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Long Wave Approximations for Water Waves

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Abstract

In this paper, we obtain new nonlinear systems describing the interaction of long water waves in both two and three dimensions. These systems are symmetric and conservative. Rigorous convergence results are provided showing that solutions of the complete free-surface Euler equations tend to associated solutions of these systems as the amplitude becomes small and the wavelength large. Using this result as a tool, a rigorous justification of all the two-dimensional, approximate systems recently put forward and analysed by Bona, Chen and Saut is obtained. In the two-dimensional context, our methods also allows a significant improvement of the convergence estimate obtained by Schneider and Wayne in their justification of the decoupled Korteweg-de Vries approximation of the two-dimensional Euler equations. It also follows from our theory that coupled models provide a better description than the decoupled ones over short time scales. Results are obtained both on an unbounded domain for solutions that evanesce at infinity as well as for solutions that are spatially periodic.

1. Introduction

1.1. Generalities

The water-wave problem for an ideal liquid consists of describing the motion of the free surface and the evolution of the velocity field of a layer of perfect, incompressible, irrotational fluid under the influence of gravity. In this paper, attention is given to both the two-dimensional case wherein the wave motion is assumed not to vary appreciably in one of the coordinate directions, say the y-direction in a standard Cartesian coordinate system, and the fully three-dimensional setting. However, consideration is restricted to the special case of a flat bottom. It is well understood that several different regimes may be obtained for this problem; attention is given here to the so-called long-wave limit. In this setting, it is assumed equations approximately satisfy the class of models S put forward by Bona, Chen and Saut. A detailed analysis of decoupled formulations is the subject of the last section.

1.2. Description of the results

There are three principal types of results in the technical elaboration of our theory, namely, consistency, existence and convergence results. To describe them, we need to discuss in a preliminary fashion the different systems involved.

The primary system on which everything developed here is based is the Euler system (1)–(4). From the Euler system, we obtain the classical Boussinesq system by making the long-wave and small-amplitude assumptions outlined above, expanding appropriately in the small parameter ε, and then dropping terms that are formally of order higher rather than linear in ε. Boussinesq's original system belongs to the wide class obtained by BONA, CHEN & SAUT [8] which is here denoted by S. All these systems have the same nonlinear structure as the original Boussinesq system; they differ from each other only in their modelling of dispersion. As mentioned above, they are all formally equivalent. The second class of systems, denoted S', is a new one. It is obtained from S by nonlinear changes of variables which renders symmetric the nonlinear hyperbolic portion (the non-dispersive part) of the system. Members of the class S' are also formally equivalent to the systems of class S, and hence formally good approximations of the full Euler equations on the long time scale characterized by ε^{-1}. The final class marked for discussion here is denoted Σ and is in fact the subclass of S' consisting of those systems for which *both* the dispersive and the nonlinear part are symmetric. The class Σ is non-empty. Indeed, the sample system displayed in Section 1.1 is a member. Systems which belong to Σ are conservative in the sense hinted at in Section 1.1, which will now be explained.

In what follows, we will prove theorems of existence for systems in Σ and convergence of their solutions to those of the full Euler equations, including bounds on the rate of convergence. We will also prove comparison theorems between solutions of the systems in Σ and solutions of the other systems under discussion.

A crucial component of our analysis will be the notion of *consistency* which is presented now. Consider the general system

$$\partial_t \mathcal{U} + A(\partial_X)\mathcal{U} + \varepsilon B(\mathcal{U})(\partial_X \mathcal{U}) + \varepsilon \left(C(\partial_X^3 \mathcal{U}) + D(\partial_X^2 \partial_t \mathcal{U}) \right) = 0, \qquad (7)$$

where $\mathcal{U}(t, X) : [0, \frac{T}{\varepsilon}] \times \mathbb{R}^d \to \mathbb{R}^p$, T is a fixed positive constant, p is an integer, $X = (x, y)$ if $d = 2$ and $X = x$ if $d = 1$, $A(\partial_X) = A_1\partial_x + A_2\partial_y$ with A_1 and A_2 constant matrices, $B(\mathcal{U})(\partial_X \mathcal{U}) = B_1(\mathcal{U})\partial_x \mathcal{U} + B_2(\mathcal{U})\partial_y \mathcal{U}$ and

$$C(\partial_X^3 \mathcal{U}) = \sum_{i,j,k=1}^{d} C_{ijk}\partial_{ijk}^3 \mathcal{U}, \qquad D(\partial_X^2 \partial_t \mathcal{U}) = \sum_{i,j=1}^{d} D_{ij}\partial_{ij}^2 \partial_t \mathcal{U}.$$

To be concrete, we can assume that B_1 and B_2 are polynomial in the components of $\mathcal{U}$ and that C_{ijk} and D_{ij} are constant matrices, though this is not necessary for some of what follows. All the systems in the classes S, S' and Σ, can be written in the form (7), so it suffices for the purposes at hand to provide a definition of consistency within the context of (7).

Definition 1. Let $s \geqq 0$, $\sigma \geqq s$, $\varepsilon_0 > 0$ and $T > 0$ be given. A family $\{\mathcal{U}^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ which is bounded in $W^{1,\infty}(0, \frac{T}{\varepsilon}; H^\sigma(\mathbb{R}^d))$ independently of ε is consistent with the system (7) if

$$\partial_t \mathcal{U}^\varepsilon + A(\partial_X)\mathcal{U}^\varepsilon + \varepsilon B(\mathcal{U}^\varepsilon)(\partial_X \mathcal{U}^\varepsilon) + \varepsilon \left(C(\partial_X^3 \mathcal{U}^\varepsilon) + D(\partial_X^2 \partial_t \mathcal{U}^\varepsilon) \right) = \varepsilon^2 \mathcal{R}^\varepsilon$$

where the family $\{\mathcal{R}^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is bounded in $L^\infty(0, \frac{T}{\varepsilon}; H^s(\mathbb{R}^d))$ for $0 < \varepsilon < \varepsilon_0$. When the values of σ and s are important, we will say that the family $\{\mathcal{U}^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is consistent with regularity σ and s.

One of the technical goals of this paper is to establish rigorous consistency results for systems of the classes S, S' and Σ and to prove related existence and convergence results. Here is a more detailed account of this objective.

Consistency results. Let $\{(\phi^\varepsilon, \eta^\varepsilon)\}$ be a family of solutions of the Euler equations (1)–(4) for some open interval of values of ε of the form $(0, \varepsilon_0)$, say, where $\varepsilon_0 > 0$. Define $V^\varepsilon := \nabla\psi^\varepsilon$ with $\psi^\varepsilon(t, X) := \phi(X, 1 + \varepsilon\eta^\varepsilon(t, X))$, the velocity potential at the free surface. If $\{(V^\varepsilon, \eta^\varepsilon)\}$ is bounded in $W^{1,\infty}(0, \frac{T}{\varepsilon}; H^\sigma(\mathbb{R}^d))$ for some σ large enough, it is established in Theorem 6 that $\{(V^\varepsilon, \eta^\varepsilon)\}$ is consistent with the Boussinesq system. This result is in the general spirit of the results of CRAIG, SCHANTZ, SULEM & SULEM in [16, 15]. A direct approach is used to prove this result which avoids use of the singular integral associated with the Dirichlet-to-Neumann operator for the flow domain.

We also prove that any net of functions consistent with any one of the systems of class S is, up to a linear change of variables, consistent with any other system of S. The linear change of variables (taken from [8]) corresponds physically to taking as a new independent variable the horizontal velocity at a different height above the bottom (see Proposition 1). Similarly, it is shown that if $\{(V^\varepsilon, \eta^\varepsilon)\}$ is consistent with a system in S then $\{(\widetilde{V^\varepsilon}, \widetilde{\eta^\varepsilon})\}$ obtained from $\{(V^\varepsilon, \eta^\varepsilon)\}$ by the aforementioned nonlinear change of variables is consistent with a system of the class S' (which has a symmetric nonlinear part). At this point, a key assumption in the three-dimensional theory is the irrotationality of the flow, which can be expressed as the vanishing of the curl of the velocity field everywhere in the flow domain (see Proposition 2). Moreover, we prove in Proposition 3 that any function consistent with a system in S' is also consistent with all the other members of S' (again, up to a linear change of variables).

Existence results. For the systems of class S, the local well posedness for the Cauchy problem has been discussed in some detail in [8, 9]. Necessary and sufficient conditions for the well posedness of the associated linear problems was given in [8]. It was posited that the systems which are linearly ill posed are unlikely to be nonlinearly well posed, and they were discarded from the discussion in [9]. In the latter reference, local well posedness was demonstrated for all the systems that were determined to be linearly well posed except for one highly degenerate case. However, when written in the scaling favoured here, the time of existence for most that the free surface may be described as the graph of a function defined over the bottom. More precisely, the motion of the fluid is described by the set of equations

$$\begin{aligned}
&\frac{h_0^2}{\lambda^2}\Delta\phi + \partial_z^2\phi = 0 && \text{for } 0 \leqq z \leqq 1 + \tfrac{a}{h_0}\eta(t, X),\\
&\partial_t\phi + \frac{1}{2}\left(|\nabla\phi|^2 + \frac{\lambda^2}{h_0^2}|\partial_z\phi|^2\right) + \frac{h_0}{a}\eta = 0 && \text{at } z = 1 + \tfrac{a}{h_0}\eta(t, X),\\
&\partial_t\eta + \nabla\phi\cdot\nabla\eta = \frac{\lambda^2}{h_0 a}\partial_z\phi && \text{at } z = 1 + \tfrac{a}{h_0}\eta(t, X),\\
&\partial_z\phi = 0 && \text{at } z = 0,
\end{aligned}$$

where the operators ∇ and Δ act on the transverse variable $X \in \mathbb{R}^d$, $d = 1$ or 2. In the case $d = 1$ (the two-dimensional case), $X = x$ is the coordinate along the primary direction of propagation and the motion is assumed not to vary appreciably in the y-direction, whilst if $d = 2$ (the three-dimensional case), then $X = (x, y)$ represents both the horizontal variables. The vertical coordinate is denoted z as usual, with gravity acting in the direction of the negative values. The equation of the featureless, horizontal bottom is $z = 0$ while the free surface is located at $z = 1 + \frac{a}{h_0}\eta(t, X)$. The equations are written in the usual non-dimensional form where a is a typical wave amplitude, h_0 is the undisturbed depth of the fluid and λ is a typical horizontal wavelength. The dependent variable ϕ is the non-dimensional velocity potential and $\eta(t, X) = (h(t, X) - h_0)/a$, where h is the total depth of the water column at the point X at time t.

The preceding equations are mathematically and numerically recalcitrant. Some results concerning the Cauchy problem wherein the free surface η is specified for all values of X and the velocity potential ϕ is specified appropriately in the resulting flow domain, both at a given instant of time, are available (see, for example, [14, 19–21, 23, 25, 26], and the references contained in these works). In many practically important situations, we rely upon simplifications of these equations to describe approximately the behavior of their solutions. Various model equations have been derived by means of formal asymptotic expansions. Historically, the initial developments in this direction were associated with works of Lagrange, St. Venant, Green, Airy and Stokes among others. A very significant step forward was made by BOUSSINESQ [13] who seems to have been the first to properly understand the long-wave regime described next.

The long-wave regime is characterized by the presumptions of long wavelength and small amplitude, *viz.*

$$\varepsilon = \frac{a}{h_0} \ll 1, \qquad \frac{\lambda}{h_0} \gg 1,$$

in conjunction with the assumption that the Stokes number

$$S = \frac{a\lambda^2}{h_0^3}$$

is of order 1. For notational simplicity, we take $S = 1$ throughout our discussion so that

$$\frac{\lambda^2}{h_0^2} = \frac{1}{\varepsilon}.$$

If we did not adhere to this presumption, the only change is that the equations would feature the value of S in various of the coefficients. With this notation and the presumption that $S = 1$, the non-dimensional water-wave equations take the form

$$\begin{aligned}
\varepsilon\Delta\phi + \partial_z^2\phi &= 0 && \text{for } 0 \leqq z \leqq 1 + \varepsilon\eta, && (1)\\
\partial_t\phi + \frac{1}{2}\left(\varepsilon|\nabla\phi|^2 + |\partial_z\phi|^2\right) + \eta &= 0 && \text{at } z = 1 + \varepsilon\eta, && (2)\\
\partial_t\eta + \varepsilon\nabla\phi\cdot\nabla\eta &= \frac{1}{\varepsilon}\partial_z\phi && \text{at } z = 1 + \varepsilon\eta, && (3)\\
\partial_z\phi &= 0 && \text{at } z = 0. && (4)
\end{aligned}$$

Perhaps the simplest of the asymptotic models which takes into account both nonlinear effects as reflected in the small but finite amplitude, and the dispersive effects coming from large but finite wavelength is the Korteweg-de Vries equation (KdV equation henceforth). It is a unidirectional, one-dimensional description in terms of the dependent variable η, with the form

$$\partial_t\eta + \partial_x\eta + \varepsilon\left(\frac{1}{6}\partial_x^3\eta + \frac{3}{2}\eta\partial_x\eta\right) = 0. \tag{5}$$

Note that in the present scaling, the variable η and its first several partial derivatives are all of order one. This regime has been analysed by CRAIG [14] starting from the Lagrangian form of (1)–(4). In terms of the variables introduced above, he showed that there exists a constant T independent of ε and a solution $(\phi^\varepsilon, \eta^\varepsilon)$ of (1)–(4) defined at least on the time interval $[0, \frac{T}{\varepsilon}]$ such that η^ε is approximated to within order ε in the L^∞ norm by an associated solution of (5). It is worth noting that according to the formal derivation of the KdV model as written in (5), we can expect the solutions to be good renditions of an associated Euler flow (1)–(4) on a time scale of order ε^{-1} and also the neglected effects to make an order-one relative contribution on a time scale of order ε^{-2}. Thus Craig's result provides theoretical justification for the use of (5), but it has nothing to say about the eventual breakdown of the model as an approximation to (1)–(4) (see [10, 11, 2, 1] for more complete discussions of these matters).

SCHNEIDER & WAYNE [23] extended Craig's result about the KdV regime by writing a theory that allowed for more general initial disturbances. They also wrote theory for wave motion in both directions. Expressed in the present variables, they showed that the solution of (1)–(4) may be approximated to within order $\varepsilon^{1/4}$ on a time scale of order ε^{-1} by the solutions of two uncoupled, counter-propagating waves, each of which satisfies a KdV-type equation, namely

$$\begin{cases}
\partial_t\eta_1 + \partial_x\eta_1 + \varepsilon\left(\frac{1}{6}\partial_x^3\eta_1 + \frac{3}{4}\partial_x(\eta_1^2)\right) = 0,\\
\partial_t\eta_2 - \partial_x\eta_2 - \varepsilon\left(\frac{1}{6}\partial_x^3\eta_2 + \frac{3}{4}\partial_x(\eta_2^2)\right) = 0.
\end{cases} \tag{6}$$

While suggestive and interesting as a principle, an error estimate of order $\varepsilon^{1/4}$ is clearly not a practically useful bound. A sharp result of this nature appears in Section 5 of this paper. A similar theory was obtained for a general class of hyperbolic systems by BEN YOUSSEF & COLIN in [4]. We will have more to say about this type of approximation presently.

