi_1,\tau_1)\bigr|\frac{|g(\xi-\xi_1,\tau-\tau_1)|}{(1+|\xi-\xi_1|)^b(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu,$$

and the latter can be bounded by

$$\sup_{h\in B}\bigl\langle A^{5/4-2b}H_{-b'}^+,F_0^+A^{-b}G_b^+\bigr\rangle\leqslant c\sup_{h\in B}\bigl\|A^{5/4-2b}H_{-b'}\bigr\|_{L_4L_2}\|f\|_{L_2L_2}\bigl\|A^{-b}G_b\bigr\|_{L_4L_\infty}\leqslant c\|f\|_{L_2L_2}\|g\|_{L_2L_2}.$$

In the region $|\xi_1|\leqslant|\xi-\xi_1|$, the quantity is dominated by

$$\sigma^{1/4}\sup_{h\in B}\int_{\mathbb{R}^4}\frac{|h(\xi,\tau)|(1+|\xi|)^{1-b}}{(1+|\tau-\xi^3|)^{-b'}}\frac{|f(\xi_1,\tau_1)|}{(1+|\xi_1|)^b}\frac{(1+|\xi-\xi_1|)^{1/4}|g(\xi-\xi_1,\tau-\tau_1)|}{(1+|\xi-\xi_1|)^b(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\,\mathrm{d}\mu.$$

The estimate continues in a similar fashion, namely

$$\begin{aligned}\sup_{h\in B}\bigl\langle A^{1-b}H_{-b'}^+,A^{-b}F_0^+A^{1/4-b}G_b^+\bigr\rangle&\leqslant c\sup_{h\in B}\bigl\|A^{1-b}H_{-b'}\bigr\|_{L_4L_2}\bigl\|A^{-b}f\bigr\|_{L_2L_2}\bigl\|A^{1/4-b}G_b\bigr\|_{L_4L_\infty}\\&\leqslant c\|f\|_{L_2L_2}\|g\|_{L_2L_2}.\end{aligned}$$

The proof in case (c) in (19) is similar to the proof in case (b). □

5. Algebraic lower bounds on σ

In this section, the algebraic decrease of σ as a function of time T is proved. The main objective is to obtain an a priori bound in $G_{\sigma(T),s}$ on the solutions of (1) for a fixed but arbitrary $T > 0$. This bound, combined with the local existence theory in [10] will enable us to prove the desired result. To obtain such a bound, a sequence of approximations to (1) is defined and it is proved that the sequence is bounded in $G_{\sigma(T),s}$ for an appropriate value for $\sigma(T)$. Consider first the following result relating the boundedness of a Sobolev-type norm to the boundedness of a Bourgain-type norm.

Lemma 4. *Let $s > -\frac{1}{2}$, $b \in [-1, 1]$, $T \geqslant 1$, $\sigma > 0$, and let u be a solution of* (1) *on the time interval* $[-2T, 2T]$.

(i) *There exists a constant c depending only on s and b such that*

$$\left\|\psi_T(t)u(\cdot,t)\right\|_{s,b} \leqslant cT^{1/2}\big(1+\alpha_T(u)\big)^{p+1} \tag{20}$$

where

$$\alpha_T(u) \equiv \sup_{t\in[-2T,2T]} \left\|u(\cdot,t)\right\|_{s+1}. \tag{21}$$

(ii) *There exists a constant c depending only on s and b such that*

$$\left\|\psi_T(t)u(\cdot,t)\right\|_{\sigma,s,b} \leqslant cT^{1/2}\big(1+\beta_T(u)\big)^{p+1} \tag{22}$$

where

$$\beta_T(u) \equiv \sup_{t\in[-2T,2T]} \left\|u(\cdot,t)\right\|_{G_{\sigma,s+1}}. \tag{23}$$

