07-01

Comparison of Quarter-plane and Two-point Boundary Value Problems: The KdV-Equation

Jerry L. Bona

Department of Mathematics, Statistics and Computer Sciences

University of Illinois at Chicago

Chicago, Illinois, 60607, USA

Hongqiu Chen

Department of Mathematical Sciences, University of Memphis

Memphis, Tennessee, 38152, USA

Shu Ming Sun

Department of Mathematics, Virginia Polytechnic Institute and State University

Blacksburg, Virginia 24061, USA

Bing-Yu Zhang

Department of Mathematical Sciences, University of Cincinnati

Cincinnati, Ohio 45221, USA

Abstract

This paper is concerned with the Korteweg-de Vries equation which models unidirectional propagation of small amplitude long waves in dispersive media. The two-point boundary value problem wherein the wave motion is specified at both ends of a finite stretch of length L of the media of propagation is considered. It is shown that the solution of the two-point boundary value problem converges as $L \to +\infty$ to the solution of the quarter-plane boundary value problem in which a semi-infinite stretch of the medium is disturbed at its finite end. In addition to its intrinsic interest, our result provides justification for the use of the two-point boundary value problem in numerical studies of the quarter plane problem for the KdV equation.

1 Introduction

Considered here are small amplitude long waves on the surface of an ideal fluid of finite depth over a flat horizontal bottom under the force of gravity. Interest is focused upon waves which propagate essentially in the horizontal x-direction and without significant variation in the transverse y-direction of a standard xyz-Cartesian frame in which gravity acts in the negative z-direction. For such waves, the full three-dimensional Euler equations can be reduced to approximate models with only one independent spatial variable. Such models go back to at least the early part of the 19^{th} century and are included in works by Airy [1] and Stokes [34] in the first half of the century. The model used in our study was developed in the work of Boussinesq [16, 17, 18] and later, Korteweg and de Vries [30]. More detailed historical accounts and formal derivations can be found in modern works (eg. Bona, Chen and Saut [7], Miura [32], Whitham [35]).

As already indicated, let x denote the coordinate whose increasing values lie in the direction of propagation and let h_0 be the undisturbed depth. The free surface is represented by $z = u(x,t) = h(x,t) - h_0$ where t is proportional to elapsed time and $h(x,t)$ is the depth of the water column over the spatial point x at time t. Under the classical small-amplitude, long wave-length assumptions which feature a balance between nonlinear and dispersive effects, the evolution equation

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.1}$$

is a formal reduction of two-dimensional Euler equations usually called the Korteweg-de Vries equation (KdV equation henceforth, first derived by Boussinesq, see [18] and [30]). The equation (1.1) is written in nondimensional laboratory coordinates, so the small amplitude, long wavelength assumptions reside implicitly in u, and hence should be explicit in the auxiliary data attached to the evolution equation if physically relevant solutions are to be considered.

Attention is now turned to the auxiliary data. It is standard in mathematical studies of the KdV equation to focus upon the pure initial value problem in which u is specified at a given value of t, say, $t = 0$. That is,

$$u(x,0) = u_0(x) \qquad \text{for} \quad x \in R \tag{1.2}$$

is specified for all values of x. Thus values of $t > 0$ represent time elapsed since the inception of the motion as described by (1.2). The formulation (1.2) does not inquire as to how the motion was truly initiated, but imagines a snapshot taken of a disturbance already generated and then uses (1.1) to predict the further evolution of the waves. The initial-value problem (1.1)-(1.2) has a distinguished history both analytically and in experimental studies and applications.

Another natural formulation for (1.1) is the quarter-plane or half-line problem. This problem, first put forward by Bona and Bryant [3], is concerned with waves propagating into an undisturbed stretch of the medium of propagation. One imagines measuring the waves as they come into the relevant portion of the medium at some fixed spatial point, say $x = 0$. This leads to the boundary condition

$$u(0,t) = g(t) \qquad \text{for} \quad t \geq 0. \tag{1.3}$$

Since (1.1) is written to describe wave propagation in the positive direction along the x-axis, it is not particularly desirable to impose a boundary condition at a finite point to the right of $x = 0$. To do so can lead to reflected waves which (1.1) is incapable of approximating accurately. (For such motions, systems of equations are useful; see, for example, Bona, Chen and Saut [7, 8].) This point leads one to pose the problem for all $x \geq 0$, thus placing the issue of a boundary condition at the right-hand end-point at ∞. The equation (1.1) along with the boundary condition (1.3) must be supplemented by an initial condition as in (1.2), *viz.*

$$u(x,0) = u_0(x) \qquad \text{for} \quad x \geq 0. \tag{1.4}$$

In practice, it is often the case that $u_0(x) \equiv 0$, corresponding to an initially undisturbed medium, but the mathematical theory does not require this. Function class restrictions on u_0 which imply at least a weak form of boundedness as $x \to +\infty$ suffice to guarantee that (1.1)-(1.3)-(1.4) constitutes a well posed problem.