Following the lead of BONA & SMITH [12], BONA, CHEN & SAUT [8] systematically took advantage of the freedom associated with the choice of the velocity variable and made full use of the lower-order relations (the wave equation written as a coupled system) in the dispersive terms to put forward a three-parameter family of Boussinesq-type systems, all of which are formally equivalent models of solutions of the two-dimensional Euler equations (1)–(4) where $X = x$. Many systems of this family were eliminated as potential models when the associated initial-value problems were shown to be ill-posed in [8]. However, there remain significant sub-families that are known to be at least locally well posed in quite reasonable smoothness classes (see [9]). These systems are reviewed in a little more detail in the next section for the reader's convenience.

In this paper, the class of systems developed by Bona, Chen and Saut is extended in an interesting and helpful way. The key to the extensions proposed here is a nonlinear change of variables that leaves the formal order of approximation unchanged, but which results in new systems with very attractive mathematical properties. In particular, we derive systems in both two and three spatial dimensions that are symmetric in their nonlinear structure and their dispersive modelling. An interesting example of the systems we derive and analyse is

$$\partial_t V + \nabla\eta + \varepsilon\left[\tfrac{1}{4}\nabla(\eta^2) + \tfrac{3}{4}\begin{pmatrix}\partial_x(V_1^2)\\ \partial_y(V_2^2)\end{pmatrix} + \tfrac{1}{4}\begin{pmatrix}\partial_x(V_2^2)\\ \partial_y(V_1^2)\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}\partial_y(V_1V_2)\\ \partial_x(V_1V_2)\end{pmatrix}\right] + \varepsilon\,(a\Delta\nabla\eta - b\Delta\partial_t V) = 0,$$

$$\partial_t\eta + \nabla\cdot V + \frac{\varepsilon}{2}\nabla\cdot(\eta V) + \varepsilon\,(c\Delta\nabla\cdot V - d\Delta\partial_t\eta) = 0,$$

where $V = (V_1, V_2)^T$ denotes the horizontal velocity field at height θ, and in our present scaling it is naturally required that $0 \leqq \theta \leqq 1$. The constants a, b, c and d appearing in the equation are

$$a = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\lambda,\; b = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\lambda),\; c = \frac{(1-\theta^2)}{2}\mu,\; d = \frac{(1-\theta^2)}{2}(1-\mu),$$

where λ and μ are any two real parameters. If we choose the parameters so that $a = c$ and $b \geqq 0$, $d \geqq 0$, then this system is symmetric and it is well posed in $W^{k,\infty}(0, \frac{T}{\varepsilon}; H^{s-3k}(\mathbb{R}^3))$ for any k and s such that $s - 3k > 2$. Moreover, we have the exact conservation law

$$\partial_t \int_{\mathbb{R}^2} V^2 + \eta^2 + \varepsilon(b|\nabla V|^2 + d|\nabla\eta|^2) = 0$$

(see Proposition 4).

Fully symmetric models turn out to be a powerful mathematical tool. Indeed, we are able to prove that appropriately smooth solutions of the full equations (1)–(4) can be approximated by solutions of these symmetric systems with an error which is at most of order $\varepsilon^2 t$, uniformly for $t \in [0, \frac{T}{\varepsilon}]$. We then show that in two dimensions, smooth solutions of any of the Bona-Chen-Saut systems differ from solutions of a symmetric system by at most a quantity of order $\varepsilon^2 t$ on the same long-time scale ε^{-1}. The latter result gives a satisfactory rigorous foundation for the use of any of the well posed versions of these models to describe two-dimensional surface water waves in the long-wave or Boussinesq regime.

D. LANNES proved recently [19, 20] that the water-waves equations in finite depth are locally in time well-posed in reasonable function classes (WU proved this earlier in the case of infinite depth [26]). However, applying this result to the present case of long waves does not yield directly that the existence time is large, of order $O(1/\varepsilon)$. Thus, our three-dimensional results are expressed in terms of solutions of the Euler equations which may exist, without the extra assertion that such solutions necessarily do persist on the relevant long time scale.

Our results apply both to the equations posed with the bottom comprising of the entire space $\mathbb{R}^d$, $d = 1$ or 2 with function-space restrictions that imply solutions decay to zero at infinity, and to the periodic initial-value problems.

Returning briefly to the issue of approximation via a decoupled system in the two-dimensional situation, we remark first that laboratory experiments and real world flows arising in geophysical contexts often show nonlinear coupling effects between counter-propagating waves. This apparent contradiction with the result of [23] quoted above appears to derive from two sources. First, in practice, the parameter ε is not so small. (Values of ε on the order of 0.3 regularly appear in situations where approximations to the Euler equations are used to model real waves.) Secondly, the Schneider-Wayne result subsists upon the assumption of a definite rate of decrease of the relevant wave motion at infinity. Many wave regimes arising in the laboratory or in field situations are of a quasi-periodic nature and certainly do not fit the approximation of tending to zero at infinity at a substantial rate, at least on the spatial and temporal scales where the models might possibly be useful. Indeed, as will become apparent in the analysis presented in Section 5, a decoupled approximation such as (6) does not present the same convergence rate to associated solutions of the Euler equations (1)–(4) as does the coupled Boussinesq equations presented here and in [8] in the situation where the initial disturbance has only function class restrictions and not a definite rate of approach to zero at infinity. For the situation where the initial data is periodic, mentioned in the last paragraph, the decoupled system may fail entirely to provide a useful approximation to the Euler equations, in contrast to the coupled systems developed here and in [8].

In the next subsection, a more detailed view of the theory developed herein is presented. The plan of the remainder of the paper is now outlined. Section 2 is devoted to the derivation of the symmetric systems to which frequent reference has just been made. The exact relationship between the symmetric systems and the full Euler equations is investigated in Section 3, while in Section 4, it is proved in the two-dimensional case that the exact solutions of the Euler equations approximately satisfy the class of models S put forward by Bona, Chen and Saut. A detailed analysis of decoupled formulations is the subject of the last section.

1.2. Description of the results

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In what follows, we will prove theorems of existence for systems in Σ and convergence of their solutions to those of the full Euler equations, including bounds on the rate of convergence. We will also prove comparison theorems between solutions of the systems in Σ and solutions of the other systems under discussion.

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where $\mathcal{U}(t,X) : [0, \frac{T}{\varepsilon}] \times \mathbb{R}^d \to \mathbb{R}^p$, T is a fixed positive constant, p is an integer, $X = (x, y)$ if $d = 2$ and $X = x$ if $d = 1$, $A(\partial_X) = A_1 \partial_x + A_2 \partial_y$ with A_1 and A_2 constant matrices, $B(\mathcal{U})(\partial_X \mathcal{U}) = B_1(\mathcal{U})\partial_x \mathcal{U} + B_2(\mathcal{U})\partial_y \mathcal{U}$ and

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Convergence results. The most fundamental of our convergence results, Theorem 1, concerns solutions of the systems within Σ. It states that corresponding to any family $\{(V^\varepsilon, \eta^\varepsilon)\}$ of functions consistent with one of the Σ-systems, there exists a family of exact solutions $\{(V^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)\}$ of the relevant system satisfying

$$|(V^\varepsilon, \eta^\varepsilon) - (V^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)|_{L^\infty(0,t;H^s(\mathbb{R}^d))} = O(\varepsilon^2 t)$$

for all $t \in [0, \frac{T}{\varepsilon}]$. This error estimate is easily established owing to the symmetry of the systems in the class Σ. From this central result, we deduce at once that the asymptotic behavior of any family of functions $\{(V^\varepsilon, \eta^\varepsilon)\}$, consistent with one of the systems of class S or S', can be described in terms of an exact solution $\{(V^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)\}$ of one of the symmetric systems of class Σ (see Corollaries 1 and 2).

The principal convergence result concerns the asymptotic behaviour of the full Euler equations (1)–(4). It is demonstrated that for any family $\{(V^\varepsilon, \eta^\varepsilon)\}$ of solutions of the Euler equations and for any system in the class Σ, there exists a family $\{(V^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)\}$ of solution of this Σ-system with the following property. Let $\{(V^\varepsilon_{app}, \eta^\varepsilon_{app})\}$ be the family of functions obtained by applying to $\{(V^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)\}$ an approximate inverse of the change of variables that arose in deriving the class S' from the class S, followed by the inverse of the transformation that arose when deriving elements of the class S from the classical Boussinesq equation. This results in a set of variables that are in principle directly comparable to the Euler variables. Using the preceding theory, we readily deduce that

$$|(V^\varepsilon, \eta^\varepsilon) - (V^\varepsilon_{app}, \eta^\varepsilon_{app})|_{L^\infty(0,t;H^s(\mathbb{R}^d))} = O(\varepsilon^2 t)$$

for all $t \in [0, \frac{T}{\varepsilon}]$ (see Theorems 2 and 3).

Combining these convergence results yields the overall objective, which is that solutions of *any* of the approximate systems discussed here yield good approximations to the full Euler equations on the long-time scale ε^{-1} where nonlinear and dispersive effects can have an order-one relative effect on the velocity field and the wave profile. In particular, these results leave us free to choose an approximate system with good mathematical properties for the modelling task at hand. This freedom can be very helpful when questions of the design of numerical schemes or the imposition of non-homogeneous boundary conditions arise. In many cases, a suitable dispersive perturbation of a symmetric hyperbolic system, the hallmark of the systems in the class Σ, appears to be mathematically very convenient, for example.

A couple of further points are worth emphasizing. Note that for the two dimensional case, for any initial data $(v_0, \eta_0) \in H^s(\mathbb{R})^2$ for suitably large s, there exists a solution of the Euler equations on the relevant time scale. Thus in the two-dimensional case, the asymptotic analysis is complete on the time scale ε^{-1}. In the three-dimensional situation, our theory applies to solutions of the full Euler equations should they exist. Finally, we refer again to Section 5 where we will analyse the approximating power of uncoupled systems as in (6). The approximation estimates mentioned above are superior to those obtained for a pair of uncoupled KdV equations in the absence of specific decay assumptions about the solutions. Moreover, the present theory remains valid in the periodic case, which is not the case for decoupled models.

2. Formal derivation of symmetric systems

2.1. The class S of Bona-Chen-Saut systems

The aim of this section is to recall the derivation of a class of model systems which, in the two-dimensional setting, were put forward recently by BONA, CHEN & SAUT in [8, 9]. We take as our starting point one of the original versions of the Boussinesq system, namely

$$\begin{aligned} &\partial_t V + \nabla\eta + \frac{\varepsilon}{2}\nabla|V|^2 = O(\varepsilon^2), \\ &\partial_t \eta + \nabla\cdot V + \varepsilon\left(\nabla\cdot(\eta V) + \frac{1}{3}\Delta\nabla\cdot V\right) = O(\varepsilon^2), \end{aligned} \tag{8}$$

as $\varepsilon \to 0$, where V denotes the horizontal velocity at the free surface and η the deviation of the free surface from its rest position as before. Henceforth, the notation (V_0, η_0) is reserved for the value of (V, η) at $t = 0$, which is to say, the initial data. As above, ε is the small parameter measuring the amplitude to depth ratio and, on account of the assumption that the Stokes number is exactly equal to 1, the square of the ratio of the depth to a typical wavelength. We give below in Proposition 1 a precise meaning of what is meant by the formal notation $O(\varepsilon^2)$ on the right-hand side of (8).

Elementary potential theory shows that the horizontal velocity of the water at height θ (recall that in the present scaling, $\theta = 1$ at the free surface and $\theta = 0$ at the bottom) is approximately given by V_θ, where

$$V_\theta = \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)^{-1} V, \tag{9}$$

or equivalently,

$$V = \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right) V_\theta.$$

Substituting the relation (9) into (8) leads to the system

$$\begin{aligned} &\partial_t V_\theta + \nabla\eta + \frac{\varepsilon}{2}\left(\nabla|V_\theta|^2 - (1-\theta^2)\Delta\partial_t V_\theta\right) = O(\varepsilon^2), \\ &\partial_t \eta + \nabla\cdot V_\theta + \varepsilon\left(\nabla\cdot(\eta V_\theta) + \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\Delta\nabla\cdot V_\theta\right) = O(\varepsilon^2) \end{aligned} \tag{10}$$

(cf. [8] in the two-dimensional case). Note that the initial data for V_θ is

$$\left(1-\frac{\varepsilon}{2}(1-\theta^2)\Delta\right)^{-1} V_0 := V_{\theta,0}.$$

To introduce the BBM-version (see [3]) of these systems, remark that at the lowest formal order,

$$\begin{aligned}\partial_t V_\theta &= -\nabla\eta + O(\varepsilon),\\ \partial_t\eta &= -\nabla\cdot V_\theta + O(\varepsilon).\end{aligned}$$

As a consequence, the dispersive terms in (10) may be rewritten in the form

$$\begin{aligned}\Delta\partial_t V_\theta &= (1-\mu)\Delta\partial_t V_\theta - \mu\Delta\nabla\eta + O(\varepsilon),\\ \Delta\nabla\cdot V_\theta &= \lambda\Delta\nabla\cdot V_\theta - (1-\lambda)\Delta\partial_t\eta + O(\varepsilon),\end{aligned} \tag{11}$$

without loss of formal accuracy in terms of powers of ε, where λ and μ are two arbitrary real parameters. Using (11) in (10) gives the system

$$\begin{cases}\partial_t V_\theta + \nabla\eta + \varepsilon\left(\frac{1}{2}\nabla|V_\theta|^2 - (1-\mu)\frac{(1-\theta^2)}{2}\Delta\partial_t V_\theta + \mu\frac{(1-\theta^2)}{2}\Delta\nabla\eta\right)\\ \quad = O(\varepsilon^2),\\ \partial_t\eta + \nabla\cdot V_\theta + \varepsilon\left(\nabla\cdot(\eta V_\theta) + \left(\frac{\theta^2}{2}-\frac{1}{6}\right)\lambda\Delta\nabla\cdot V_\theta - \left(\frac{\theta^2}{2}-\frac{1}{6}\right)(1-\lambda)\Delta\partial_t\eta\right)\\ \quad = O(\varepsilon^2).\end{cases}$$

The class S is just all the systems of the form $S_{\theta,\lambda,\mu}$ where $0\leqq\theta\leqq 1$ and μ and λ are arbitrary real numbers. These are written in the compact form

$$S_{\theta,\lambda,\mu}\quad\begin{cases}\partial_t V + \nabla\eta + \varepsilon\left(\frac{1}{2}\nabla|V|^2 + a\Delta\nabla\eta - b\Delta\partial_t V\right) = 0,\\ \partial_t\eta + \nabla\cdot V + \varepsilon\left(\nabla\cdot(\eta V) + c\Delta\nabla\cdot V - d\Delta\partial_t\eta\right) = 0\end{cases}$$

with

$$\begin{aligned} a &= \frac{(1-\theta^2)}{2}\mu, \quad b = \frac{(1-\theta^2)}{2}(1-\mu)\\ c &= \left(\frac{\theta^2}{2}-\frac{1}{6}\right)\lambda, \quad d = \left(\frac{\theta^2}{2}-\frac{1}{6}\right)(1-\lambda)\end{aligned} \tag{12}$$

as in [8] in the two-dimensional case. Note that $a+b+c+d=\frac{1}{3}$, so the collection S really is a three-parameter family.

The first consistency result (in the sense of Definition 1) is provided in the following proposition.