Proof. Changing variables in the definition of the norm, it follows immediately that

$$\begin{aligned}\left\|\psi_T(t)u(x,t)\right\|_{s,b}^2 &= \int_{-\infty}^{\infty}\big(1+|\xi|\big)^{2s}\int_{-\infty}^{\infty}\left|\Lambda^b\big(\psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t)\big)\right|^2 \mathrm{d}t\,\mathrm{d}\xi\\ &\leqslant c\int_{-\infty}^{\infty}\big(1+|\xi|\big)^{2s}\int_{-\infty}^{\infty}\left|\psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t)\right|^2 \mathrm{d}t\,\mathrm{d}\xi\\ &\quad + c\int_{-\infty}^{\infty}\big(1+|\xi|\big)^{2s}\int_{-\infty}^{\infty}\left|\partial_t\big(\psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t)\big)\right|^2 \mathrm{d}t\,\mathrm{d}\xi.\end{aligned}$$

Differentiating with respect to t, the second integrand is seen to be

$$\frac{1}{T}\psi_T'(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t) + \psi_T(t)(-\mathrm{i}\xi^3)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t) + \psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u_t(\xi,t).$$

Using the equation $u_t = -u^p u_x - u_{xxx}$, the last term can be replaced by

$$-\frac{1}{p+1}\psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\,\mathrm{i}\xi\mathcal{F}_x(u^{p+1})(\xi,t) - \psi_T(t)(-\mathrm{i}\xi^3)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t).$$

Notice that the terms containing the third derivative cancel. Thus there appears the inequality

$$
\begin{aligned}
\left\|\psi_T(t)u(x,t)\right\|_{s,b}^2 &\leqslant c\int_{-\infty}^{\infty}\left(1+|\xi|\right)^{2s}\int_{-\infty}^{\infty}\left|\psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t)\right|^2 \mathrm{d}t\,\mathrm{d}\xi \\
&\quad + c\int_{-\infty}^{\infty}\left(1+|\xi|\right)^{2s}\int_{-\infty}^{\infty}\left|\frac{1}{T}\psi_T'(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathcal{F}_x u(\xi,t)\right|^2 \mathrm{d}t\,\mathrm{d}\xi \\
&\quad + c\int_{-\infty}^{\infty}\left(1+|\xi|\right)^{2s}\int_{-\infty}^{\infty}\left|\frac{1}{p+1}\psi_T(t)\,\mathrm{e}^{-\mathrm{i}\xi^3 t}\mathrm{i}\xi\mathcal{F}_x(u^{p+1})(\xi,t)\right|^2 \mathrm{d}t\,\mathrm{d}\xi \\
&\leqslant 2c\int_{-\infty}^{\infty}\left(1+|\xi|\right)^{2s}\int_{-2T}^{2T}\left|\mathcal{F}_x u(\xi,t)\right|^2 \mathrm{d}t\,\mathrm{d}\xi + c\int_{-\infty}^{\infty}\left(1+|\xi|\right)^{2s}\int_{-2T}^{2T}\left|\xi\mathcal{F}_x(u^{p+1})(\xi,t)\right|^2 \mathrm{d}t\,\mathrm{d}\xi \\
&\leqslant 8cT\sup_{t\in[-2T,2T]}\left\|u(\cdot,t)\right\|_{H^s}^2 + 4cT\sup_{t\in[-2T,2T]}\left\|u^{p+1}(\cdot,t)\right\|_{H^{s+1}}^2.
\end{aligned}
$$

It is now clear that the inequality (20) holds. The proof of part (ii) is obtained by adding the exponential weight $\mathrm{e}^{2\sigma(1+|\xi|)}$ to the ξ-integral in the proof of (i). □

Next, define a sequence of approximations to (1) as follows. Consider the initial-value problems

$$
\left.
\begin{aligned}
&u_t^n + u_{xxx}^n = -\frac{1}{p+1}\partial_x\left[(\eta_n * \psi_S u^n)^{p+1}\right], \\
&u^n(x,0) = u_0(x),
\end{aligned}
\right\} \tag{24}
$$

for n in $\mathbb{N}$ and $S > 0$ where η_n is defined via its Fourier transform to be

$$
\hat{\eta}_n(\xi) = \begin{cases} 0, & |\xi| \geqslant 2n, \\ 1, & |\xi| \leqslant n, \end{cases}
$$

and $\hat{\eta}_n$ is smooth and monotone on $(-2n,-n)$ and $(n,2n)$. Each η_n is therefore an entire function of exponential type. The following properties of $\{u_n\}$ are evident since $\mathcal{F}_x[\partial_x(\eta_n * \psi_S u^n)^{p+1}]$ is a smooth function with compact support for $(\xi,t) \in \mathbb{R}\times\mathbb{R}$.