The initial boundary value problem (1.1)-(1.3)-(1.4), sometimes in a modified form that includes some kind of dissipation, has been used to study the relevance of (1.1) to wave motions in laboratory settings (see, for example, Hammack [26], Hammack and Segur [27] and

Bona, Pritchard and Scott [10]). When comparison between experimentally produced waves are made with model predictions, one usually has to depend upon numerical approximation of its solution. For this, a bounded domain is normally used (but see the work of Guo and Shen [25]). Analytical theory for the two-point boundary value problem for (1.1) posed on a finite spatial interval with an initial condition and suitable boundary conditions is also available.

Study of the KdV equation posed on a finite interval appears to have begun with the work of B. A. Bubnov [19]. A review of developments following Bubnov's work may be found in the recent paper of Bona, Sun and Zhang [13] (also see [22, 23, 24, 28, 29]). For the KdV equation (1.1), well-posedness holds for the auxiliary specifications

$$\begin{aligned} u(x,0) = u_0(x), \qquad &\text{for} \quad 0 \le x \le L, \\ u(0,t) = g(t), \quad u(L,t) = g_1(t), \quad u_x(L,t) = g_2(t), \qquad &\text{for } t \ge 0, \end{aligned} \tag{1.5}$$

for example, where u_0, g, g_1 and g_2 are drawn from reasonable function classes. It is also the case that the problem (1.1) with the auxiliary conditions

$$\begin{aligned} u(x,0) = u_0(x), \qquad &\text{for} \quad 0 \le x \le L, \\ u(0,t) = g(t), \quad u_x(L,t) = g_1(t), \quad u_{xx}(L,t) = g_2(t), \qquad &\text{for } t \ge 0 \end{aligned} \tag{1.6}$$

is well posed in reasonable function classes as Colin and Ghidaglia [20] showed.

A natural question arises within the context of the above discussion. What is the relationship between the two-point boundary value problem for (1.1) and the quarter-plane problem for the same equation? It has been assumed, in using a finite interval for numerical simulations, that these problems yield essentially the same answer in the appropriate part of space-time, if $g_1 = g_2 \equiv 0$. However, the only theory that has come to our attention is the work of Colin and Gisclon [21] connected with (1.6).

It is our purpose to bring forward exact theory comparing the two types of problems in view here. Consider the following two initial boundary value problems (IBVP): the KdV equation in a bounded domain,

$$u_t + u_x + uu_x + u_{xxx} = 0\,, \qquad x \in [0, L]\,, \quad t > 0\,, \tag{1.7}$$

$$u(0,t) = g(t)\,, \quad u(L,t) = u_x(L,t) = 0\,, \tag{1.8}$$

$$u(x,0) = u_0(x)\,, \tag{1.9}$$

and the KdV equation in a semi-infinite domain,

$$u_t + u_x + uu_x + u_{xxx} = 0\,, \qquad x \ge 0\,, \quad t > 0 \tag{1.10}$$

$$u(0,t) = g(t)\,, \qquad u(x,0) = u_0(x)\,. \tag{1.11}$$

We will prove that if the initial and boundary conditions are chosen from function classes that include suitable decay of $u_0(x)$ as $x \to +\infty$, then for fixed $T > 0$ the solution of (1.7)-(1.9) will converge to the solution of (1.10)-(1.11) as L goes to infinity. To provide a more precise goal, a version of our main result is stated here.