Proposition 1. *Let λ and μ be given and suppose $\theta,\theta_1\in[0,1]$. Let $(V^\varepsilon,\eta^\varepsilon)$ be consistent with the system $S_{\theta,\lambda,\mu}$ and let V_1^ε be defined by*

$$V_1^\varepsilon = \left(1-\frac{\varepsilon}{2}(1-\theta_1^2)\Delta\right)^{-1}\left(1-\frac{\varepsilon}{2}(1-\theta^2)\Delta\right)V^\varepsilon.$$

Then $(V_1^\varepsilon,\eta^\varepsilon)$ is consistent with $S_{\theta_1,\lambda,\mu}$. Moreover, for all $(\lambda_1,\mu_1)\in\mathbb{R}^2$, $(V_1^\varepsilon,\eta^\varepsilon)$ is consistent with $S_{\theta_1,\lambda_1,\mu_1}$.

Proof. This is clear thanks to relation (9). □

As discussed already, for a long wave model for the water-wave problem to be useful, it must have a local existence theory for order-one initial data that provides smooth solutions at least on time intervals of the form $[0,\frac{T}{\varepsilon}]$. Moreover, solutions with fixed initial data in the variables in force here must be bounded with respect to ε in $W^{1,\infty}(0,\frac{T}{\varepsilon};H^s(\mathbb{R}^d))$ for suitably large values of s. For most of the members of the class S, even those which are locally well posed, there is no theory of this nature. (There is a sparse subclass of the systems in S that have a global existence theory and these of course conform to the time-scale requirements. However, even in these cases, there is no analysis showing that the solutions are bounded independently of small values of ε).

This unsatisfactory situation will be rectified by making use of the symmetric systems which are the subject of the next subsection.

2.2. *The class S' of nonlinearly symmetric systems*

In the last subsection, we reviewed how to obtain formally equivalent model systems from the Boussinesq system by making changes of variables that had the effect of modifying the dispersive part of the model. In this section, attention is given to the nonlinear part. The formal zero-dispersion limit of the Boussinesq system (8) is the system

$$\begin{cases}\partial_t V + \nabla\eta + \frac{\varepsilon}{2}\nabla|V|^2 = 0,\\ \partial_t\eta + \nabla\cdot V + \varepsilon\nabla\cdot(\eta V) = 0,\end{cases} \tag{13}$$

of hyperbolic conservation laws. It is handy to write (13) in the form

$$\partial_t\begin{pmatrix}V\\ \eta\end{pmatrix} + A_1(V,\eta)\partial_x\begin{pmatrix}V\\ \eta\end{pmatrix} + A_2(V,\eta)\partial_y\begin{pmatrix}V\\ \eta\end{pmatrix} = 0,$$

where

$$A_1(V,\eta) = \begin{pmatrix}\varepsilon V_1 & \varepsilon V_2 & 1\\ 0 & 0 & 0\\ 1+\varepsilon\eta & 0 & \varepsilon V_1\end{pmatrix}$$

and

$$A_2(V,\eta) = \begin{pmatrix} 0 & 0 & 0 \\ \varepsilon V_1 & \varepsilon V_2 & 1 \\ 0 & 1+\varepsilon\eta & \varepsilon V_2 \end{pmatrix},$$

in the three-dimensional case. In two dimensions, this becomes

$$\partial_t \begin{pmatrix} v \\ \eta \end{pmatrix} + A(v,\eta)\partial_x \begin{pmatrix} v \\ \eta \end{pmatrix} = 0,$$

where

$$A(v,\eta) = \begin{pmatrix} \varepsilon v & 1 \\ 1+\varepsilon\eta & \varepsilon v \end{pmatrix}.$$

Obviously, whatever the space dimension is, these systems of conservation laws are not symmetric. However, a symmetrizer in the fully three-dimensional case is

$$\begin{pmatrix} 1+\varepsilon\eta & 0 & \varepsilon V_1 \\ 0 & 1+\varepsilon\eta & \varepsilon V_2 \\ \varepsilon V_1 & \varepsilon V_2 & 1 \end{pmatrix}.$$

Note that in two dimensions, this symmetrizer specializes to

$$\begin{pmatrix} 1+\varepsilon\eta & \varepsilon v \\ \varepsilon v & 1 \end{pmatrix},$$

but that

$$\begin{pmatrix} 1+\varepsilon\eta & 0 \\ 0 & 1 \end{pmatrix}$$

can also be used.

Independent of the dimension, these symmetrizers are not compatible with the dispersion terms. Therefore, they cannot be used for adducing solutions of the Cauchy problem. This disappointing observation leads us to search for another strategy for obtaining equivalent systems, in the sense of the preceding subsection, that are symmetric with regards to their nonlinear portion. Consider the nonlinear change of variables

$$\tilde V = V\left(1+\frac{\varepsilon}{2}\eta\right) \tag{14}$$

and formally compute the equations satisfied by $\tilde V$ and η:

$$\begin{aligned}
\partial_t \tilde V &= \partial_t V\left(1+\frac{\varepsilon}{2}\eta\right) + \tilde V \frac{\varepsilon}{2}\partial_t\eta + O(\varepsilon^2)\\
&= -[\nabla\eta + \frac{\varepsilon}{2}\nabla|V|^2]\left(1+\frac{\varepsilon}{2}\eta\right) - \frac{\varepsilon}{2}\tilde V\nabla\cdot\tilde V + O(\varepsilon^2)\\
&= -\nabla\eta - \varepsilon\left(\frac{1}{2}\nabla|\tilde V|^2 + \frac{1}{4}\nabla|\eta|^2 + \frac{1}{2}\tilde V\nabla\cdot\tilde V\right) + O(\varepsilon^2).
\end{aligned}$$

It follows that

$$\partial_t \tilde V + \nabla\eta + \varepsilon\left(\frac{1}{2}\nabla|\tilde V|^2 + \frac{1}{4}\nabla|\eta|^2 + \frac{1}{2}\tilde V\nabla\cdot\tilde V\right) = O(\varepsilon^2). \tag{15}$$

Similarly, we compute

$$\begin{aligned}
\partial_t\eta &= -\nabla\cdot V - \varepsilon\nabla\cdot(\eta V) + O(\varepsilon^2)\\
&= -\nabla\cdot\left[\tilde V\left(1-\frac{\varepsilon}{2}\eta\right)\right] - \varepsilon\nabla\cdot(\eta V) + O(\varepsilon^2)\\
&= -\nabla\cdot\tilde V - \varepsilon\left(\nabla\cdot(\eta\tilde V) - \frac{1}{2}\nabla(\eta\tilde V)\right) + O(\varepsilon^2),
\end{aligned}$$

and so it transpires that

$$\partial_t\eta + \nabla\cdot\tilde V + \frac{\varepsilon}{2}\nabla\cdot(\eta\tilde V) = O(\varepsilon^2). \tag{16}$$

The system formed by (15) and (16) is symmetric in the two-dimensional case, but not in three dimensions where $X = (x, y)$. Indeed, it can be written (omitting the tilde) as

$$\begin{aligned}
&\partial_t V_1 + \partial_x\eta + \varepsilon\left(V_1\partial_x V_1 + V_2\partial_x V_2 + \frac{1}{2}\eta\partial_x\eta + \frac{1}{2}V_1(\partial_x V_1 + \partial_y V_2)\right) = O(\varepsilon^2),\\
&\partial_t V_2 + \partial_y\eta + \varepsilon\left(V_1\partial_y V_1 + V_2\partial_y V_2 + \frac{1}{2}\eta\partial_y\eta + \frac{1}{2}V_2(\partial_x V_1 + \partial_y V_2)\right) = O(\varepsilon^2),\\
&\partial_t\eta + \partial_x V_1 + \partial_y V_2 + \frac{\varepsilon}{2}\left(\partial_x\eta V_1 + \eta\partial_x V_1 + \partial_y\eta V_2 + \eta\partial_y V_2\right) = O(\varepsilon^2),
\end{aligned}$$

or equivalently,

$$\partial_t\begin{pmatrix} V_1 \\ V_2 \\ \eta\end{pmatrix} + \begin{pmatrix} \frac{3\varepsilon}{2}V_1 & \varepsilon V_2 & 1+\frac{\varepsilon}{2}\eta \\ \frac{\varepsilon}{2}V_2 & 0 & 0 \\ 1+\frac{\varepsilon}{2}\eta & 0 & \frac{\varepsilon}{2}V_1 \end{pmatrix}\partial_x\begin{pmatrix} V_1 \\ V_2 \\ \eta\end{pmatrix}$$
$$+ \begin{pmatrix} 0 & \frac{\varepsilon}{2}V_1 & 0 \\ \varepsilon V_1 & \frac{3\varepsilon}{2}V_2 & 1+\frac{\varepsilon}{2}\eta \\ 0 & 1+\frac{\varepsilon}{2}\eta & \frac{\varepsilon}{2}V_2 \end{pmatrix}\partial_y\begin{pmatrix} V_1 \\ V_2 \\ \eta\end{pmatrix} = O(\varepsilon^2). \tag{17}$$

At this stage, we apply a 0−curl condition. Indeed, even if curl $\tilde{V} = O(\varepsilon)$ and not exactly 0, we can replace $\frac{\varepsilon}{2}V_2\partial_x V_2$ by $\frac{\varepsilon}{2}V_2\partial_y V_1$ in the first equation of (17) and $\frac{\varepsilon}{2}V_1\partial_y V_1$ by $\frac{\varepsilon}{2}V_1\partial_x V_2$ in the second one. With these substitutions, the system becomes

$$\begin{cases} \partial_t V_1 + \partial_x \eta + \varepsilon\left(\frac{1}{4}\partial_x(\eta^2) + \frac{3}{4}\partial_x(V_1^2) + \frac{1}{4}\partial_x(V_2^2) + \frac{1}{2}\partial_y(V_1V_2)\right) = O(\varepsilon^2), \\ \partial_t V_2 + \partial_y \eta + \varepsilon\left(\frac{1}{4}\partial_y(\eta^2) + \frac{1}{4}\partial_y(V_1^2) + \frac{3}{2}\partial_y(V_2^2) + \frac{1}{2}\partial_x(V_1V_2)\right) = O(\varepsilon^2), \\ \partial_t \eta + \partial_x V_1 + \partial_y V_2 + \frac{\varepsilon}{2}\left(\partial_x(\eta V_1) + \partial_y(\eta V_2)\right) = O(\varepsilon^2). \end{cases} \tag{18}$$

Introduce the new class $S' = S'_{\theta,\lambda,\mu}$ of systems having the form

$$S'_{\theta,\lambda,\mu} \quad \begin{cases} \partial_t V + \nabla\eta + \varepsilon\left(\frac{1}{4}\nabla\eta^2 + \frac{3}{4}\begin{pmatrix}\partial_x(V_1^2)\\ \partial_y(V_2^2)\end{pmatrix}\right. \\ \qquad \left. +\frac{1}{4}\begin{pmatrix}\partial_x(V_2^2)\\ \partial_y(V_1^2)\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\partial_y(V_1V_2)\\ \partial_x(V_1V_2)\end{pmatrix} + a\Delta\nabla\eta - b\Delta\partial_t V\right) = 0, \\ \partial_t\eta + \nabla\cdot V + \varepsilon\left(\frac{1}{2}\nabla(\eta V) + c\Delta\nabla\cdot V - d\Delta\partial_t\eta\right) = 0, \end{cases} \tag{19}$$

where a, b, c, d are as in (12). The previous computations allow us to write the following proposition.

Proposition 2. *If $\{(V^\varepsilon, \eta^\varepsilon)\}$ is consistent with $S_{\theta,\lambda,\mu}$ and nearly irrotational in the sense that* curl $V^\varepsilon = O(\varepsilon)$, *then $\{(\tilde{V}^\varepsilon, \eta^\varepsilon)\}$ is consistent with $S'_{\theta,\lambda,\mu}$, where $\tilde{V}^\varepsilon = V^\varepsilon\left(1 + \frac{\varepsilon}{2}\eta^\varepsilon\right)$.*

The following result is clear at this point.

Proposition 3. *Proposition 1 remains true for systems of class S' instead of S.*

2.3. The symmetric class.

Considered here is the subclass of S' for which $a = c$, $b \geqq 0$ and $d \geqq 0$. This (non-empty!) class is denoted by Σ.

Proposition 4. *Fix θ, λ, μ such that system (19) is in the class Σ. For all $s > \frac{d}{2} + 1$ and all $(V_0, \eta_0) \in (H^s(\mathbb{R}^d))^{d+1}$, there exist $T_0 > 0$ independent of ε and a unique*

$(V, \eta) \in \mathcal{C}\left(\left[0, \frac{T_0}{\varepsilon}\right]; (H^s(\mathbb{R}^d))^{d+1}\right)$ solution of (19) such that $(V,\eta)(t = 0) = (V_0, \eta_0)$.
Moreover, there exists $C_0 > 0$ independent of ε such that

$$|(V,\eta)|_{W^{k,\infty}(0,\frac{T_0}{\varepsilon};H^{s-3k})} \leqq C_0$$

for all k such that $s - 3k > d/2 + 1$.

Furthermore, the quantity

$$\int V^2 + \eta^2 + \varepsilon b|\nabla V|^2 + d\varepsilon|\nabla\eta|^2$$

is independent of t (the integral is taken over all of $\mathbb{R}^d$).