Lemma 5. (i) *Let $r \geqslant 0$ and $u_0 \in H^r$, and let u be a solution of* (1) *with initial data u_0 that lies in $C([-2S,2S],H^r)$ for some $S > 0$. For $n = 1,2,\ldots$, let u^n be the solution of* (24) *with initial data u_0. Then each u^n lies in $C((-2S,2S),H^r)$, and the sequence $\{u^n\}$ converges to u in $C([-S,S],H^r)$. In addition, the bounds in Lemma* 4 *hold for each u^n, uniformly in n.*

(ii) *An identical result holds in $C([-S,S],G_{\sigma,r})$ provided $u_0 \in G_{\sigma,r}$ for some $\sigma > 0$.*

Henceforth, it is assumed that u is a solution of (1) in $C([-4T,4T],H^{s+1})$ with initial data u_0 in $G_{\sigma_0,s+1}$ for some $\sigma_0 > 0$ and $s > \frac{3}{2}$. Note that Lemma 5 and (24) with $T = S$ imply that

$$
\psi_T(t)u^n = \psi_T(t)W(t)u_0 - \frac{1}{p+1}\psi_T(t)\int_0^t W(t-s)\partial_x\left(\left[\eta_n * (\psi_T u^n)\right]^{p+1}\right)\mathrm{d}s \tag{25}
$$

holds for all t in $(-\infty,\infty)$. This representation will reveal that $\psi_T u^n$ is in $X_{\sigma,s,b}$ for all n in $\mathbb{N}$. Our goal now is to show that there exists a $\sigma(T)$ and a suitable $R(T)$ such that the sequence $\{\psi_T u^n\}$ lies in the ball $B_{R(T)} \subset X_{\sigma(T),s,b}$ of radius $R(T)$ for n large enough.

Proposition 2. *Let $T \geqslant 1, p \geqslant 2, \sigma_0 > 0, s > \frac{3}{2}$ and $b = \frac{1}{2} + \epsilon$ for some ϵ in the range $0 < \epsilon < \frac{1}{4}$. Suppose u is a solution of* (1) *in $C([-4T, 4T], H^{s+1})$ with initial data $u_0 \in G_{\sigma_0,s+1}$. Then there exist constants $\sigma_1 < \sigma_0$ and $K > 0$ depending on $s, b, p, \|u_0\|_{G_{\sigma_0,s+1}}$ and $\alpha_T(u)$ (see* (21)) *such that the sequence $\{\psi_T u^n\}$ is bounded in $X_{\sigma(T),s,b}$ as long as*

$$\sigma(T) \leqslant \min\{\sigma_1, KT^{-(p^2+3p+2)}\}. \tag{26}$$

For the proof of this proposition, use will be made of the following inequality which was proved by two of the authors (Theorem 11 in [3]). It is worth note that the hypothesis $s > \frac{3}{2}$ goes back to the seminal work of Kato [13,14].

Theorem 3 (Bona–Grujić). *Let u be the solution of* (1) *corresponding to the initial data $u_0 \in G_{\sigma_0,s+1}$ for some $\sigma_0 > 0$ and $s > \frac{3}{2}$, and let $\tau > 0$. Then*

$$\sup_{t\in[-2\tau,2\tau]} \|u(\cdot,t)\|_{G_{\sigma(t),s+1}} \leqslant \|u_0\|_{G_{\sigma_0,s+1}} + C\tau^{1/2} \sup_{t\in[-2\tau,2\tau]} \|u(\cdot,t)\|_{H^{s+1}}^{(p+2)/2}, \tag{27}$$

with $\sigma(t) = \sigma_0 \mathrm{e}^{-\gamma(t)}$, and

$$\gamma(t) = \frac{1}{p+1} \int_0^t \left[d_1 + d_2 \int_0^{t'} \|u(\cdot,t'')\|_{H^{s+1}}^{p+2} \,\mathrm{d}t'' \right]^p \mathrm{d}t',$$

where $d_1 = \|u_0\|^2_{G_{\sigma_0,s+1}}$, and d_2 is a constant depending only on s and p.