Theorem 1.1 *Let $u_\infty(x,t)$ be the solution of the IBVP (1.10) and (1.11) for the KdV equation posed for $x, t \geq 0$ with the initial condition $u_0(x) \in H^s(R^+)$ and the boundary condition $g(t) \in H^{\frac{s+1+\epsilon}{3}}(0,T)$ for some s in $[0,3]$, where $\epsilon > 0$ is any positive constant. Assume u_0 is supported on $[0,N]$, say. Let $u_L(x,t)$ be the solution of the two-point boundary-value problem (1.7)-(1.9) for the KdV-equation posed for $0 \leq x \leq L$ and $t \geq 0$ with the same initial condition and the boundary conditions indicated in (1.8), where $L > N$. Assume that the compatibility condition $u_0(0) = g(0)$ is satisfied if $1/2 < s \leq 3$. Then, $u_\infty(x,t)$ and $u_L(x,t)$ exist for $t \in [0,T]$ and the inequality*

$$\sup_{t\in[0,T]} \|u_\infty(\cdot,t) - u_L(\cdot,t)\|_{H^s(0,L)} \leq Ce^{-bL}, \tag{1.12}$$

holds, where C only depends on the corresponding norms of $u_0(x)$ and $g(t)$ and of the form $e^{\gamma T}$. The constant γ, which is dependent of norms of $u_0(x)$ and $g(t)$, is of order one, as is the constant b. Moreover, if $u_0(x) \in H^s(R^+)$ and $g(t) \in H^{\frac{s+1}{3}}(0,T)$, the above statement holds for $t \in [0,T^*]$ for some $T^* \in (0,T]$.

Note that the fractional-order Sobolev classes $H^s(R^+)$ have been explicitly defined in [12], and such Sobolev classes on finite intervals can be defined similarly. A good reference for the fractional-order Sobolev spaces is Lions and Magenes [31].

A similar result for the BBM equation was obtained in [5].

Remark 1.2 *From the theorem, we see that if solutions u_L on a time interval $[0,T]$ are in question and the data is physically relevant, then L must be chosen to be of the form*

$$L \gtrsim O(T) + |\log \delta|$$

to have an approximation to the solution of the quarter-plane problem of error at most δ, uniformly on $[0,T]$. Notice that once $L \gtrsim O(T)$ the error decays exponentially with larger values of L, a very satisfactory result from a practical perspective.

To prove Theorem 1.1, it is convenient to use a change of the dependent variable by writing $u(x,t) = w(x,t)e^{(b+b^3)t-bx}$ with $b > 0$, which transforms the equation in (1.7) or (1.10) to a KdV-Burgers type equation,

$$w_t + (1+3b^2)w_x - 3bw_{xx} + w_{xxx} + e^{(b+b^3)t-bx}(w_x - bw)w = 0\ , \quad x,t > 0\ , \tag{1.13}$$

with initial condition

$$w(x,0) = e^{bx}u_0(x) = \phi(x)$$

and boundary condition at $x = 0$

$$w(0,t) = e^{-(b+b^3)t}g(t) = h(t)\,, \qquad t > 0\,.$$

The boundary conditions at $x = L$ remain the same for (1.8). The proof of Theorem 1.1 is based upon bounds on the solutions of (1.13) in certain Banach spaces. It is worthy mentioning that the change of variables from u to w is equivalent to considering u in a weighted space with weight e^{-bx} in the x-variable. This gives the solution w of (1.13) a global Kato-smoothing effect in $L_2(0,L)$- or $L_2(0,\infty)$-based spaces, which carries over to the solution u of (1.1). Such global smoothing effects are necessary to obtain estimates of w independent of L.

In outline, the paper proceeds as follows. Section 2 is devoted to linear estimates for (1.13) when considered on $[0,L]$. Similar linear estimates for (1.13) when posed on all of R^+ are given in Section 3. Local well-posedness of the problems is discussed in Section 4 and global well-posedness is presented in Section 5. The proof of Theorem 1.1 is presented in Section 6.

2 Linear estimates for (1.13) on $[0, L]$

Here, for fixed $L > 0$, consideration is given to the linear problem

$$w_t + (1+3b^2)w_x - 3bw_{xx} + w_{xxx} = 0\,, \qquad x \in [0,L]\,, \quad t > 0\,, \tag{2.1}$$

$$w(0,t) = h_1(t)\,, \qquad w(L,t) = h_2(t)\,, \qquad w_x(L,t) = h_3(t), \quad t > 0\,, \tag{2.2}$$

$$w(x,0) = \phi(x)\,, \qquad x \in [0,L]\,. \tag{2.3}$$

Let A be the operator defined by

$$Aw = -w_{xxx} + 3bw_{xx} - (1+3b^2)w_x$$

with domain $D(A) = \{w(x) \in H^3(0,L), \quad w(0) = w(L) = w'(L) = 0\}$. It is straightforward to check that A is the infinitesimal generator of a C_0-semigroup $W_L(t)$ in $L_2(0,L)$. Therefore, the solution u of