Proof. This is established straightforwardly using standard energy estimates for quasi-linear, symmetric hyperbolic systems. Note that the dispersive part does not interact with these estimates because of the relationships $a = c$, $b \geqq 0$ and $d \geqq 0$. The conservation of the energy follows from the following standard computation. Multiply the system by $\begin{pmatrix}V\\ \eta\end{pmatrix}$ and integrate over all of the space. The result for the nonlinear part of this calculation is zero as the following calculation shows:

$$\begin{aligned} &\int \frac{1}{2}\eta\partial_x\eta V_1 + 3V_1^2\partial_x V_1 + \frac{1}{2}V_2\partial_x V_2 V_1 + \frac{1}{2}\partial_y V_1 V_2 V_1 + \frac{1}{2}V_1\partial_y V_2 V_1 \\ &\quad + \frac{1}{2}\eta\partial_y\eta V_2 + \frac{1}{2}V_1\partial_y V_1 V_2 + 3V_2\partial_y V_2 V_2 + \frac{1}{2}\partial_x V_1 V_2^2 \\ &\quad + \frac{1}{2}V_1\partial_x V_2 V_2 + \frac{1}{2}\eta^2\partial_x V_1 + \frac{1}{2}\partial_x\eta V_1\eta + \frac{1}{2}\eta^2\partial_y V_2 + \frac{1}{2}\partial_y\eta V_2\eta \\ &= \int \eta\partial_x\eta V_1 + \frac{1}{2}\eta^2\partial_x V_1 + \eta\partial_y\eta V_2 + \frac{1}{2}\eta^2\partial_y V_2 \\ &\quad + V_2\partial_x V_2 V_1 + \frac{1}{2}\partial_x V_1 V_2^2 + \partial_y V_1 V_2 V_1 + \frac{1}{2}V_1\partial_y V_2 V_1, \\ &= \frac{1}{2}\int \partial_x(\eta^2)V_1 + \eta^2\partial_x V_1 + \partial_y(\eta^2)V_2 + \eta^2\partial_y V_2 \\ &\quad + \partial_x(V_2^2)V_1 + \partial_x V_1 V_2^2 + \partial_y(V_1^2)V_2 + V_1^2\partial_y V_2 \\ &= 0. \end{aligned}$$

The result follows. □

3. Error estimates

3.1. The symmetric systems

In this section, we establish the fundamental estimates ensuring that any family consistent with a symmetric system is near a solution of that system on time intervals of size $O\left(\frac{1}{\varepsilon}\right)$. We take θ, λ, μ such that the system $S'_{\theta,\lambda,\mu}$ given by (19) is in Σ, that is, it has symmetric dispersive and nonlinear parts. Such a system is written as

$$\partial_t V + \nabla\eta + \varepsilon\left(\frac{1}{4}\nabla\eta^2 + \frac{3}{4}\begin{pmatrix}\partial_x(V_1^2)\\ \partial_y(V_2^2)\end{pmatrix} + \frac{1}{4}\begin{pmatrix}\partial_x(V_2^2)\\ \partial_y(V_1^2)\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\partial_y(V_1V_2)\\ \partial_x(V_1V_2)\end{pmatrix} + a\Delta\nabla\eta - b\Delta\partial_t V\right) = 0, \quad (20)$$

$$\partial_t\eta + \nabla\cdot V + \varepsilon\left(\frac{1}{2}\nabla(\eta V) + a\Delta\nabla\cdot V - d\Delta\partial_t\eta\right) = 0, \quad (21)$$

with a, b, c and d as in (12) and where

$$a = \frac{(1-\theta^2)}{2}\mu = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\lambda = c, \quad b \geqq 0, \quad d \geqq 0. \quad (22)$$

Theorem 1. *Fix λ, μ, θ satisfying (22) and $s \geqq 0$. Let $\{(V^\varepsilon, \eta^\varepsilon)\}$ be a family of functions bounded with respect to ε in $W^{1,\infty}\left(0, \frac{T}{\varepsilon}; H^{s+3}(\mathbb{R}^d)\right)$ for some $T > 0$. If this family is consistent with (20)–(21) in the sense of Definition 1, then there exists a family $\{(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)\}$ of exact solutions of (20)–(21) defined on $\left[0, \frac{T}{\varepsilon}\right]$ and such that*

$$|V^\varepsilon - V_\Sigma^\varepsilon|_{L^\infty(0,t;H^s(\mathbb{R}^d))} + |\eta^\varepsilon - \eta_\Sigma^\varepsilon|_{L^\infty(0,t;H^s(\mathbb{R}^d))} \leqq C\varepsilon^2 t \textit{ for all } t \in \left[0, \frac{T}{\varepsilon}\right].$$

Proof. Write the system (20)–(21) as a dispersive perturbation of an hyperbolic quasi-linear symmetric system, *viz.*

$$\mathcal{L}\begin{pmatrix}V\\ \eta\end{pmatrix} := \partial_t\begin{pmatrix}V\\ \eta\end{pmatrix} + \begin{pmatrix}\nabla\eta\\ \nabla\cdot V\end{pmatrix} + \varepsilon\left(A(V,\eta)\partial_x\begin{pmatrix}V\\ \eta\end{pmatrix} + B(V,\eta)\partial_y\begin{pmatrix}V\\ \eta\end{pmatrix} + a\begin{pmatrix}\Delta\nabla\eta\\ \Delta\nabla\cdot V\end{pmatrix} - \Delta\partial_t\begin{pmatrix}bV\\ d\eta\end{pmatrix}\right) = 0. \quad (23)$$

The assumption of consistency made on $(V^\varepsilon, \eta^\varepsilon)$ means that

$$\mathcal{L}\begin{pmatrix}V^\varepsilon\\ \eta^\varepsilon\end{pmatrix} = \varepsilon^2\begin{pmatrix}r_1^\varepsilon\\ r_2^\varepsilon\end{pmatrix},$$

with $(r_1^\varepsilon, r_2^\varepsilon)$ bounded in $L^\infty\left(0, \frac{T}{\varepsilon}; H^s(\mathbb{R}^d)\right)$. Now let $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ be the solution of (23) such that $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)(t=0) = (V^\varepsilon, \eta^\varepsilon)(t=0)$. This solution is defined at least on $\left[0, \frac{T_0}{\varepsilon}\right]$ with $T_0 > 0$ by Proposition 4. Writing the system satisfied by the difference $(V^\varepsilon - V_\Sigma^\varepsilon, \eta^\varepsilon - \eta_\Sigma^\varepsilon)$ and performing standard energy estimates on it leads to the error given in theorem 1 on $\left[0, \frac{T_1}{\varepsilon}\right]$ where $T_1 = Min\{T_0, T\}$. By the usual scaling arguments, we can take $T_1 = T$. □

3.2. Corollaries

The first corollary shows that the asymptotic behavior of any family $\{(V^\varepsilon, \eta^\varepsilon)\}$ consistent with a system of S' can be described in terms of the solution of one of the systems of Σ, *via* a pseudo-differential change of variables.

Corollary 1. *Let $s \geqq 0$ and $(\underline{\theta}, \underline{\lambda}, \underline{\mu})$ fixed satisfying (22). Assume that there exist a set of parameters (θ, λ, μ) and a smooth enough family $\{(V^\varepsilon, \eta^\varepsilon)\}$ consistent with the system $S'_{\theta,\lambda,\mu} \in S'$ and defined on $\left[0, \frac{T}{\varepsilon}\right]$ for some $T > 0$. Then the system $S'_{\underline{\theta},\underline{\lambda},\underline{\mu}} \in \Sigma$ admits a unique family of solutions $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ defined on $\left[0, \frac{T}{\varepsilon}\right]$ with initial conditions $(V_{\Sigma,0}^\varepsilon, \eta_{\Sigma,0}^\varepsilon)$ defined as*

$$V_{\Sigma,0}^\varepsilon = \left(1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\Delta\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)V^\varepsilon(t=0), \quad \eta_{\Sigma,0}^\varepsilon = \eta^\varepsilon(t=0).$$

Moreover, the net $\{(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)\}$ satisfies the error estimate

$$\left|V^\varepsilon - \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\Delta\right)V_\Sigma^\varepsilon\right|_{L^\infty(0,t;H^s(\mathbb{R}^d))} + |\eta^\varepsilon - \eta_\Sigma^\varepsilon|_{L^\infty(0,t;H^s(\mathbb{R}^d))} \leqq C\varepsilon^2 t$$

for all $t \in [0, \frac{T}{\varepsilon}]$.

Proof. From Proposition 3, it follows that

$$(\widetilde{V}^\varepsilon, \widetilde{\eta}^\varepsilon) := \left(\left(1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\Delta\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)V^\varepsilon, \eta^\varepsilon\right)$$

is consistent with $S'_{\underline{\theta},\underline{\lambda},\underline{\mu}} \in \Sigma$. From Proposition 4 we can deduce the existence of $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ and from Theorem 1, the error estimate between $(\widetilde{V}^\varepsilon, \widetilde{\eta}^\varepsilon)$ and $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$. Inverting the pseudo-differential change of variables then yields the result. □

The same kind of property holds for systems of the original class S, but we must also perform a nonlinear change of variables.

Corollary 2. *Let $s \geqq 0$ and $(\underline{\theta}, \underline{\lambda}, \underline{\mu})$ fixed satisfying (22). Assume that there exist a set of parameters (θ, λ, μ) and a smooth enough family $\{(V^\varepsilon, \eta^\varepsilon)\}$ consistent with the system $S_{\theta,\lambda,\mu} \in S$ and defined on $\left[0, \frac{T}{\varepsilon}\right]$ for some $T > 0$. If, for each ε, V^ε is nearly irrotational in the sense that* curl $V^\varepsilon = O(\varepsilon)$*, then the system $S'_{\underline{\theta},\underline{\lambda},\underline{\mu}} \in \Sigma$ admits a unique family of solutions $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ defined on $\left[0, \frac{T}{\varepsilon}\right]$ with initial conditions $(V_{\Sigma,0}^\varepsilon, \eta_{\Sigma,0}^\varepsilon)$ defined as*

$$(V_{\Sigma,0}^\varepsilon, \eta_{\Sigma,0}^\varepsilon) = \left(\widetilde{V}_0^\varepsilon\left(1 + \frac{\varepsilon}{2}\eta^\varepsilon(t=0)\right), \eta^\varepsilon(t=0)\right),$$

with

$$\widetilde{V}_0^\varepsilon = \left(1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\Delta\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right) V^\varepsilon(t=0).$$

Moreover, there exists $\varepsilon_0 > 0$ *such that for all* ε *with* $0 < \varepsilon < \varepsilon_0$,

$$\left|V^\varepsilon - \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\Delta\right)\left(V_\Sigma^\varepsilon(1 - \frac{\varepsilon}{2}\eta_\Sigma^\varepsilon)\right)\right|_{L^\infty(0,t;H^s(\mathbb{R}^d))} + \left|\eta^\varepsilon - \eta_\Sigma^\varepsilon\right|_{L^\infty(0,t;H^s(\mathbb{R}^d))} \leqq C\varepsilon^2 t$$

for all $t \in [0, \frac{T}{\varepsilon}]$.

Proof. From Proposition 1, it follows that $\{(\widetilde{V}^\varepsilon, \widetilde{\eta}^\varepsilon)\}$, as defined in the proof of Corollary 1, is consistent with $S_{\underline{\theta},\underline{\lambda},\underline{\mu}} \in S$. From Proposition 9, we then deduce that one family $\{(\widetilde{\widetilde{V}}^\varepsilon, \widetilde{\widetilde{\eta}}^\varepsilon)\}$ defined as

$$\widetilde{\widetilde{V}}^\varepsilon = \widetilde{V}\left(1 + \frac{\varepsilon}{2}\widetilde{\eta}\right), \qquad \widetilde{\widetilde{\eta}}^\varepsilon = \widetilde{\eta},$$

is consistent with $S'_{\underline{\theta},\underline{\lambda},\underline{\mu}} \in \Sigma$. From Proposition 4 we can deduce the existence of $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ and from Theorem 1, the error estimate between $(\widetilde{\widetilde{V}}^\varepsilon, \widetilde{\widetilde{\eta}}^\varepsilon)$ and $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$. Inverting the nonlinear change of variables (which is possible for ε small enough) and then the pseudo-differential one yields the result. □

Remark 1. The irrotationality condition imposed in Corollary 2 is not necessary in the one-dimensional situation in which there is a single horizontal variable since the nonlinear change of variables symmetrizes the hyperbolic part of the Boussinesq system without using the fact that the curl of the velocity field vanishes (as seen in Section 2.2).

3.3. The main result

The aim of this section is to state and prove the theorem concerning the approximation of solutions of the Euler equations. We choose to work with a formulation of the Euler equations alternative to (1)–(4). This new formulation is a system of partial differential equations coupling η, the deviation of the free surface from its rest position, to $V = \nabla\psi$, where ψ denotes the values of the velocity potential at the free surface, $\psi(t,X) = \phi(t,X,1+\varepsilon\eta)$. This formulation, which involves a Dirichlet-to-Neumann operator G_ε, is derived in Section 4.1. The system in question is

$$\partial_t V - \varepsilon\nabla\left(\partial_t\eta G_\varepsilon(\eta)\psi\right) + \nabla\eta + \frac{\varepsilon}{2}\nabla\left(|V - \varepsilon\nabla\eta G_\varepsilon(\eta)\psi|^2\right) + \nabla|G_\varepsilon(\eta)\psi|^2 = 0 \tag{24}$$

for $X \in \mathbb{R}^d$ and $t > 0$, and

$$\partial_t\eta + \varepsilon\nabla\eta\cdot(V - \varepsilon\nabla\eta G_\varepsilon(\eta)\psi) = \frac{1}{\varepsilon}\nabla(G_\varepsilon(\eta)\psi), \tag{25}$$

where $G_\varepsilon(\eta)\psi$ is given by

$$G_\varepsilon(\eta)\psi = \partial_z\phi(t,X,1+\varepsilon\eta).$$

Of course, once the free surface is assumed to be known, the velocity potential ϕ satisfies the elliptic system

$$\begin{cases} \varepsilon\Delta\phi + \partial_z^2\phi = 0, & X\in\mathbb{R}^d,\quad 0 < z < 1+\varepsilon\eta, \\ \partial_z\phi = 0 & \text{at } z = 0,\quad X\in\mathbb{R}^d, \\ \phi(X, 1+\varepsilon\eta) = \psi(t,X). \end{cases} \tag{26}$$

If $(V^\varepsilon, \eta^\varepsilon)$ is a solution of (24)–(25) with initial data $(V_0^\varepsilon, \eta_0^\varepsilon)$, we construct what will be called an approximate solution $(V_{app}^\varepsilon, \eta_{app}^\varepsilon)$ of (24)–(25) in the following way. First, consider $(\widetilde{V_0}^\varepsilon, \widetilde{\eta_0}^\varepsilon)$ given by

$$\widetilde{V_0}^\varepsilon = \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)^{-1} V_0^\varepsilon, \qquad \widetilde{\eta_0}^\varepsilon = \eta_0^\varepsilon.$$

Then, take $(V_{\Sigma,0}^\varepsilon, \eta_{\Sigma,0}^\varepsilon)$ to be

$$(V_{\Sigma,0}^\varepsilon, \eta_{\Sigma,0}^\varepsilon) = \left(\widetilde{V_0}^\varepsilon(1 + \frac{\varepsilon}{2}\eta_0^\varepsilon), \eta_0^\varepsilon\right). \tag{27}$$

Choose the parameters $(\underline{\theta}, \underline{\lambda}, \underline{\mu})$ such that the system $S'_{\underline{\theta},\underline{\lambda},\underline{\mu}}$ belongs to the class Σ (i.e., it is completely symmetric); in other words, choose the parameters in such a way that (20)–(21)–(22) are satisfied. Let $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)$ be its solution with initial data $(V_\Sigma^\varepsilon, \eta_\Sigma^\varepsilon)(t=0) = (V_{\Sigma,0}^\varepsilon, \eta_{\Sigma,0}^\varepsilon)$. This solution exists and is bounded with respect to ε in $L^\infty\left(0, \frac{T}{\varepsilon}; H^s\right)$ for some $T > 0$ by Proposition 4.

From this family of solutions of the symmetric system $S'_{\underline{\theta},\underline{\lambda},\underline{\mu}}$, we obtain our approximate solution of the Euler equations by inverting approximately the nonlinear change of variables, and then the pseudo-differential one, *viz.*

$$\begin{cases} V_{app}^\varepsilon = \left(1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\Delta\right)\left[V_\Sigma^\varepsilon\left(1 - \frac{\varepsilon}{2}\eta_\Sigma^\varepsilon\right)\right], \\ \eta_{app}^\varepsilon = \eta_\Sigma^\varepsilon. \end{cases} \tag{28}$$

Theorem 2. (The two–dimensional case). *Let* $s \geqq 0$ *and let* $\{(v_0^\varepsilon, \eta_0^\varepsilon)\}$ *be a bounded family in* $H^\sigma(\mathbb{R})^2$ ($\sigma \geqq s$ *large enough). There exist* $T > 0$ *and* $\varepsilon_0 > 0$ *such that the following holds. There is a unique family of solutions* $\{(v^\varepsilon, \eta^\varepsilon)\}$ *of the Euler equations* (24)–(25) *with initial conditions* $(v_0^\varepsilon, \eta_0^\varepsilon)$, *and for all* ε *with* $0 < \varepsilon < \varepsilon_0$,

$$\left|v^\varepsilon - v_{app}^\varepsilon\right|_{L^\infty(0,t;H^s)} + \left|\eta^\varepsilon - \eta_{app}^\varepsilon\right|_{L^\infty(0,t;H^s)} \leqq c\varepsilon^2 t$$

for all $t \in [0, \frac{T}{\varepsilon}]$, *where* $(v_{app}^\varepsilon, \eta_{app}^\varepsilon)$ *is given by* (28).