Notice that this theorem implies that

$$\sup_{t\in[-2\tau,2\tau]} \gamma(t) \leqslant d_3\tau\|u_0\|^{2p}_{G_{\sigma_0,s+1}} + d_4\tau^p \sup_{t\in[-2\tau,2\tau]} \|u(\cdot,t)\|_{H^{s+1}}^{p(p+2)} \tag{28}$$

where $d_3 = 2^p/(p+1)$ and $d_4 = 2^{2p}d_2^p/(p+1)$. With this estimate in hand, we can mount a direct attack on proving the foregoing proposition.

Proof of Proposition 2. From Eq. (25), the linear estimates (5) and (7), and the multilinear estimate in Theorem 1, it follows that

$$\begin{aligned}
\|\psi_T u^n\|_{\sigma,s,b} &\leqslant \|\psi_T W(t)u_0\|_{\sigma,s,b} + \frac{1}{p+1} \left\| \psi_T \int_0^t W(t-s)\partial_x\big[\big(\eta_n * \psi_T u^n(s)\big)^{p+1}\big]\,\mathrm{d}s \right\|_{\sigma,s,b} \\
&\leqslant cT^{1/2}\|u_0\|_{G_{\sigma,s}} + cT\big\|\partial_x\big[(\eta_n * \psi_T u^n)^{p+1}\big]\big\|_{\sigma,s,b'} \\
&\leqslant cT^{1/2}\|u_0\|_{G_{\sigma,s}} + cT\big\{\|\psi_T u^n\|_{s,b}^{p+1} + \sigma^{1/2}\|\psi_T u^n\|_{\sigma,s,b}^{p+1}\big\}
\end{aligned}$$

for any $0 < \sigma \leqslant \sigma_0$, where $b' = b - 1 + \epsilon'$ for some $\epsilon' > 0$ small enough, and for some large enough constant c depending only on s, b and b'. Next, note that (see Lemmas 4 and 5)

$$\|\psi_T u^n\|_{s,b} \leqslant cT^{1/2}\big(1 + \alpha_T(u^n)\big)^{p+1} \leqslant 2cT^{1/2}\big(1 + \alpha_T(u)\big)^{p+1}$$

for n and c large enough. Here, $\alpha_T(u)$ is as in (21). Thus, the first inequality in the proof may be extended to read

$$\|\psi_T u^n\|_{\sigma,s,b} \leqslant cT^{1/2}\|u_0\|_{G_{\sigma,s}} + cT^{(p+3)/2}\big(1 + \alpha_T(u)\big)^{(p+1)^2} + cT\sigma^{1/2}\|\psi_T u^n\|_{\sigma,s,b}^{p+1} \tag{29}$$

for n large and an appropriate constant c. The relation (29) holds under the presumption that $T \geqslant 1$. Additional information about the boundedness of the sequence at $T = 1$ will now be provided by Theorem 3. To be more specific, using (22), (23), Lemma 5, and the bounds (27) and (28), the norm can be estimated at $T = 1$ as follows.