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = 0\,, \qquad x \in [0,L]\,, \quad t > 0\,, \tag{2.4}$$

$$u(0,t) = 0\,, \qquad u(L,t) = 0\,, \qquad u_x(L,t) = 0, \quad t > 0\,, \tag{2.5}$$

$$u(x,0) = \phi(x)\,, \quad x \in [0,L] \tag{2.6}$$

can be expressed in the form

$$u(t) = W_L(t)\phi \in C_b([0,T]; L_2(0,L)) \,, \tag{2.7}$$

where $C_b([0,T]; L_2(0,L))$ is the bounded continuous function space from $[0,T]$ to $L_2(0,L)$ with supreme norm. Because $b > 0$, the parabolic nature of the equation insures that u is smooth for $0 < x < L$ and $t > 0$. Upon multiplying (2.4) by $2u$ and integrating the result from 0 to L, using the boundary conditions (2.5), there obtains the following energy-type estimate (or global Kato smoothing)

Proposition 2.1 *For any $\phi \in L_2([0,L])$, $u(t) = W_L(t)\phi$ satisfies*

$$\|u(\cdot,t)\|^2_{L_2(0,L)} + 6b\int_0^t\int_0^L u_x^2 dxdt + \int_0^t u_x^2(0,\tau)d\tau = \|\phi\|^2_{L_2([0,L])} \,.$$

Next, our attention is turned to the inhomogeneous linear problem

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = f(x,t)\,, \qquad x \in [0,L]\,, \quad t > 0\,, \tag{2.8}$$

$$u(0,t) = 0\,, \qquad u(L,t) = 0\,, \qquad u_x(L,t) = 0, \quad t > 0\,, \tag{2.9}$$

$$u(x,0) = 0\,, \quad x \in [0,L]\,. \tag{2.10}$$

In terms of the operator A defined above, one may write (2.8) as an initial-value problem for an abstract non-homogeneous evolution equation, *viz.*

$$\frac{du}{dt} = Au + f, \qquad u(0) = 0. \tag{2.11}$$

By standard semigroup theory (see [33]), for any $f \in L_{1,loc}(R^+; L_2(0,L))$,

$$u(t) = \int_0^t W_L(t-\tau)f(\tau)d\tau \tag{2.12}$$

belongs to the space $C(R^+; L_2(0,L))$ and is called a mild solution of (2.8)-(2.10). It is also a weak solution in the sense of distributions. In addition, if $f(t) \in \mathcal{D}(A)$ for $t > 0$ and $Af \in L_{1,loc}(R^+; L_2(0,L))$, then $u(t)$ given by (2.12) solves (2.8)-(2.10) for each $t > 0$ in the $L_2(0,L)$ sense and is called a strong solution.

Proposition 2.2 *For any $f \in L_{1,loc}(R^+; L_2(0,L))$ and any $T > 0$, the solution $u(t) = \int_0^t W_L(t-\tau)f(\tau)d\tau$ of (2.8)-(2.10) satisfies*

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|^2_{L_2(0,L)} + 6b\int_0^T\int_0^L u_x^2 dxdt + \int_0^T u_x^2(0,\tau)d\tau \le 8\|f\|^2_{L_1([0,T];L_2(0,L))} \,.$$

Proof: Without loss of generality, we assume that u is a strong solution. The general case follows using a standard limiting procedure. Multiply the equation in (2.8) by $2u$ and integrate over $(0, L)$ with respect to x and $(0, t)$ with respect to t. Integration by parts leads to

$$\|u(\cdot,t)\|^2_{L_2(0,L)} + 6b\int_0^t\int_0^L u_x^2 dx dt + \int_0^t u_x^2(0,\tau)d\tau \leq 2\int_0^t \|f(\cdot,\tau)\|_{L_2(0,L)}\|u(\cdot,\tau)\|_{L_2(0,L)}d\tau . \tag{2.13}$$

Assume that $\|u(\cdot,t)\|^2_{L_2(0,L)}$ takes its maximum value on $[0, L]$ at the point t_0. Because

$$2\int_0^{t_0} \|f(\cdot,t)\|_{L_2(0,L)}\|u(\cdot,t)\|_{L_2(0,L)}dt \leq 2\|u(\cdot,t_0)\|_{L_2(0,L)}\int_0^{t_0}\|f(\cdot,t)\|_{L_2(0,L)}dt \leq 2\left(\int_0^{t_0}\|f(\cdot,t)\|_{L_2(0,L)}dt\right)^2 + \frac{1}{2}\|u(\cdot,t_0)\|^2_{L_2(0,L)},$$