Theorem 3. (The three–dimensional case). *Let $s \geqq 0$ and let $\{(V_0^\varepsilon, \eta_0^\varepsilon)\}$ be a bounded family in $H^\sigma(\mathbb{R}^2)^2$ ($\sigma \geqq s$ large enough) such that* curl $V_0^\varepsilon = 0$. *Let $\{(V^\varepsilon, \eta^\varepsilon)\}$ be a family of solutions of the Euler equations* (24)–(25) *with initial conditions $(V_0^\varepsilon, \eta_0^\varepsilon)$ and bounded with respect to ε in $W^{1,\infty}\left(0, \frac{T}{\varepsilon}; H^\sigma(\mathbb{R}^2)\right)$. Then, there exist $T > 0$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$\left|V^\varepsilon - V^\varepsilon_{app}\right|_{L^\infty(0,t;H^s)} + \left|\eta^\varepsilon - \eta^\varepsilon_{app}\right|_{L^\infty(0,t;H^s)} \leqq c\varepsilon^2 t$$

for all $t \in [0, \frac{T}{\varepsilon}]$, where $(V^\varepsilon_{app}, \eta^\varepsilon_{app})$ is given by (28).

Proof. In the two-dimensional case, the existence part follows from [14] or [23]. The key point of the proof is that the solutions of the Euler system are consistent with the Boussinesq system (8). This fact is proved in Theorem 6. Since this latter system belongs to the class S (it can be written in the form $S_{\theta,\lambda,\mu}$ with $\theta = \lambda = 1$), the results of both theorems can be deduced from Corollary 2. □

Comments:

1. This result is not only a consistency result but a true convergence result with improved error bounds. The available error estimates for KdV-type decoupled systems such as in [23] lead only to errors of order $O(\varepsilon^{1/4})$ in the scaling adopted here. We improve this result in Section 5 by showing that the error estimate for KdV-type uncoupled approximations is $O(\varepsilon)$, and that this estimate is sharp. It is clear that for short times, the estimate given in Theorems 2 and 3 is more precise. Moreover, both theorems remain valid in the periodic framework (provided of course that solutions of the Euler equations exist over the relevant time scales), which is not the case for uncoupled approximations.

2. Among all the systems of class S and S', only those belonging to the class Σ are well understood in terms of the Cauchy problem on a time interval of size $\left[0, O\left(\frac{1}{\varepsilon}\right)\right]$. This is why we constructed our approximate solution $(V^\varepsilon_{app}, \eta^\varepsilon_{app})$ in terms of the solution $(V^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)$ of such a symmetric system $S'_{\theta,\lambda,\mu}$. However, an approximate solution can be constructed from any family of solutions consistent with one of the systems of S, or S'. Indeed, if $\{(V^\varepsilon_1, \eta^\varepsilon_1)\}$ is consistent with $S_{\theta,\lambda,\mu}$ and coincides with the family of solutions $\{(V^\varepsilon, \eta^\varepsilon)\}$ of the Euler equations at $t = 0$, then we can define $(V^\varepsilon_{app,1}, \eta^\varepsilon_{app,1})$ to be

$$V^\varepsilon_{app,1} = \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right) V^\varepsilon_1, \quad \eta^\varepsilon_{app,1} = \eta^\varepsilon_1.$$

From Proposition 1, $\{(V^\varepsilon_{app,1}, \eta^\varepsilon_{app,1})\}$ is consistent with the Boussinesq system (8). Therefore, thanks to Corollary 2, we can replace $(V^\varepsilon, \eta^\varepsilon)$ by $(V^\varepsilon_{app,1}, \eta^\varepsilon_{app,1})$ in the statement of Theorem 1. By the triangle inequality, this result, together with Theorems 2 and 3 yields the following corollary.

Corollary 3. *If the family $\{(V^\varepsilon, \eta^\varepsilon)\}$, defined on $[0, \frac{T}{\varepsilon}]$ for some $T > 0$, solves the Euler equations and if, for all the relevant $\varepsilon > 0$,* curl $V^\varepsilon(t = 0) = 0$, *then*

$$\left|V^\varepsilon - V^\varepsilon_{app,1}\right|_{L^\infty(0,t;H^s)} + \left|\eta^\varepsilon - \eta^\varepsilon_{app,1}\right|_{L^\infty(0,t;H^s)} \leqq c\varepsilon^2 t$$

for all $t \in [0, \frac{T}{\varepsilon}]$.

Similarly, if $\{(V^\varepsilon_2, \eta^\varepsilon_2)\}$ is consistent with $S'_{\theta,\lambda,\mu}$ and coincides with the solutions $\{(V^\varepsilon, \eta^\varepsilon)\}$ of the Euler equations at $t = 0$, then for each $\varepsilon > 0$, we can define $(V^\varepsilon_{app,2}, \eta^\varepsilon_{app,2})$ as

$$V^\varepsilon_{app,2} = \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\Delta\right)\left(V^\varepsilon_2\left(1 - \frac{\varepsilon}{2}\eta^\varepsilon_2\right)\right), \qquad \eta^\varepsilon_{app,2} = \eta^\varepsilon_2.$$

With the same method as above, we obtain the following corollary.

Corollary 4. *If $\{(V^\varepsilon, \eta^\varepsilon)\}$, defined on $[0, \frac{T}{\varepsilon}]$ for some $T > 0$, solves the Euler equations and if* curl $V^\varepsilon(t = 0) = 0$, *then*

$$\left|V^\varepsilon - V^\varepsilon_{app,2}\right|_{L^\infty(0,t;H^s)} + \left|\eta^\varepsilon - \eta^\varepsilon_{app,2}\right|_{L^\infty(0,t;H^s)} \leqq c\varepsilon^2 t$$

for all $t \in [0, \frac{T}{\varepsilon}]$.

3. It follows from Corollaries 3 and 4 that all the formal approximate systems of the class S of Bona *et al.* and of the new class S' are justified rigorously. This comment can sometimes lead to a spectacular conclusion. For example, the original system due to Boussinesq is ill-posed, but nonetheless, we have a convergence result! More precisely, we can prove that any family of solutions of the Euler equations existing over times $O(1/\varepsilon)$, is well approximated as $\varepsilon \to 0$ by any family of functions consistent with the Boussinesq system and having appropriate initial values.

Theorem 4. *Let $\{(V^\varepsilon, \eta^\varepsilon)\}$ and $\{(V^\varepsilon_{Bous}, \eta^\varepsilon_{Bous})\}$ be two sufficiently smooth, appropriately bounded families defined on $[0, \frac{T}{\varepsilon}]$, with V^ε_{Bous} and V^ε irrotational functions. Suppose that these families are consistent with the Euler equations and the Boussinesq system* (8) *respectively. Then for all $t \in [0, \frac{T}{\varepsilon}]$,*

$$\left|V^\varepsilon - V^\varepsilon_{Bous}\right|_{L^\infty(0,t;H^s)} + \left|\eta^\varepsilon - \eta^\varepsilon_{Bous}\right|_{L^\infty(0,t;H^s)} \leqq c\varepsilon^2 t.$$

4. Note that the way we prove the error estimates is a little unusual. Indeed, the method used is the opposite of the standard approach in that we consider the solution of the Euler equations (that is the "exact solution") as an approximate solution of the symmetric Boussinesq system $\Sigma_{\theta,\lambda,\mu}$, rather than the other way around. The error estimate which is the center of our analysis is obtained by referring to this Boussinesq system. The crux of the matter is that it is not so difficult to analyze approximate solutions to a completely symmetric system.

5. Another approach to improving the error estimates for the uncoupled KdV approximation (6) for the water-wave problem consists of computing the next order terms in the WKB expansion. This method, studied by WAYNE & WRIGHT [24] for a model problem (which is in fact a Boussinesq system) has the advantage of giving an $O(\varepsilon^2)$ error term, but appears not to extend to the periodic case. Of course, higher-order Boussinesq approximations are also available and would have the same type of formal error estimates when compared with solutions of the full Euler equations (see, e.g., [7]).

4. Consistency of the Euler equations with the Boussinesq system

4.1. Statement of the problem

The aim of this section is to prove the consistency of any solution of the Euler equations with the Boussisnesq systems. For convenience, let us recall equations (1)–(4), which are the Euler equations in the long-wave small-amplitude scaling, with Stokes number equal to 1, namely

$$\varepsilon\Delta\phi + \partial_z^2\phi = 0 \qquad \text{for } 0 \leqq z \leqq 1+\varepsilon\eta, \tag{29}$$

$$\partial_z\phi = 0 \qquad \text{at } z = 0, \tag{30}$$

$$\partial_t\phi + \frac{1}{2}\left(\varepsilon|\nabla\phi|^2 + |\partial_z\phi|^2\right) + \eta = 0 \qquad \text{at } z = 1+\varepsilon\eta, \tag{31}$$

$$\partial_t\eta + \varepsilon\nabla\phi\cdot\nabla\eta = \frac{1}{\varepsilon}\partial_z\phi \qquad \text{at } z = 1+\varepsilon\eta, \tag{32}$$

where Δ and ∇ denote the usual Laplace and gradient operators in the transverse variable $X \in \mathbb{R}^d$, $X = (x, y)$ if $d = 2$ and $X = x$ if $d = 1$. By the earlier works of CRAIG [14] and SCHNEIDER & WAYNE [23], we know that for $d = 1$ there exists a family of smooth solutions of (29)–(32) defined on $\left[0, \frac{T}{\varepsilon}\right]$ such that $(\nabla_{X,z}, \phi, \zeta)$ is bounded (with respect to ε) in $W^{k,\infty}\left(0, \frac{T}{\varepsilon}; H^s(\mathbb{R})\right)$ for k and s large enough. Following [15], we take $\psi_\varepsilon(t, X) := \phi(t, X, 1+\varepsilon\eta(t, X))$ as a new unknown; as mentioned before, this quantity is the velocity potential at the free surface. A standard approach (see, e.g., [21, 15, 16, 25, 23]), is to write the equations satisfied by η and ψ_ε. To this end, we need to use the Dirichlet-to-Neumann operator which basically expresses the normal velocity at the free surface in terms of the value of the potential at the free surface. Since the normal velocity can be deduced from $\partial_z\phi$, it is natural to consider the operator which maps ψ to $\partial_z\phi|_{z=1+\varepsilon\eta}$. More precisely, for any $f \in (C^1 \cap W^{1,\infty})(\mathbb{R}^d)$ and for any ε such that $0 < 1 - \varepsilon|f|_\infty$, we define the operator $G_\varepsilon(f)$, which acts on $H^{3/2}(\mathbb{R}^d)$ and has values in $H^{1/2}(\mathbb{R}^d)$, to be

$$G_\varepsilon(f)g = \partial_z u(X, 1+\varepsilon f), \tag{33}$$

where u is the solution of

$$\varepsilon\Delta u + \partial_z^2 u = 0 \qquad \text{for } 0 \leqq z \leqq 1+\varepsilon f, \quad X \in \mathbb{R}^d, \tag{34}$$

$$\partial_z u = 0 \qquad \text{at } z = 0, \quad X \in \mathbb{R}^d, \tag{35}$$

$$u(X, 1+\varepsilon f) = g, \quad X \in \mathbb{R}^d. \tag{36}$$

Before rewriting equations (29)–(32) using this operator, we need to compute the derivatives of ϕ in terms of ψ and η. Simple computations yield

$$\partial_t\phi|_{z=1+\varepsilon\eta} = \partial_t\psi - \varepsilon\partial_t\eta G_\varepsilon(\eta)\psi, \tag{37}$$

$$\nabla\phi|_{1+\varepsilon\eta} = \nabla\psi - \varepsilon\nabla\eta G_\varepsilon(\eta)\psi. \tag{38}$$

Thanks to (37)–(38), it is straightforward to rewrite (29)–(32) as

$$\partial_t\psi - \varepsilon\partial_t\eta G_\varepsilon(\eta)\psi + \eta + \frac{\varepsilon}{2}|\nabla\psi - \varepsilon\nabla\eta G_\varepsilon(\eta)\psi|^2 + \frac{1}{2}|G_\varepsilon(\eta)\psi|^2 = 0, \tag{39}$$

$$\partial_t\eta + \varepsilon\nabla\eta\cdot[\nabla\psi - \varepsilon\nabla\eta G_\varepsilon(\eta)\psi] = \frac{1}{\varepsilon}G_\varepsilon(\eta)\psi, \tag{40}$$

with $X \in \mathbb{R}^d$ and $t \in [0, \frac{T}{\varepsilon}]$.

The next step is to obtain an asymptotic expansion of the operator G_ε as $\varepsilon \to 0$. This is the goal of the next theorem.

Theorem 5. *For any f and g smooth enough, define $G_1(f)g = -\Delta g$ and $G_2(f)g = -\frac{1}{3}\Delta^2 g - f\Delta g$. For any $s \in \mathbb{N}$, there exist $\sigma \in \mathbb{N}$, $\sigma \geqq s$ and $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$, if $(f, \nabla g) \in H^\sigma(\mathbb{R}^d)^{1+d}$, then*

$$|G_\varepsilon(f)g - \varepsilon G_1(f)g - \varepsilon^2 G_2(f)g|_{H^s(\mathbb{R}^d)} \leqq \varepsilon^3 C(|f|_{H^\sigma}, |\nabla g|_{H^\sigma}),$$

where C is a continuous function of its arguments.

Remark 2. This theorem could certainly be obtained using the estimates of the Dirichlet to Neumann operators in the works of CRAIG [14] and SCHNEIDER & WAYNE [23]. However our proof is interesting since it is simpler and uses directly the elliptic equations. We postpone it to Section 4.3. A similar approach is used in [22].

4.2. Asymptotic expansion of solutions of the Euler equations

The aim of this section is to give an asymptotic expansion of the solutions of (39)–(40). This is accomplished in the following theorem, the gist of which is that any solution of (39)–(40) is consistent with the Boussinesq system.

Theorem 6. *Let $s \geqq 0$. Let $\{(\psi^\varepsilon, \eta^\varepsilon)\}$ be a family of solutions of (39)–(40). There exists $\sigma \in \mathbb{N}$ such that if $(\nabla\psi^\varepsilon, \eta^\varepsilon)$ is bounded in $W^{1,\infty}(0, \frac{T}{\varepsilon}; H^\sigma(\mathbb{R}^d))$, then*

$$\partial_t V^\varepsilon + \nabla\eta^\varepsilon + \frac{\varepsilon}{2}\nabla(|V^\varepsilon|^2) = \varepsilon^2 r_1^\varepsilon,$$

$$\partial_t\eta^\varepsilon + \nabla\cdot V^\varepsilon + \varepsilon\left(\nabla\cdot(\eta^\varepsilon V^\varepsilon) + \frac{1}{3}\Delta\nabla\cdot V^\varepsilon\right) = \varepsilon^2 r_2^\varepsilon,$$

where $V^\varepsilon := \nabla\psi^\varepsilon$, and $r_1^\varepsilon, r_2^\varepsilon$ are bounded in $L^\infty(0, \frac{T}{\varepsilon}; H^s(\mathbb{R}^d))$, independently of $\varepsilon > 0$.