$$\begin{aligned}\|\psi_1 u^n\|_{\sigma_1,s,b} &\leqslant c\Big(1 + \sup_{t\in[-2,2]} \big\|u^n(\cdot,t)\big\|_{G_{\sigma_1,s+1}}\Big)^{p+1} \leqslant 2c\Big(1 + \sup_{t\in[-2,2]} \big\|u(\cdot,t)\big\|_{G_{\sigma_1,s+1}}\Big)^{p+1} \\ &\leqslant 2cl\Big(1 + \|u_0\|_{G_{\sigma_1,s+1}}^{p+1} + C \sup_{t\in[-2,2]} \big\|u(\cdot,t)\big\|_{H^{s+1}}^{((p+1)(p+2))/2}\Big) \equiv M_1 \end{aligned} \tag{30}$$

for n large enough where $\sigma_1 = \sigma_0 \mathrm{e}^{-\gamma(1)}$ and

$$\gamma(1) = d_3\|u_0\|_{G_{\sigma_0,s+1}}^{2p} + d_4 \sup_{t\in[-2,2]} \big\|u(\cdot,t)\big\|_{H^{s+1}}^{p(p+2)}.$$

Consider a slightly weakened version of (29), namely

$$\|\psi_T u^n\|_{\sigma(T),s,b} \leqslant M_1 + cT^{1/2}\|u_0\|_{G_{\sigma_0,s}} + cT^{(p+3)/2}\big(1+\alpha_T(u)\big)^{(p+1)^2} + cT\sigma(T)^{1/2}\|\psi_T u_n\|_{\sigma(T),s,b}^{p+1} \tag{31}$$

for $T \geqslant 1, \sigma(T) \leqslant \sigma_1 < \sigma_0$ and n large enough. Fix n large enough so that (30) holds and define dependent variables z, a and d by

$$\begin{aligned} z &= z(T) = \|\psi_T u^n\|_{\sigma(T),s,b}, \\ a &= a(T) = M_1 + cT^{1/2}\|u_0\|_{G_{\sigma_0,s}} + cT^{(p+3)/2}\big(1+\alpha_T(u)\big)^{(p+1)^2} \quad \text{and} \\ d &= d(T) = cT. \end{aligned}$$

With this notation, (31) becomes

$$z \leqslant a + d\sigma(T)^{1/2} z^{p+1}. \tag{32}$$

If $\sigma(T)$ is defined to be

$$\sigma(T) = \frac{\delta^2}{d^2 a^{2p} 2^{2p}},$$

then (32) becomes

$$y(1 - \delta y^p) \leqslant \frac{1}{2},$$

where $y = y(T) = \frac{z}{2a}$. It follows that by choosing δ small enough for a given p, there are constants m^* and M^* with $\frac{1}{2} < m^* < 1 < M^*$ such that either $y \leqslant m^*$ or $y \geqslant M^*$. Because of (30) and the definition of a, $z(1) \leqslant a$, so that $y(1) \leqslant \frac{1}{2} < m^*$. Because $\|\psi_T u^n\|_{\sigma(T),s,b}$ is a continuous function of $T \geqslant 1$, it follows that $y \leqslant m^* < 1$ for all $T \geqslant 1$, which means that $z(T) \leqslant 2a$ for $T \geqslant 1$. This yields the desired estimate with a constant K depending on p, s, b, b', $\|u_0\|_{G_{\sigma_0,s+1}}$ and $\alpha_T(u)$. □

Proposition 3. *Let $T \geqslant 1$, $p = 1$, $\sigma_0 > 0$, $s > \frac{3}{2}$ and $b = \frac{1}{2} + \epsilon$ for some ϵ in the range $0 < \epsilon < \frac{1}{8}$. Suppose u is a solution of* (1) *in $C([-4T, 4T], H^{s+1})$ with initial data $u_0 \in G_{\sigma_0,s+1}$. Then there exist constants $\sigma_1 < \sigma_0$ and $K > 0$ depending on s, b, $\|u_0\|_{G_{\sigma_0,s+1}}$ and $\alpha_T(u)$ such that the sequence $\{\psi_T u^n\}$ is bounded in $X_{\sigma(T),s,b}$ as long as*

$$\sigma(T) \leqslant \min\{\sigma_1, KT^{-12}\}. \tag{33}$$

Proof. Eq. (25), the linear estimates (5) and (7), and the bilinear estimate in Theorem 2 yield the inequality

$$\|\psi_T u^n\|_{\sigma,s,b} \leqslant cT^{1/2}\|u_0\|_{G_{\sigma_0,s}} + cT\big\{\|\psi_T u^n\|_{s,b}^2 + \sigma^{1/4}\|\psi_T u^n\|_{\sigma,s,b}^2\big\}$$

for an appropriate constant c. The proof now follows along the same lines as the proof of Proposition 2. □

The estimates (26) and (33) provide the basis for the proof of the main theorem of the paper which is stated next.