(2.13) implies that

$$\|u(\cdot,t_0)\|^2_{L_2(0,L)} \leq 4\left(\int_0^{t_0}\|f(\cdot,\tau)\|_{L_2(0,L)}d\tau\right)^2 \leq 4\left(\int_0^{T}\|f(\cdot,\tau)\|_{L_2(0,L)}d\tau\right)^2 .$$

The last inequality combined with (2.13) yields

$$6b\int_0^T\int_0^L u_x^2 dx dt + \int_0^T u_x^2(0,\tau)d\tau \leq 2\int_0^T \|f(\cdot,\tau)\|_{L_2(0,L)}\|u(\cdot,\tau)\|_{L_2(0,L)}d\tau \leq 4\left(\int_0^T\|f(\cdot,\tau)\|_{L_2(0,L)}d\tau\right)^2,$$

thus completing the proof. □

Finally, consider the nonhomogeneous boundary value problem

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = 0, \qquad x \in [0, L], \quad t > 0, \tag{2.14}$$

$$u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \quad t > 0, \tag{2.15}$$

$$u(x,0) = 0, \quad x \in [0, L]. \tag{2.16}$$

Here, we use the method put forward in [13], which treats the KdV equation itself in a finite domain. Applying the Laplace transform with respect to t, (2.14) becomes

$$s\hat{u} + (1+3b^2)\hat{u}_x - 3b\hat{u}_{xx} + \hat{u}_{xxx} = 0, \qquad x \in [0, L], \quad s > 0, \tag{2.17}$$

$$\hat{u}(0,s) = \hat{h}_1(s), \quad \hat{u}(L,s) = \hat{h}_2(s), \quad \hat{u}_x(L,s) = \hat{h}_3(s), \tag{2.18}$$

where

$$\hat{u}(x,s) = \int_0^{+\infty} e^{-st}u(x,t)dt \quad \text{and} \quad \hat{h}_j(s) = \int_0^{+\infty} e^{-st}h_j(t)dt\,, \quad j = 1,2,3. \tag{2.19}$$

The solution u of (2.14) is obtained from the inverse Laplace transform

$$u(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{u}(x,s)ds \tag{2.20}$$

where $r > 0$ is arbitrary. The solution $\hat{u}(x,s)$ of (2.17) can be written in the form

$$\hat{u}(x,s) = \sum_{j=1}^{3} c_j(s)e^{\lambda_j(s)x} \tag{2.21}$$

where for $j = 1,2,3$, the λ_j are the three roots of

$$s + (1+3b^2)\lambda - 3b\lambda^2 + \lambda^3 = 0 \tag{2.22}$$

and $c_j = c_j(s)$ are solutions of the linear system

$$\left.\begin{array}{l} c_1 + c_2 + c_3 = \hat{h}_1(s)\,, \\ c_1e^{\lambda_1(s)L} + c_2e^{\lambda_2(s)L} + c_3e^{\lambda_3(s)L} = \hat{h}_2(s)\,, \\ c_1\lambda_1e^{\lambda_1(s)L} + c_2\lambda_2e^{\lambda_2(s)L} + c_3\lambda_3e^{\lambda_3(s)L} = \hat{h}_3(s)\,. \end{array}\right\} \tag{2.23}$$

By Cramer's rule, c_j has the form

$$c_j = \frac{\Delta_j(s,L)}{\Delta(s,L)} \tag{2.24}$$

where $\Delta(s,L)$ is the determinant of the coefficient matrix of the linear system (2.23) and $\Delta_j(s,L)$ is the determinant of the matrix that is obtained by replacing the j-th column of the coefficient matrix by $(\hat{h}_1,\hat{h}_2,\hat{h}_3)^T$. Hence, (2.20), (2.21) and (2.24) imply

$$u(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\sum_{j=1}^{3}\frac{\Delta_j(s,L)}{\Delta(s,L)}e^{\lambda_j(s)x}ds\,. \tag{2.25}$$