Remark 3. From previous works [14, 23] it is known when $d = 1$ that equations (39)–(40) are well posed. Therefore the assumption made in Theorem 6 on the existence and regularity of $(\nabla\psi^\varepsilon, \eta^\varepsilon)$ reduces to a simple assumption of regularity on the initial conditions (ψ_0, η_0) taken for (39)–(40).

Proof. Let $\{(\psi^\varepsilon, \eta^\varepsilon)\}$ be as in the statement of the theorem. Using Theorem 5, we have

$$G_\varepsilon(\eta^\varepsilon)\psi^\varepsilon + \varepsilon\Delta\psi^\varepsilon + \varepsilon^2\left(\frac{1}{3}\Delta^2\psi^\varepsilon + \eta^\varepsilon\Delta\psi^\varepsilon\right) = O(\varepsilon^3), \tag{41}$$

where the $O(\varepsilon^3)$ error term is taken relative to the $L^\infty\left(0, \frac{T}{\varepsilon}; H^s(\mathbb{R}^d)\right)$ norm. Substituting (41) into (39), taking the gradient of the equation thus obtained, and keeping only the terms of order $O(\varepsilon)$ leads to the conclusion that

$$\partial_t V^\varepsilon + \nabla\eta^\varepsilon + \frac{\varepsilon}{2}\nabla(|V^\varepsilon|^2) = O(\varepsilon^2), \qquad X \in \mathbb{R}^d, \quad t \in \left[0, \frac{T}{\varepsilon}\right], \tag{42}$$

with $V^\varepsilon := \nabla\psi^\varepsilon$. Similarly, substituting (41) into (40) leads to

$$\partial_t\eta^\varepsilon + \nabla\cdot V^\varepsilon + \varepsilon\left(\nabla\cdot(\eta^\varepsilon V^\varepsilon) + \frac{1}{3}\Delta\nabla\cdot V^\varepsilon\right) = O(\varepsilon^2). \tag{43}$$

Note that the error terms $O(\varepsilon^2)$ are relative to the space $L^\infty\left(0, \frac{T}{\varepsilon}; H^s(\mathbb{R}^d)\right)$ for any chosen s, provided that σ is large enough. □

4.3. Expansion of G_ε

This subsection is devoted to the proof of Theorem 5. Recall that we are concerned with the operator G_ε defined by (33)–(36). Applying a change of variables, the problem may be reduced to one set on an horizontal strip: let $\widetilde{u}$ be defined on $\mathcal{S} := \{X \in \mathbb{R}^d, z \in [0, 1]\}$ by

$$\widetilde{u}(X, z) := u(X, z(1 + \varepsilon f)), \tag{44}$$

where u and f are as in (33)–(36). It follows immediately that for all $X \in \mathbb{R}^d$ and all $z \in [0, 1 + \varepsilon f]$, we have $u(X, z) = \widetilde{u}\left(X, \frac{z}{1+\varepsilon f}\right)$. We now deduce from (33)–(36) the equations satisfied by $\widetilde{u}$. Equations (35), (36) and (33) yield, respectively,

$$\partial_z\widetilde{u}(X, 0) = 0, \qquad X \in \mathbb{R}^d, \tag{45}$$

$$\widetilde{u}(X, 1) = g(X), \qquad X \in \mathbb{R}^d, \tag{46}$$

$$G_\varepsilon(f)(g) = \frac{1}{1+\varepsilon f}\partial_z\widetilde{u}(X, 1), \qquad X \in \mathbb{R}^d. \tag{47}$$

To use (34), we must compute $\partial_z^2 u$ and Δu in terms of $\widetilde{u}$. Introducing $\chi_\varepsilon := \dfrac{1}{1+\varepsilon f}$, we see that

$$\partial_z^2 u = \chi_\varepsilon^2\partial_z^2\widetilde{u},$$
$$\Delta u = \Delta\widetilde{u} + z\Delta\chi_\varepsilon\partial_z\widetilde{u} + 2z\nabla\partial_z\widetilde{u}\cdot\nabla\chi_\varepsilon + z^2|\nabla\chi_\varepsilon|^2\partial_z^2\widetilde{u},$$

and equation (34) therefore becomes

$$\varepsilon\Delta\widetilde{u} + \left(\chi_\varepsilon^2 + \varepsilon z^2|\nabla\chi_\varepsilon|^2\right)\partial_z^2\widetilde{u} + 2\varepsilon z\nabla\chi_\varepsilon\cdot\nabla\partial_z\widetilde{u} + \varepsilon z\Delta\chi_\varepsilon\partial_z\widetilde{u} = 0. \tag{48}$$

Now, define

$$\begin{aligned}\widetilde{u}_0 &:= g,\\ \widetilde{u}_1 &:= -\frac{(z^2-1)}{2}\Delta g,\\ \widetilde{u}_2 &:= \frac{\Delta^2 g}{4}\left(\frac{z^4}{6} - z^2 + \frac{5}{6}\right) - f\Delta g(z^2-1).\end{aligned} \tag{49}$$

The principal obstacle to establishing Theorem 5 is overcome via the following preparatory result.

Proposition 5. *For any $s \in \mathbb{N}$, there exist $\sigma \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for all positive $\varepsilon < \varepsilon_0$ and all $(f, g) \in H^\sigma(\mathbb{R}^d)^2$, the solution $\widetilde{u}$ of* (45), (46) *and* (48) *satisfies*

$$\widetilde{u} - \left(\widetilde{u}_0 + \varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2\right) = O(\varepsilon^3) \quad \text{in } H^s(\mathcal{S}),$$

where $\widetilde{u}_0$, $\widetilde{u}_1$ and $\widetilde{u}_2$ are as in (49).

Proof. To prove this proposition, we proceed as follows:

(i) In a first step, we construct an approximate solution of (45), (46) and (48), up to order $O(\varepsilon^3)$;
(ii) In the second step, we perform energy estimates on the difference between the exact and approximate solutions.

Step 1. Construction of the approximate solutions

The energy estimates which will be used in Step 2 use the principal part of (48), which is $\varepsilon\Delta\widetilde{u} + \left(\chi_\varepsilon^2 + \varepsilon z^2|\nabla\chi_\varepsilon|^2\right)\partial_z^2\widetilde{u}$. Because of the coefficient ε in front of $\Delta\widetilde{u}$, the error estimate on X derivatives are one order worse (in terms of ε) than those on z-derivatives. For this reason, it is helpful to construct approximate solutions which are good to one order further than $O(\varepsilon^2)$, that is, to order $O(\varepsilon^3)$. We therefore make the ansatz

$$\widetilde{u}_a = \widetilde{u}_0 + \varepsilon\widetilde{u}_1 + \varepsilon^2\widetilde{u}_2 + \varepsilon^3\widetilde{u}_3,$$

for the approximate solutions, and substitute it into (48) assuming that $\partial_z\widetilde{u}_i = 0$ at $z = 0$ for $i = 0\ldots3$, $\widetilde{u}_0 = g$ at $z = 1$ and $\widetilde{u}_i = 0$ at $z = 1$ for $i = 1\ldots3$. At order ε^0, we find $\partial_z^2\widetilde{u}_0 = 0$, which, with the boundary conditions on $\widetilde{u}_0$, yields $\widetilde{u}_0 = g$.

To expand (48) in powers of ε up to order 3, we need to expand χ_ε and its derivatives, *viz.*

$$\begin{aligned}\chi_\varepsilon &= 1 - \varepsilon f + \varepsilon^2 f^2 + O(\varepsilon^3),\\ |\chi_\varepsilon|^2 &= 1 - 2\varepsilon f + 3\varepsilon^2 f^2 + O(\varepsilon^3),\\ |\nabla\chi_\varepsilon|^2 &= \varepsilon^2|\nabla f|^2 + O(\varepsilon^3),\\ \Delta\chi_\varepsilon &= -\varepsilon\Delta f + O(\varepsilon^2).\end{aligned}$$

Using these expressions and neglecting the terms of order $O(\varepsilon^4)$ and higher in (48) gives an equation of the form $\varepsilon P_1+\varepsilon^2 P_2+\varepsilon^3 P_3 = O(\varepsilon^4)$. Choosing $\widetilde{u}_1$, $\widetilde{u}_2$ and $\widetilde{u}_3$ to cancel P_1, P_2 and P_3 yields three equations. The first one, namely $P_1 = 0$, yields

$$\Delta\widetilde{u}_0 + \partial_z^2\widetilde{u}_1 = 0; \tag{50}$$

the second one, $P_2 = 0$, gives

$$\Delta\widetilde{u}_1 + \partial_z^2\widetilde{u}_2 - 2f\partial_z^2\widetilde{u}_1 = 0; \tag{51}$$

and finally $P_3 = 0$ is the same as

$$\Delta\widetilde{u}_2 + \partial_z^2\widetilde{u}_3 - 2f\partial_z^2\widetilde{u}_2 + 3f^2\partial_z^2\widetilde{u}_1 - 2z\nabla f\partial_z\nabla\widetilde{u}_1 - z^2\Delta f\partial_z\widetilde{u}_1 = 0. \tag{52}$$

Using the boundary conditions, these equations can be solved; for (50), it transpires that

$$\widetilde{u}_1 = -\frac{\Delta g}{2}(z^2-1). \tag{53}$$

For $\widetilde{u}_2$, (51) gives

$$\partial_z^2\widetilde{u}_2 = \frac{\Delta^2\widetilde{u}_0}{2}(z^2-1) - 2f\Delta\widetilde{u}_0, \tag{54}$$

and hence

$$\widetilde{u}_2 = \frac{\Delta^2 g}{4}\left(\frac{z^4}{6} - z^2 + \frac{5}{6}\right) - f\Delta g(z^2-1). \tag{55}$$

Equation (52) also allows the determination of the value of $\widetilde{u}_3$. However, this quantity is of no interest in the present context, and consequently the computation is omitted.

The next task is to understand just how close the families of approximate solutions just constructed are to the exact solutions.

Step 2. Error estimates

Introduce now the difference between the exact and approximate solutions of (48), namely $w := \widetilde{u} - \widetilde{u}_a$. Then w satisfies

$$\varepsilon\Delta w + \underbrace{\left(\chi_\varepsilon^2 + \varepsilon z^2|\nabla\chi_\varepsilon|^2\right)}_{:=a_\varepsilon(X,z)}\partial_z^2 w + 2\varepsilon\underbrace{(z\nabla\chi_\varepsilon)}_{:=\varepsilon b_\varepsilon(X,z)}\cdot\partial_z\nabla w + \varepsilon\underbrace{(z\Delta\chi_\varepsilon)}_{:=\varepsilon c_\varepsilon(X,z)}\partial_z w = O(\varepsilon^4), \tag{56}$$

with $\partial_z w = 0$ at $z = 0$, $w = 0$ at $z = 1$. Remark that a_ε, b_ε and c_ε are uniformly bounded in $W^{k,\infty}(\mathcal{S})$, k being as large as we wish provided that σ is large enough. Moreover, in $W^{k,\infty}(\mathcal{S})$, we have $a_\varepsilon = 1+O(\varepsilon)$. As above, $\mathcal{S}$ is the strip $\mathbb{R}^d\times[0,1]$.

We first prove the estimate of Proposition 5 for $s = 1$ and $s = 2$, and then prove the general result by induction. Multiply (56) by w and integrate the result over the strip $\mathcal{S} = \{X\in\mathbb{R}^d, z\in[0,1]\}$ to reach the formula

$$\begin{aligned}-\varepsilon\int|\nabla w|^2 - \int a_\varepsilon|\partial_z w|^2 - \int\partial_z a_\varepsilon\partial_z ww - 2\varepsilon^2\int b_\varepsilon\cdot\nabla w\partial_z w\\ -2\varepsilon^2\int\nabla\cdot b_\varepsilon\partial_z ww + \varepsilon^2\int c_\varepsilon\partial_z ww = \int O(\varepsilon^4)w,\end{aligned} \tag{57}$$

where the boundary conditions on w have been used to eliminate the boundary terms in the integration by parts performed with respect to the z variable. Since $a_\varepsilon = 1+O(\varepsilon)$, the second term of the above equation can easily be controlled, and the third one satisfies

$$\left|\int\partial_z a_\varepsilon\partial_z ww\right| \leqq O(\varepsilon)|\partial_z w|\,|w|,$$

where $|\cdot|$ denotes here the L^2 norm on $\mathcal{S}$. From Poincaré's inequality, it follows that

$$\left|\int\partial_z a_\varepsilon\partial_z ww\right| \leqq O(\varepsilon)|\partial_z w|^2. \tag{58}$$

For the fourth term of the left-hand side of (57), remark that

$$\begin{aligned}\left|2\varepsilon^2\int b_\varepsilon\cdot\nabla w\partial_z w\right| &\leqq O(\varepsilon^2)|\partial_z w|\,|\nabla w|\\ &\leqq O(\varepsilon^2)\left(|\partial_z w|^2 + |\nabla w|^2\right).\end{aligned} \tag{59}$$

Similarly, the fifth and sixth terms are bounded above as follows:

$$\left|2\varepsilon^2\int\nabla\cdot b_\varepsilon\partial_z ww + \varepsilon^2\int c_\varepsilon\partial_z ww\right| \leqq O(\varepsilon^2)|\partial_z w|^2, \tag{60}$$

where Poincaré's inequality has been used again. Substituting (58)–(60) into (57), it is determined that

$$\varepsilon\int|\nabla w|^2 + \int|\partial_z w|^2 = O(\varepsilon^4)\left(\int|w|^2\right)^{1/2},$$

and hence, by Poincaré's inequality,

$$\varepsilon\int|\nabla w|^2 + \int|\partial_z w|^2 = O(\varepsilon^8).$$

This last equations yields

$$|\partial_z w| = O(\varepsilon^4), \qquad |\nabla w| = O(\varepsilon^{7/2}), \tag{61}$$

which proves the proposition for $s = 1$.