Theorem 4. (i) *Let $p \geqslant 2$, and suppose that $u_0 \in G_{\sigma,s+1}$ for some $s > \frac{3}{2}$ and $\sigma > 0$. Let $T \geqslant 1$ and assume that the solution u of* (1) *corresponding to the initial value u_0 lies in $C([-4T, 4T], H^{s+1})$. Then $u \in C([-T, T], G_{\sigma(T)/2,s})$ where $\sigma(T)$ is given by* (26).

(ii) *Let $p = 1$, and suppose that $u_0 \in G_{\sigma,s+1}$ for some $s > \frac{3}{2}$ and $\sigma > 0$. Let $T \geqslant 1$ and assume that the solution u of* (1) *corresponding to the initial value u_0 lies in $C([-4T, 4T], H^{s+1})$. Then $u \in C([-T, T], G_{\sigma(T)/2,s})$ where $\sigma(T)$ is given by* (33).

Proof. It follows from Propositions 2 or 3 and inequality (2) that the sequence $\{u^n\}$ associated with u_0 as in (24) is bounded in $G_{\sigma(T),s}$, uniformly on $[-T, T]$. Proposition 1 then implies that all the spatial derivatives of u^n are bounded on the strip $S_{\sigma(T)/2,s}$. Since each u^n satisfies Eq. (24), the time derivatives of u^n are also uniformly bounded on the strip $S_{\sigma(T)/2,s}$.

Thus, in particular, $\{\partial_t u^n\}$ and $\{\partial_x^k u^n\}$ for $k = 0, 1, 2, 3$ are equicontinuous families on $(-T, T) \times S_{\sigma(T)/2}$ and we can therefore extract a subsequence (call it $\{u^n\}$ again) converging uniformly on compact subsets of $(-T, T) \times S_{\sigma(T)/2}$ – along with the sequences $\{\partial_t u^n\}$, $\{\partial_x u^n\}$ and $\{\partial_x^3 u^n\}$ – to a smooth function $\tilde{u}$. Passing to the limit in (24) reveals that $\tilde{u}$ is a smooth extension of u to $(-T, T) \times S_{\sigma(T)/2}$. Moreover, since for every $t \in (-T, T)$, $u^n(\cdot, t)$ converges uniformly on compact subsets in $S_{\sigma(T)/2}$ to $\tilde{u}(\cdot, t)$, and each $u^n(\cdot, t)$ is analytic on $S_{\sigma(T)/2}$, $\tilde{u}(\cdot, t)$ is also analytic on $S_{\sigma(T)/2}$. In addition, since the sequence $\{u^n\}$ is bounded in $G_{\sigma(T)/2,s}$, uniformly on $[-T, T]$, it follows that $u \equiv \tilde{u} \in L_\infty((-T, T), G_{\sigma(T)/2,s})$. This combined with the local-in-time well-posedness obtained in [10] yields $u \in C([-T, T], G_{\sigma(T)/2,s})$, as advertised. □

This theorem has some interesting consequences. First, suppose that $\|u(\cdot, t)\|_{H^{s+1}}$ is bounded for all time. Then Theorem 3 can be strengthened to yield the following.

Corollary 1. *Let $p \geqslant 2$, and suppose that $u_0 \in G_{\sigma,s+1}$ for some $s > \frac{3}{2}$ and $\sigma > 0$. If $\sup_{t\in(-\infty,\infty)} \|u(\cdot, t)\|_{H^{s+1}} \leqslant C$, then for all $T \geqslant 1$, $u \in C([-T, T], G_{\sigma(T)/2,s})$, where $\sigma(T)$ is given by* (26). *The same result holds for $p = 1$, but $\sigma(T)$ is given by* (33).