It will be shown in a moment that $\Delta(s,L) \neq 0$ whenever $\operatorname{Re} s \geq 0$. It is easy to see that the terms in the integral in (2.25) are analytic in s and the left-hand side of (2.25) is independent of r. It follows that we may take $r = 0$ in the formula, thereby obtaining the tidy representation

$$u(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{\Delta_j(s,L)}{\Delta(s,L)}e^{\lambda_j(s)x}ds\,. \tag{2.26}$$

Here, if s is written as $s = i\rho^3$, then the λ_j, $j = 1, 2, 3$, are the roots of

$$i\rho^3 + (1 + 3b^3)\lambda - 3b\lambda^2 + \lambda^3 = 0 \tag{2.27}$$

or, equivalently,

$$(\lambda - b)^3 + (\lambda - b) + (i\rho^3 + b + b^3) = 0\,. \tag{2.28}$$

Notice that the λ_j are independent of L, but certainly depend on b. To be definite, the λ_j's are arranged so that $\mathrm{Re}\,\lambda_1 \le 0$, $\mathrm{Re}\,\lambda_2 > 0$, $\mathrm{Im}\,\lambda_2 > 0$ and $\mathrm{Re}\,\lambda_3 > 0$, $\mathrm{Im}\,\lambda_3 < 0$. Thus, if $\rho = 0$, then $\lambda_1 = 0$ and $\lambda_{2,3} = (3b \pm \sqrt{9b^2 - 4(1 + 3b^2)})/2$. Moreover, from (2.27) and (2.28), it is straightforward to check that for nonzero ρ, there are no λ_j that are either purely real or purely imaginary, nor can they lie on the axis $\mathrm{Re}\,\lambda = b$. This implies that for nonzero ρ in R,

$$\mathrm{Re}\,\lambda_1 < 0\,,\ \mathrm{Im}\,\lambda_1 \neq 0\,, \qquad \mathrm{Re}\,\lambda_2 > b\,,\ \mathrm{Im}\,\lambda_2 > 0\,, \qquad \mathrm{Re}\,\lambda_3 > b\,,\ \mathrm{Im}\,\lambda_3 < 0. \tag{2.29}$$

Also, the three curves $\lambda_j = \lambda_j(\rho)$ for $\rho \in R$ are symmetric about the real axis in the λ-plane, which is to say, $\mathrm{Im}\,\lambda_1(-\rho) = -\mathrm{Im}\,\lambda_1(\rho)$ and $\mathrm{Im}\,\lambda_2(-\rho) = -\mathrm{Im}\,\lambda_3(\rho)$.

Isolate the terms in (2.26) containing $h_j(t)$, $j = 1, 2, 3$ as follows. Let $\Delta_{j,m}(s, L)$ be obtained from $\Delta_j(s, L)$ by letting $\hat{h}_m(s) = 1$, $\hat{h}_k(s) = 0$ for $k, m = 1, 2, 3$, $k \neq m$. Then, the solution $u(x, t)$ in (2.26) can be rewritten as

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) \tag{2.30}$$

where $u_m(x, t)$ only involves $h_m(t)$ and is defined by

$$\begin{aligned} u_m(x, t) =& \sum_{j=1}^{3} \frac{1}{2\pi i} \int_0^{i\infty} e^{st} \frac{\Delta_{j,m}(s, L)}{\Delta(s, L)} e^{\lambda_j(s)x} \hat{h}_m(s) ds \\ &+ \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{-i\infty}^{0} e^{st} \frac{\Delta_{j,m}(s, L)}{\Delta(s, L)} e^{\lambda_j(s)x} \hat{h}_m(s) ds \\ =& \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{\infty} e^{i\rho^3 t} \frac{\Delta^+_{j,m}(\rho, L)}{\Delta^+(\rho, L)} e^{\lambda_j^+(\rho)x} \hat{h}^+_m(\rho) 3\rho^2 d\rho \\ &+ \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^{\infty} e^{-i\rho^3 t} \frac{\Delta^-_{j,m}(\rho, L)}{\Delta^-(\rho, L)} e^{\lambda_j^-(\rho)x} \hat{h}^-_m(\rho) 3\rho^2 d\rho \\ =& I^+_m + I^-_m \end{aligned} \tag{2.31}$$

where $\hat{h}^+_m(\rho) = \hat{h}_m(i\rho^3)$ and $\Delta^\pm, \Delta^\pm_{j,m}$ are obtained from $\Delta, \Delta_{j,m}$ by replacing s with $i\rho^3$ for $\rho \ge 0$. Note that $\Delta^-(\rho) = \overline{\Delta^+(\rho)}$, $\Delta^-_{j,m}(\rho) = \overline{\Delta^+_{j,m}(\rho)}$, $j = 1, 2, 3$ and $\hat{h}^-_m = \overline{\hat{h}^+_m}$.