We now prove it for $s = 2$. Taking the L^2 scalar product of (56) with Δw yields

$$\varepsilon \int |\Delta w|^2 + \int a_\varepsilon \partial_z^2 w \Delta w + 2\varepsilon^2 \int b_\varepsilon \cdot \partial_z \nabla w \Delta w + \varepsilon^2 \int c_\varepsilon \partial_z w \Delta w = \int O(\varepsilon^4) \Delta w, \quad (62)$$

and taking the L^2 scalar product of (56) with $\partial_z^2 w$ gives

$$\varepsilon \int \Delta w \partial_z^2 w + \int a_\varepsilon |\partial_z^2 w|^2 + 2\varepsilon^2 \int b_\varepsilon \cdot \partial_z \nabla w \partial_z^2 w + \varepsilon^2 \int c_\varepsilon \partial_z w \partial_z^2 w = \int O(\varepsilon^4) \partial_z^2 w. \quad (63)$$

Attention is first given to the two terms in the above equations which give control of the cross derivatives, namely $\int a_\varepsilon \partial_z^2 w \Delta w$ and $\varepsilon \int \Delta w \partial_z^2 w$. Notice that

$$\int \Delta w \partial_z^2 w = -\int \partial_z \Delta w \partial_z w + \left[\int_{\mathbb{R}^d} \Delta w \partial_z w \right]_{z=0}^{z=1} = \int |\partial_z \nabla w|^2, \quad (64)$$

since the boundary terms are zero, thanks to the boundary conditions satisfied by w. The other term can be treated in the following manner:

$$\int a_\varepsilon \partial_z^2 w \Delta w = -\int \partial_z a_\varepsilon \partial_z w \Delta w - \int a_\varepsilon \partial_z w \partial_z \Delta w + \left[\int_{\mathbb{R}^d} a_\varepsilon \partial_z w \Delta w \right]_{z=0}^{z=1}$$
$$= -\int \partial_z a_\varepsilon \partial_z w \Delta w + \int \nabla a_\varepsilon \partial_z w \partial_z \nabla w + \int a_\varepsilon |\partial_z \nabla w|^2.$$

Using the fact that $a_\varepsilon = 1 + O(\varepsilon)$ and (61), it follows that

$$\int a_\varepsilon \partial_z^2 w \Delta w = O(\varepsilon^5)\left(|\Delta w| + |\partial_z \nabla w|\right) + \int a_\varepsilon |\partial_z \nabla w|^2. \quad (65)$$

Then, using (64) and (65) in (62) and (63), it is determined that

$$\varepsilon \int |\Delta w|^2 + \int a_\varepsilon |\partial_z^2 w|^2 + \varepsilon \int |\partial_z \nabla w|^2 + \int a_\varepsilon |\partial_z \nabla w|^2$$
$$= O(\varepsilon^5)\left(|\Delta w| + |\partial_z \nabla w|\right) + O(\varepsilon^2)|\partial_z \nabla w|\,|\Delta w| + O(\varepsilon^2)|\partial_z w|\,|\Delta w|$$
$$+ O(\varepsilon^2)|\partial_z \nabla w|\,|\partial_z^2 w| + O(\varepsilon^2)|\partial_z w|\,|\partial_z^2 w| + O(\varepsilon^4)\left(|\Delta w| + |\partial_z^2 w|\right).$$

Multiple use of the inequality $ab \leqq \frac{1}{2}(a^2 + b^2)$ and of the fact that $a_\varepsilon = 1 + O(\varepsilon)$ then yields

$$\varepsilon \int |\Delta w|^2 + \int |\partial_z^2 w|^2 + \int |\partial_z \nabla w|^2 = O(\varepsilon^8),$$

which implies the proposition for $s = 2$.

Attention is now turned to proving the general case of the proposition by induction. Assume that for some integer $s \geqq 1$ and for any horizonatal derivative ∂, both $\sqrt{\varepsilon}|\partial^s \partial_z w|$ and $|\partial^{s-1} \partial_z^2 w|$ are of size $O(\varepsilon^4)$. Apply ∂^s to (56) and take the L^2 scalar product with $\partial^s \partial_z^2 w$; this leads to the relation

$$\varepsilon \int \Delta \partial^s w \partial^s \partial_z^2 w + \int \partial^s (a_\varepsilon \partial_z^2 w) \partial^s \partial_z^2 w + 2\varepsilon^2 \int \partial^s (b_\varepsilon \partial_z \nabla w) \partial^s \partial_z^2 w$$
$$+ \varepsilon^2 \int \partial^s (c_\varepsilon \partial_z w) \partial^s \partial_z^2 w = \int O(\varepsilon^4) \partial^s \partial_z^2 w.$$

Rewrite this equation, *viz.*

$$\varepsilon \int |\nabla \partial^s \partial_z w|^2 + \int \left([\partial^s, a_\varepsilon] \partial_z^2 w\right) \partial_z^2 \partial^s w + \int a_\varepsilon |\partial^s \partial_z^2 w|^2$$
$$+ \varepsilon^2 \int \left([\partial^s, b] \partial_z \nabla w\right) \partial^s \partial_z^2 w + 2\varepsilon^2 \int b_\varepsilon \partial^s \partial_z \nabla w \partial^s \partial_z^2 w$$
$$+ \varepsilon^2 \int \left([\partial^s, c_\varepsilon] \partial_z w\right) \partial^s \partial_z^2 w + \varepsilon^2 \int c_\varepsilon \partial^s \partial_z w \partial^s \partial_z^2 w = \int O(\varepsilon^4) \partial^s \partial_z^2 w, \quad (66)$$

where $[P, Q]$ denotes the commutator of the operators P and Q, which is to say,

$$[P, Q] = P \circ Q - Q \circ P.$$

The terms containing commutators are bounded above in simple ways:

$$\left| \int \left([\partial^s, a_\varepsilon] \partial_z^2 w\right) \partial_z^2 \partial^s w \right| \leqq \text{Const.}\, |\partial_z^2 w|_{H^{s-1}(\mathcal{S})} |\partial_z^2 \partial^s w|,$$
$$\left| \varepsilon^2 \int \left([\partial^s, b_\varepsilon] \partial_z \nabla w\right) \partial_z^2 \partial^s w \right| \leqq \text{Const.}\, \varepsilon^2 |\partial_z w|_{H^s(\mathcal{S})} |\partial^s \partial_z^2 w|,$$
$$\left| \varepsilon^2 \int \left([\partial^s, c_\varepsilon] \partial_z w\right) \partial_z^2 \partial^s w \right| \leqq \text{Const.}\, \varepsilon^2 |\partial_z w|_{H^{s-1}(\mathcal{S})} |\partial^s \partial_z^2 w|,$$

where we have used $\left|[\partial^s, a_\varepsilon] w\right|_{L^2} \leqq \text{Const.}\, |w|_{H^{s-1}}$. The other terms of (66) are estimated using the inequalities

$$\left| 2\varepsilon^2 \int b_\varepsilon \partial^s \partial_z \nabla w \partial^s \partial_z^2 w \right| \leqq O(\varepsilon^2) |\partial_z w|_{H^{s+1}} |\partial_z^2 w|_{H^s},$$
$$\left| \varepsilon^2 \int c_\varepsilon \partial^s \partial_z w \partial^s \partial_z^2 w \right| \leqq O(\varepsilon^2) |\partial_z w|_{H^s} |\partial_z^2 w|_{H^s}.$$

Combining these estimates appropriately in (66) leads to the inequality

$$\varepsilon \int |\nabla \partial^s \partial_z w|^2 + \int |\partial^s \partial_z^2 w|^2$$
$$\lesssim |\partial_z^2 w|_{H^{s-1}} |\partial_z^2 \partial^s w| + \varepsilon^2 |\partial_z w|_{H^s} |\partial^s \partial_z^2 w| + \varepsilon^2 |\partial_z w|_{H^{s-1}} |\partial^s \partial_z^2 w|$$
$$+ \varepsilon^2 |\partial_z w|_{H^{s+1}} |\partial_z^2 w|_{H^s} + \varepsilon^2 |\partial_z w|_{H^s} |\partial_z^2 w|_{H^s} + \int O(\varepsilon^4) \partial^s \partial_z^2 w,$$

where by $\lesssim$, we mean that the inequality holds up to a constant which is independent of ε. Recalling that $\sqrt{\varepsilon}|\partial_z w|_{H^s}$ and $|\partial_z^2 w|_{H^{s-1}}$ are of size $O(\varepsilon^4)$, it is then clear that

$$\varepsilon|\partial_z w|^2_{H^{s+1}} + |\partial_z^2 w|^2_{H^s} = O(\varepsilon^8),$$

which completes the induction. The proposition is deduced from the induction property using Poincaré's inequality once again.

Attention is turned back to the proof of Theorem 5. Recall that the operator $G_\varepsilon(f)(g)$ is given in (47) as $G_\varepsilon(f)(g) = \frac{1}{1+\varepsilon f}\partial_z\widetilde{u}(z=1)$. It is therefore necessary for us to compute $\partial_z\widetilde{u}(z=1)$. From Proposition 5, it is deduced that in the $H^s(\mathbb{R}^d)$ norm, we have $\partial_z\widetilde{u}|_{z=1} = (\partial_z\widetilde{u}_0 + \varepsilon\partial_z\widetilde{u}_1 + \varepsilon^2\partial_z\widetilde{u}_2)|_{z=1} + O(\varepsilon^3)$. Using the explicit expressions (49) for the $\widetilde{u}_i$, $i = 0, 1, 2$, we arrive at the conclusion

$$\partial_z\widetilde{u}(z=1) = -\varepsilon\Delta g - \frac{\varepsilon^2}{3}\Delta^2 g - 2\varepsilon^2 f\Delta g + O(\varepsilon^3),$$

and a straightforward Taylor expansion of (47) then implies the Theorem. □

5. Uncoupled approximations

Many uncoupled models exist for the water-wave equations in the case of two-dimensional motions (one-dimensional surfaces). SCHNEIDER & WAYNE [23] proved that the uncoupled KdV-KdV approximation is indeed an approximation of the full equations in certain circumstances. Our goal here is to justify a whole class of uncoupled models (including the KdV-KdV and BBM-BBM models) obtained formally from the water-wave equations. Moreover, we present sharp error estimates, sensibly better than those of [23], and also comment on the validity of the uncoupled models in the case of spatially periodic motion: in particular, we show that the uncoupled models are not good approximations of the full equations in the periodic setting if we are unwilling to make a zero-mass assumption on the initial data. Such an assumption is certainly not natural in coastal engineering applications where the mean water level may rise as much as a meter in storm environments (see, e.g., [6, 5]).

5.1. From the symmetric systems to uncoupled approximations

Constructed here are approximate solutions of the symmetric systems in the class Σ, namely,

$$\begin{cases} \partial_t v + \partial_x\eta + \varepsilon\left(\frac{1}{4}\partial_x\eta^2 + \frac{3}{4}\partial_x v^2 + a\partial_x^3\eta - b\partial_x^2\partial_t v\right) = 0, \\ \partial_t\eta + \partial_x v + \varepsilon\left(\frac{1}{2}\partial_x(\eta v) + a\partial_x^3 v - d\partial_x^2\partial_t\eta\right) = 0, \end{cases} \tag{67}$$

where a, b and d are as given in (22).

The analysis commences by diagonalizing these systems. Introducing the unknowns $U = v + \eta$ and $N = v - \eta$ and writing (67) in terms of U and N leads to the system

$$\begin{cases} \partial_t U + \partial_x U + \varepsilon\Big(\frac{1}{8}\partial_x\left(3U^2 + N^2 + 2UN\right) \\ \qquad + a\partial_x^3 U - \frac{b}{2}\partial_x^2\partial_t(U+N) - \frac{d}{2}\partial_x^2\partial_t(U-N)\Big) = 0 \\ \partial_t N - \partial_x N + \varepsilon\Big(\frac{1}{8}\partial_x\left(U^2 + 3N^2 + 2UN\right) \\ \qquad - a\partial_x^3 N - \frac{b}{2}\partial_x^2\partial_t(U+N) + \frac{d}{2}\partial_x^2\partial_t(U-N)\Big) = 0. \end{cases} \tag{68}$$

As is usual when positing long-wave WKB expansions, we seek approximate solutions (U_a, N_a) of (68) in the form

$$\begin{aligned} U_a(t,x) &= U_0(\varepsilon t, x-t) + \varepsilon U_1(\varepsilon t, t, x), \\ N_a(t,x) &= N_0(\varepsilon t, x+t) + \varepsilon N_1(\varepsilon t, t, x), \\ (U_a, N_a)|_{t=0} &= (U, N)|_{t=0}. \end{aligned} \tag{69}$$

Substituting this ansatz into (68), cancelling the first powers of ε appearing in the expressions thus obtained, and using the usual decoupling tools (see [17, 4]), we attain the following *uncoupled* equations for U_0 and N_0:

$$\begin{cases} \partial_T U_0 + a\partial_x^3 U_0 - \frac{(b+d)}{2}\partial_x^2\partial_t U_0 + \frac{3}{8}\partial_x U_0^2 = 0, \\ \partial_T N_0 - a\partial_x^3 N_0 - \frac{(b+d)}{2}\partial_x^2\partial_t N_0 + \frac{3}{8}\partial_x N_0^2 = 0, \end{cases} \tag{70}$$

where T stands for εt. By the same token, the equations determining the correctors U_1 and N_1 are

$$\begin{cases} (\partial_t + \partial_x)U_1 = -\left(\frac{1}{8}\partial_x(N_0^2 + 2U_0N_0) - \frac{(b-d)}{2}\partial_x^2\partial_t N_0\right), \\ (\partial_t - \partial_x)N_1 = -\left(\frac{1}{8}\partial_x(U_0^2 + 2U_0N_0) - \frac{(b-d)}{2}\partial_x^2\partial_t U_0\right). \end{cases} \tag{71}$$

5.2. Estimates of the correctors

Equations (71) give (U_1, N_1) in terms of (U_0, N_0). These equations can be solved explicitly:

$$\begin{aligned} U_1(T,t,x) = &-\frac{1}{16}\left(N_0^2(T, x+t) - N_0^2(T, x-t)\right) \\ &-\frac{(b-d)}{2}\left(\partial_x\partial_t N_0(T, x+t) - \partial_x\partial_t N_0(T, x-t)\right) \\ &-\frac{1}{4}\partial_x U_0(T, x-t)\int_0^t N_0(T, x-t+2s)ds \\ &-\frac{1}{8}U_0(T, x-t)\left(N_0(T, x+t) - N_0(T, x-t)\right), \end{aligned} \tag{72}$$

with a similar expression holding for N_1. For all $s \in \mathbb{R}$, and with one possible exception, the terms which appear on the right-hand side of (72) are obviously bounded in $L^\infty([0, T_0] \times \mathbb{R}_t; H^s(\mathbb{R}))$ provided that $(U_0, N_0) \in L^\infty([0, T_0]; H^\sigma(\mathbb{R}))^2$ for some σ big enough. The possible exception is the term

$$W_1(T, t, x) := -\frac{1}{4}\partial_x U_0(T, x-t) \int_0^t N_0(T, x-t+2s)ds. \tag{73}$$

While not quite obvious, this latter term is under control as well, due to the following lemma.

Lemma 1. *Let $s \in \mathbb{N}$. Then for all σ large enough, the following results hold:*
(i) *If $(U_0, N_0) \in L^\infty([0, T_0]; H^\sigma(\mathbb{R}))^2$ then $W_1 \in L^\infty_{loc}([0, T_0] \times \mathbb{R}_t; H^s(\mathbb{R}))$ and*

$$\sup_{T\in[0,T_0]} |W_1(T, t, \cdot)|_{H^s(R)} \leqq \text{Const. } \sqrt{t}, \qquad \forall t \geqq 0;$$

(i′) *If, moreover, N_0 satisfies the following decay assumption: there exists $\alpha > \frac{1}{2}$ such that*

$$\sup_{(T,x)\in[0,T_0]\times\mathbb{R}} \left|(1+x^2)^\alpha \partial_x^\beta N_0(T, x)\right| < \infty, \qquad \beta = 0, \dots, s,$$

then

$$\sup_{T\in[0,T_0]} |W_1(T, t, \cdot)|_{H^s(R)} \leqq \text{Const.}, \qquad \forall t \geqq 0;$$

(ii) *In the periodic case, i.e., if $(U_0, N_0) \in L^\infty([0, T_0]; H^\sigma(\mathbb{T}))^2$, then*

$$W_1(T, t, x) = -\frac{t}{8\pi}\partial_x U_0(T, x-t) \int_0^{2\pi} N_0(T, x)dx + O(1) \text{ as } t \to \infty.$$

In particular, W_1 is bounded in $L^\infty([0, T_0]; H^s(\mathbb{T}))$ when $N_0(T, \cdot)$ has zero mean value for all $T \in [0, T_0]$. Otherwise, it grows linearly in t.