In fact, for $p = 1$ or $p = 2$, all the integer Sobolev norms remain bounded owing to the well known infinite sequence of polynomial conservation laws. Thus the following corollary emerges.

Corollary 2. (i) *For $p = 1$, suppose that $u_0 \in G_{\sigma,k+1}$ for some integer $k \geqslant 2$ and $\sigma > 0$. Then $u \in C([-T, T], G_{\sigma(T)/2,k})$ for any $T \geqslant 1$ where $\sigma(T)$ is given by*

$$\sigma(T) \leqslant \min\{\sigma_1, KT^{-12}\}.$$

(ii) *For $p = 2$, suppose that $u_0 \in G_{\sigma,k+1}$ for some integer $k \geqslant 2$ and $\sigma > 0$. Then $u \in C([-T, T], G_{\sigma(T)/2,k})$ for any $T \geqslant 1$ where $\sigma(T)$ is given by*

$$\sigma(T) \leqslant \min\{\sigma_1, KT^{-24}\}.$$

Note that in both cases, the constant K depends only on k, p and $\|u_0\|_{G_{\sigma,k+1}}$, as well as on the choice of b and b'.

Time-independent bounds on Sobolev norms of solutions of (1) in case $p = 3$ are known only for H^1. However, solutions are globally defined. On the other hand, for $p = 4$ there is a finite-time blow-up [19]. Strong numerical evidence supplemented with scaling arguments indicate that some solutions may lose regularity in finite time for $p > 4$ [1,2,4]. Consequently, assuming finiteness of a certain Sobolev norm when $p \geqslant 3$ seems necessary for

studying global-in-time analyticity of solutions. According to Corollary 1, uniform-in-time H^r-boundedness for some $r > \frac{5}{2}$ suffices. The following theorem due to Staffilani implies that this can be scaled down to uniform-in-time H^1-boundedness.

Theorem 5 (Staffilani). *Let $p \geqslant 3$ and $s > 1$. Assume that for a solution u of* (1) $\sup_{t\in(-\infty,\infty)} \|u(\cdot,t)\|_{H^1} \leqslant C$. *Then there exists a constant $c(s,p)$ such that the estimate*

$$\|u(\cdot,t)\|_{H^s} \leqslant c(s,p)\big(1+|t|\big)^{s-1}$$

holds for all t in $(-\infty,\infty)$.

More precisely, Corollary 1 and Theorem 5 yield the following result.

Corollary 3. *Let $p \geqslant 3$ and suppose that $u_0 \in G_{\sigma,s+1}$ for some $s > \frac{3}{2}$ and $\sigma > 0$. Assume that for a solution u of* (1) *emanating from u_0, $\sup_{t\in(-\infty,\infty)} \|u(\cdot,t)\|_{H^1} \leqslant C$. Then $u \in C([-T,T], G_{\sigma(T)/2,s})$ for any $T \geqslant 1$ where $\sigma(T)$ is given by*

$$\sigma(T) \leqslant \min\big\{\sigma_1, KT^{-\mu(s,p)}\big\},$$

with $\mu(s,p) = (p^2+3p+2)+2p(p+1)^2 s$, and the constant K depending only on s, p and $\|u_0\|_{G_{\sigma,s+1}}$, and on the choice of b and b'.

Noting that indeed the H^1-norm stays bounded for all time if $p = 3$, the final corollary emerges.

Corollary 4. *Let $p = 3$ and suppose that $u_0 \in G_{\sigma,s+1}$ for some $s > \frac{3}{2}$ and $\sigma > 0$. Then there exists a solution $u \in C([-T,T], G_{\sigma(T)/2,s})$ of* (1) *for any $T \geqslant 1$, where $\sigma(T)$ is given by*

$$\sigma(T) \leqslant \min\big\{\sigma_1, KT^{-20-96s}\big\}.$$

Also in this case, the constant K depends only on s and $\|u_0\|_{G_{\sigma,s+1}}$, as well as on the choice of b and b'.