Lemma 2.3 *For any $\rho \in R^+$, $\Delta^+(\rho) \neq 0$ and there is a constant C_1 such that $|\Delta^+(\rho)| \leq C_1|\rho|e^{(\sqrt{3}\rho L)/2}$. Moreover, there is a constant C such that*

$$\Delta^+(\rho) = \rho\left(\frac{\sqrt{3}}{2} - \frac{3}{2}i\right)e^{(\sqrt{3}\rho L)/2} + \Delta_0^+(\rho)$$

where $|\Delta_0^+(\rho)| \leq C|\rho|$ as $\rho \to +\infty$. The constants C and C_1 may be chosen to be independent of L.

Proof: If $\Delta^+(\rho) = \Delta(i\rho^3, L) = 0$ for some $\rho \geq 0$, then there is a nontrivial solution of

$$\begin{aligned} &i\rho^3 u + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = 0, \\ &u(0) = 0, \quad u(L) = u_x(L) = 0\,. \end{aligned} \tag{2.32}$$

Multiply (2.32) by $\bar{u}$ and integrate it from 0 to L to obtain

$$\int_0^L (i\rho^3|u|^2 + (1+3b^2)u_x\bar{u} + 3b|u_x|^2 - u_{xx}\bar{u}_x)dx = 0 \tag{2.33}$$

where integration by parts has been used. The elementary identity

$$\int_0^L u_x\bar{u}dx + \int_0^L \bar{u}_x u dx = 0 = 2\,\mathrm{Re}\int_0^L u_x\bar{u}dx$$

implies that $\int_0^L u_x\bar{u}dx$ is purely imaginary. Also,

$$\int_0^L u_{xx}\bar{u}_x dx + \int_0^L u_x\bar{u}_{xx}dx = u_x\bar{u}_x|_0^L = -|u_x(0)|^2\,.$$

which implies 2Re $\int_0^L u_{xx}\bar{u}_x dx = -|u_x(0)|^2$. Therefore, the real part of (2.33) has to be of the form

$$\int_0^L 3b|u_x|^2dx + \frac{1}{2}|u_x(0)|^2 = 0$$

which yields $u_x \equiv 0$ or $u = 0$. This contradiction leaves only the conclusion $\Delta(i\rho^3, L) \neq 0$ for all $\rho \in R^+$. Moreover, it is straightforward to show that

$$\begin{aligned} \Delta(s) = &e^{(\lambda_2(s)+\lambda_3(s))L}(\lambda_3(s) - \lambda_2(s)) + (\lambda_2(s) - \lambda_1(s))e^{(\lambda_1(s)-\lambda_3(s))L} \\ &+ (\lambda_1(s) - \lambda_3(s))e^{(\lambda_1(s)-\lambda_2(s))L}\,. \end{aligned} \tag{2.34}$$

As $\rho \to +\infty$, the $\lambda_j^+(\rho)$ have the asymptotic forms

$$\lambda_1^+(\rho) \sim -\frac{\sqrt{3}}{2}\rho - \frac{1}{2}\rho i\,, \quad \lambda_3^+(\rho) \sim \frac{\sqrt{3}}{2}\rho - \frac{1}{2}\rho i\,, \quad \lambda_2^+(\rho) \sim b + \rho i\,. \tag{2.35}$$

Substitution of the asymptotic forms in (2.35) into (2.34) gives the asymptotic form of $\Delta^+(\rho, L)$, namely

$$\Delta^+(i\rho^3, L) = \left(\frac{\sqrt{3}}{2}\rho - \frac{3}{2}\rho i\right) e^{\left(\frac{\sqrt{3}}{2}\rho + \frac{1}{2}\rho i\right)L} - \sqrt{3}\rho e^{-\rho i L}$$

$$+ \left(\frac{\sqrt{3}}{2}\rho + \frac{3}{2}\rho i\right) e^{\left(-\frac{\sqrt{3}}{2}\rho + \frac{1}{2}\rho i\right)L} + O(|\rho|) = \rho\left(\frac{\sqrt{3}}{2} - \frac{3}{2}i\right) e^{\left(\frac{\sqrt{3}}{2}\rho + \frac{1}{2}\rho i\right)L} + \Delta_0^+(\rho)$$

as $\rho \to +\infty$. This implies that $|\Delta^+(i\rho^3, L)| \le C_1|\rho| e^{\left(\frac{\sqrt{3}}{2}\rho\right)L}$ and $|\Delta_0^+(\rho)| \le C\rho$, where C and C_1 can be chosen to be independent of L. Similar estimates hold for $\rho < 0$. The proof is complete. □