Proof. Point **(i)** is classical (see, e.g., [18, 4]). We recall the proof in the case $s = 0$. From (73), we deduce

$$\begin{aligned} |W_1(T, t, \cdot)|_{L^2(\mathbb{R})} &\leqq \frac{1}{4}|\partial_x U_0(T)|_{L^2(\mathbb{R})} \left| \int_0^t N_0(T, x-t+2s)ds \right|_{L^\infty(\mathbb{R})} \\ &\leqq \text{Const. } \sqrt{t}|\partial_x U_0(T)|_{L^2(\mathbb{R})}|N_0(T)|_{L^2(\mathbb{R})}, \end{aligned}$$

which easily yields the desired estimate.
Point **(i′)** is in the spirit of Lemma 5.5 of [23] and Proposition 3.5 of [18], and its proof is quite obvious in the present case. Finally, **(ii)** is deduced easily from (73) by expanding N_0 into Fourier series. □

5.3. Validity of the uncoupled approximations for the diagonalized symmetric system

We first consider the error made when approximating the exact solution (U, N) of (68) with initial conditions (U^0, N^0) by (U_a, N_a) as given by (69)–(71).

Proposition 6. *Let $s \in \mathbb{N}$. Then for all σ large enough, there is a $T_0 > 0$ so that:*
(i) *If $(U_0, N_0) \in L^\infty([0, T_0]; H^\sigma(\mathbb{R}))^2$ then for all $t \in [0, \frac{T_0}{\varepsilon}]$,*

$$|(U, N) - (U_a, N_a)|_{L^\infty([0,t], H^s(\mathbb{R}))^2} \leqq \text{Const. } \varepsilon^2 t^{3/2}.$$

(i′) *If, moreover, U_0 and N_0 satisfy the following decay assumption: there exists $\alpha > 1/2$ such that*

$$\sup_{(T,x)\in[0,T_0]\times\mathbb{R}} \left|(1+x^2)^\alpha \left(\partial_x^\beta U_0(T, x), \partial_x^\beta N_0(T, x)\right)\right| < \infty, \qquad \beta = 0, \dots, s,$$

then for all $t \in [0, \frac{T_0}{\varepsilon}]$,

$$|(U, N) - (U_a, N_a)|_{L^\infty([0,t], H^s(\mathbb{R}))^2} \leqq \text{Const. } \varepsilon^2 t.$$

(ii) *In the periodic case, i.e. if $(U_0, N_0) \in L^\infty([0, T_0]; H^\sigma(\mathbb{T}))^2$ then for every $t \in [0, \frac{T_0}{\varepsilon}]$,*

$$|(U, N) - (U_a, N_a)|_{L^\infty([0,t], H^s(\mathbb{R}))^2} \leqq \text{Const. } \varepsilon^2 t^2.$$

(ii′) *If, in addition, the initial conditions U^0 and V^0 satisfy a zero mean value condition, which is to say, $\int_0^{2\pi} U^0 = \int_0^{2\pi} N^0 = 0$, then for $t \in [0, \frac{T_0}{\varepsilon}]$,*

$$|(U, N) - (U_a, N_a)|_{L^\infty([0,t], H^s(\mathbb{R}))^2} \leqq \text{Const. } \varepsilon^2 t.$$

Proof. The approximate solution (U_a, N_a) satisfies (68) with an error term of size $O(\varepsilon^2\sqrt{t})$, $O(\varepsilon^2)$, $O(\varepsilon^2 t)$ and $O(\varepsilon^2)$ in cases **(i)**, **(i′)**, **(ii)** and **(ii′)** respectively, by Lemma 1. Standard energy estimates applied to (68) imply the desired results. □

Remark 4. To use Lemma 1 in case **(i′)** (or **(ii′)**), we must know that $U_0(T, \cdot)$ and $N_0(T, \cdot)$ satisfy the decay condition (respectively the zero mean value condition) for all $T \in [0, T_0]$ and not only at the initial value (U^0, N^0). It is classical that these properties are propagated by the KdV–type equations (70) (see, e.g., [12] and the references therein for periodicity and zero mean conditions; see Proposition 6.3 of [23] for the propagation by the usual KdV equation of the decay condition arising in **(i′)**).

Proposition 7. *In the periodic case, if the initial data $U^0(x)$ or $N^0(x)$ has nonzero mean value, there exist $T_0 \geqq T_1 > 0$ and $C > 0$ such that for all $t \in [0, \frac{T_1}{\varepsilon}]$,*

$$|(U, N) - (U_0, N_0)|_{L^\infty([0,t], H^s(\mathbb{R}))^2} \geqq C\varepsilon t.$$

Remark 5. This means that in the periodic case, without the zero mean value condition, the decoupled KdV approximation is not a good one on the long-time scales of interest in the use of such models.

Proof. The second term of the approximate solution (U_a, N_a) has in this case a linear growth in time as shown by **(ii)** of Lemma 1. Therefore $U - U_0$ is bounded from below by $C_1\varepsilon t - C_2\varepsilon^2 t^2$. The result follows. □

5.4. Validity of the uncoupled approximation to the Euler equations

In the previous sections, we derived a class of uncoupled KdV-like equations (70) from systems (67) of the class Σ. We now put forward a set of two uncoupled KdV equations starting from a one-dimensional version of Boussinesq's original system (8), namely

$$\begin{cases} \partial_t v + \partial_x \eta + \frac{\varepsilon}{2}\partial_x v^2 = 0, \\ \partial_t \eta + \partial_x v + \varepsilon\left(\partial_x(\eta v) + \frac{1}{3}\partial_x^3 v\right) = 0. \end{cases} \tag{74}$$

Exactly as in Section 5.1, we can diagonalize this system by introducing the unknowns $F = v + \eta$ and $G = v - \eta$, and search for an approximation (F_a, G_a) of (F, G) in the form

$$\begin{aligned} F_a(t,x) &= F_0(\varepsilon t, x-t) + \varepsilon F_1(\varepsilon t, t, x), \\ G_a(t,x) &= G_0(\varepsilon t, x+t) + \varepsilon G_1(\varepsilon t, t, x), \\ (F_a, G_a)|_{t=0} &= (F, G)|_{t=0}. \end{aligned}$$

It turns out that the equations which F_0 and G_0 must satisfy are uncoupled and exactly the same as equations (70) which already appeared in the symmetric case, provided we take $b = d = 0$ and $a = \frac{1}{6}$. This is to say that $(f^\varepsilon, g^\varepsilon)$, defined as

$$f^\varepsilon(t,x) = F_0(\varepsilon t, x-t), \qquad g^\varepsilon(t,x) = G_0(\varepsilon t, x+t),$$

solves the set of equations

$$\begin{cases} (\partial_t + \partial_x) f^\varepsilon + \varepsilon\left(\frac{3}{8}\partial_x f^{\varepsilon 2} + \frac{1}{6}\partial_x^3 f^\varepsilon\right) = 0, \\ (\partial_t - \partial_x) g^\varepsilon + \varepsilon\left(\frac{3}{8}\partial_x g^{\varepsilon 2} - \frac{1}{6}\partial_x^3 g^\varepsilon\right) = 0. \end{cases} \tag{75}$$

Construction of the KdV approximation for water-waves.
Consider initial data (v_0, η_0) for the Euler equations in the form (24)–(25) and denote by $\{(v^\varepsilon, \eta^\varepsilon)\}$ the associated family of solutions. The KdV approximation of $(v^\varepsilon, \eta^\varepsilon)$ is constructed as follows. Let $f_0 := v_0 + \eta_0$ and $g_0 := v_0 - \eta_0$ and denote by $(f^\varepsilon, g^\varepsilon)$ the family of solution of (75) with initial condition (f_0, g_0). The KdV approximation $(v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})$ is

$$v^\varepsilon_{KdV} = \frac{f^\varepsilon + g^\varepsilon}{2}, \qquad \eta^\varepsilon_{KdV} = \frac{f^\varepsilon - g^\varepsilon}{2}.$$

For any $k \in \mathbb{N}$ and $s \in \mathbb{R}$, note that for sufficiently smooth initial data, there exists $T_0 > 0$ such that both $\{(v^\varepsilon, \eta^\varepsilon)\}$ and $\{(v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})\}$ are bounded families in $W^{k,\infty}\left([0, \frac{T_0}{\varepsilon}], H^s(\mathbb{R})\right)$.

Choice of a reference symmetric system.
As mentioned above, if we choose $a = c = 1/6$ and $b = d = 0$ in (67) (which can be obtained by taking $\lambda = \mu = 1$ and $\theta^2 = 2/3$), the associated uncoupled equations (70) coincide with the uncoupled KdV equations satisfied by F_0 and G_0. It is therefore natural to consider the symmetric system $S'_{\sqrt{2/3},1,1}$ determined by (70) with this choice of parameters. From Theorem 2 the approximation $(v^\varepsilon_{app}, \eta^\varepsilon_{app})$ given by (27) and (28) (with $\underline{\lambda} = \underline{\mu} = 1$ and $\underline{\theta}^2 = 2/3$) satisfies

$$\left|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{app}, \eta^\varepsilon_{app})\right|_{L^\infty(0,t,H^s(\mathbb{R}))^2} \leq \text{Const. } \varepsilon^2 t \quad \text{for } t \in \left[0, \frac{T_0}{\varepsilon}\right]. \tag{76}$$

Error estimate for the KdV approximation.
We can now estimate the error made when approximating the solution $(v^\varepsilon, \eta^\varepsilon)$ of the Euler equations by the KdV approximation $(v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})$ constructed above. According to (76), we have

$$\begin{aligned} (v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV}) &= (v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{app}, \eta^\varepsilon_{app}) + (v^\varepsilon_{app}, \eta^\varepsilon_{app}) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV}) \\ &= O(\varepsilon^2 t) + (v^\varepsilon_{app}, \eta^\varepsilon_{app}) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV}). \end{aligned}$$

Inverting approximatively the nonlinear pseudo-differential change of variables, it is observed that

$$\begin{aligned} &\left|(v^\varepsilon_{app}, \eta^\varepsilon_{app}) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})\right|_{L^\infty(0,t,H^s(\mathbb{R}))^2} \\ &= \left|(v^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma) - (\widetilde{v^\varepsilon_{KdV}}, \widetilde{\eta^\varepsilon_{KdV}})\right|_{L^\infty(0,t,H^s(\mathbb{R}))^2} + O(\varepsilon^2), \end{aligned}$$

with

$$\widetilde{v^\varepsilon_{KdV}} = \left(1 - \frac{\varepsilon}{6}\partial_x^2\right)^{-1} v^\varepsilon_{KdV}\left(1 + \frac{\varepsilon}{2}\eta^\varepsilon_{KdV}\right), \qquad \widetilde{\eta^\varepsilon_{KdV}} = \eta^\varepsilon_{KdV},$$

and where $(v^\varepsilon_\Sigma, \eta^\varepsilon_\Sigma)$ denotes the solution to the symmetric system $S'_{\sqrt{2/3},1,1}$ with initial conditions (27). In accordance with the notation of Section 5.1, we write $U = v^\varepsilon_\Sigma + \eta^\varepsilon_\Sigma$ and $N = v^\varepsilon_\Sigma - \eta^\varepsilon_\Sigma$ so that (U_a, N_a) as constructed in Section 5.1 gives a good asymptotic description of (U, N). Now remark that $\widetilde{F} := \widetilde{v^\varepsilon_{KdV}} + \widetilde{\eta^\varepsilon_{KdV}}$ and $\widetilde{G} := \widetilde{v^\varepsilon_{KdV}} - \widetilde{\eta^\varepsilon_{KdV}}$ solve the uncoupled KdV equations (70) up to a term of order $O(\varepsilon^2)$. It follows that if we replace (U_0, N_0) by $(\widetilde{F}, \widetilde{G})$ in the ansatz (69), the results of Proposition 6 are not altered.

The outcome of these ruminations is that the error made by the KdV approximation can be evaluated in the $L^\infty(0, t, H^s(\mathbb{R}))$ norm to be

$$\|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})\| = O(\varepsilon^2 t) + \|(U, N) - (U_a, N_a)\|.$$

Our final result is now a simple consequence of Proposition 6.

Theorem 7. *Let $s \in \mathbb{R}$. For σ large enough, if $(v_0, \eta_0) \in (H^\sigma(\mathbb{R}))^2$, then there exists $T_0 > 0$ such that for all $t \in [0, \frac{T_0}{\varepsilon}]$,*
(i)

$$|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})|_{L^\infty([0,t],H^s(\mathbb{R}))^2} \leq \text{Const. } \varepsilon^2 t^{3/2}.$$

(i′) *If, moreover, v_0 and η_0 satisfy the decay assumption: there exists $\alpha > \frac{1}{2}$ such that*

$$\sup_{x\in\mathbb{R}} \left|(1+x^2)^\alpha \left(\partial_x^\beta v_0(x), \partial_x^\beta \eta_0(x)\right)\right| < \infty, \qquad \beta = 0, \dots, s,$$

then for every $t \in [0, \frac{T_0}{\varepsilon}]$,

$$|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})|_{L^\infty([0,t],H^s(\mathbb{R}))^2} \leqq \text{Const. } \varepsilon^2 t.$$

(ii) *In the periodic case, i.e., if $(v_0, \eta_0) \in H^\sigma(\mathbb{T})^2$, then for $t \in [0, \frac{T_0}{\varepsilon}]$,*

$$|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})|_{L^\infty([0,t],H^s(\mathbb{T}))^2} \leqq \text{Const. } \varepsilon^2 t^2.$$

(ii′) *If, in addition, $\int_0^{2\pi} v_0 = \int_0^{2\pi} \eta_0 = 0$, then for every $t \in [0, \frac{T_0}{\varepsilon}]$,*

$$|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})|_{L^\infty([0,t],H^s(\mathbb{T}))^2} \leqq \text{Const. } \varepsilon^2 t.$$

(iii) *In the periodic case, if v_0 or η_0 have nonzero mean value, there exist $T_0 \geqq T_1 > 0$ and $C > 0$ such that for all $t \in \left[0\frac{T_1}{\varepsilon}\right]$,*

$$|(v^\varepsilon, \eta^\varepsilon) - (v^\varepsilon_{KdV}, \eta^\varepsilon_{KdV})|_{L^\infty([0,t],H^s(\mathbb{T}))^2} \geqq C\varepsilon t.$$

Remark 6. **(i)** In [23], it is proved that under the decay assumption featured in **(i′)**, the decoupled KdV model furnishes an approximate solution of the Euler equations with an error estimate of size $O(\varepsilon^{1/4})$ which was shown to be valid on the long-time scale of order ε^{-1}. The error estimates **(i′)** and even **(i)** improve considerably this bound. In the case **(i′)**, our estimate is sharp (since we have constructed the next term in the asymptotic expansion). In the periodic case, our results show that the uncoupled KdV approximation diverges from the exact solution of the Euler equation unless a zero-mean-value assumption is made on the initial data.
(ii) It is worth pointing out that there is not currently an existence result for the Euler equations in the periodic framework. So, just as for the three-dimensional case with Sobolev class initial date, we assume the existence of a family of solutions $\{(v^\varepsilon, \eta^\varepsilon)\}$ of the Euler equations, over times $O(\frac{1}{\varepsilon})$ and with initial condition (v_0, η_0).

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