This paper has been concerned with the question of lower bounds on the uniform radius of spatial analyticity σ. The question of upper bounds is currently being studied by the authors using a completely different method, based on finite-time blow-up results for certain complex-valued solutions for a large class of nonlinear dispersive wave equations presented in [5]. If such upper bounds could indeed be proved, the complex singularities would be confined between the lower and the upper bounds. This would yield a much more precise description of the dynamics of complex singularities in time.

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References

[1] J.L. Bona, V.A. Dougalis, O.A. Karakashian, W.R. McKinney, Conservative high-order numerical schemes for the generalized Korteweg–de Vries equation, Philos. Trans. Roy. Soc. London Ser. A 351 (1995) 107–164.

[2] J.L. Bona, V.A. Dougalis, O.A. Karakashian, W.R. McKinney, Numerical simulation of singular solutions of the generalized Korteweg–de Vries equation, in: F. Dias, J.-M. Ghidaglia, J.-C. Saut (Eds.), Contemp. Math., vol. 200, Amer. Math. Soc., Providence, RI, 1996, pp. 17–29.

[3] J.L. Bona, Z. Grujić, Spatial analyticity for nonlinear waves, Math. Models Methods Appl. Sci. 13 (2003) 1–15.

[4] J.L. Bona, F.B. Weissler, Similarity solutions of the generalized Korteweg–de Vries equation, Math. Proc. Cambridge Philos. Soc. 127 (1999) 323–351.
[5] J.L. Bona, F.B. Weissler, Blow-up of spatially periodic complex-valued solutions of nonlinear dispersive equations, Indiana Univ. Math. J. 50 (2001) 759–782.
[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, Geom. Funct. Anal. 3 (1993) 107–156, 209–262.
[7] A. de Bouard, N. Hayashi, K. Kato, Gevrey regularizing effect for the (generalized) Korteweg–de Vries equation and nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1995) 673–715.
[8] C. Foias, R. Temam, Gevrey class regularity for the solutions of the Navier–Stokes equations, J. Funct. Anal. 87 (1989) 359–369.
[9] J. Ginibre, Y. Tsutsumi, G. Velo, On the Cauchy problem for the Zakharov system, J. Funct. Anal. 151 (1997) 384–436.
[10] Z. Grujić, H. Kalisch, Local well-posedness of the generalized Korteweg–de Vries equation in spaces of analytic functions, Differential Integral Equations 15 (2002) 1325–1334.
[11] N. Hayashi, Analyticity of solutions of the Korteweg–de Vries equation, SIAM J. Math. Anal. 22 (1991) 1738–1743.
[12] N. Hayashi, Solutions of the (generalized) Korteweg–de Vries equation in the Bergman and Szegö spaces on a sector, Duke Math. J. 62 (1991) 575–591.
[13] T. Kato, Quasilinear equations of evolution with applications to partial differential equations, in: Lecture Notes in Math., vol. 448, Springer-Verlag, 1975, pp. 25–70.
[14] T. Kato, On the Korteweg–deVries equation, Manuscripta Math. 28 (1979) 89–99.
[15] T. Kato, K. Masuda, Nonlinear evolution equations and analyticity I, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986) 455–467.
[16] K. Kato, T. Ogawa, Analyticity and smoothing effect for the Korteweg–de Vries equation with a single point singularity, Math. Ann. 316 (2000) 577–608.
[17] C.E. Kenig, G. Ponce, L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J. 40 (1991) 33–69.
[18] C.E. Kenig, G. Ponce, L. Vega, On the Cauchy problem for the Korteweg–deVries equation in Sobolev spaces of negative indices, Duke Math. J. 71 (1993) 1–20.
[19] Y. Martel, F. Merle, Blow up in finite time and dynamics of blow up solutions for the L_2-critical generalized KdV equation, J. Amer. Math. Soc. 15 (2002) 617–664.
[20] G. Staffilani, On the growth of high Sobolev norms of solutions for KdV and Schrödinger equations, Duke Math. J. 86 (1997) 109–142.