Remark 2.4 *The proof of Lemma 2.3 is easily extended to show that $\Delta(s, L) \neq 0$ for any $s = r + i\rho^3$ with $r > 0$ and $\rho \in R$*

Next are recounted a couple of technical lemmas that will find frequent use in this section. The proofs can be found in [13].

Lemma 2.5 *For any $f \in L_2(0, +\infty)$, let Kf be the function defined by*

$$Kf(x) = \int_0^{+\infty} e^{\nu(\mu)x} f(\mu)\, d\mu$$

where $\nu(\mu)$ is a continuous, complex-valued function on $(0, +\infty)$ satisfying

(i) $\sup_{0<\mu\le\delta} \frac{1}{\mu}|Re\,\nu(\mu)| \ge b > 0$,

(ii) $\lim_{\mu\to+\infty} \frac{1}{\mu}\nu(\mu) = \alpha + i\beta$ with $\alpha^2 + \beta^2 \neq 0$,

(iii) $Re\,\nu(\mu) \neq 0$ for $\mu > 0$.

Then, there exists a constant C independent of L such that for all $f \in L_2(R^+)$,

$$||Kf||_{L_2(0,L)} \le C\left(||e^{Re\,\nu(\cdot)L} f(\cdot)||_{L_2(R^+)} + ||f(\cdot)||_{L_2(R^+)}\right).$$

Lemma 2.6 *Let $a > 0$ be given. For any $f \in L_2(0, a)$, let Gf be the function defined by*

$$Gf(x) = \int_0^a e^{i\xi(\mu)x} f(\mu)\, d\mu$$

where ξ is a continuous real-valued function defined on the interval $[0, a]$ which is C^1 on the open interval $(0, a)$ and such that there is a constant C_1 for which $\frac{1}{|\xi'(\mu)|} \le C_1$ for $0 < \mu < a$. Then there exists a constant C such that for all $f \in L_2(0, a)$,

$$||Gf||_{L_2(0,a)} \le C||f||_{L_2(0,a)}.$$

The following propositions provide estimates for u_1, u_2 and u_3, respectively.

Proposition 2.7 *There exists a constant C independent of L but which may depend on b, such that for all $h_1 \in H^{\frac{1}{3}}(R^+)$*

$$\sup_{0\le t<+\infty} \|u_1(\cdot,t)\|_{L_2(0,L)} + \|u_1\|_{L_2(R^+;H^1(0,L))} \le C\|h_1\|_{H^{\frac{1}{3}}(R^+)}$$

and

$$\sup_{x\in[0,L]} \left(\|\partial_x u_1(x,\cdot)\|_{L_2(R^+)} + \|u_1(x,\cdot)\|_{H_t^{1/3}(R^+)}\right) \le C\|h_1\|_{H^{\frac{1}{3}}(R^+)}\,.$$

Proof: First assume that $h_1(t)$ is smooth with compact support in R^+. From the definitions of $\Delta_{i,1}$, $i=1,2,3$, it is determined that

$$\Delta_{1,1} = (\lambda_3-\lambda_2)e^{(\lambda_2+\lambda_3)L}\,, \qquad \Delta_{2,1} = (\lambda_1-\lambda_3)e^{(\lambda_1+\lambda_3)L}\,, \qquad \Delta_{3,1} = (\lambda_2-\lambda_1)e^{(\lambda_1+\lambda_2)L}\,.$$

Because of the asymptotic forms (2.35) of the λ_i, $i=1,2,3$, it thus follows readily that

$$\frac{\Delta^+_{1,1}(\rho,L)}{\Delta^+(\rho,L)} \sim 1\,, \qquad \frac{\Delta^+_{2,1}(\rho,L)}{\Delta^+(\rho,L)} \sim e^{-\frac{\sqrt{3}}{2}\rho L}\,, \qquad \frac{\Delta^+_{1,1}(\rho,L)}{\Delta^+(\rho,L)} \sim e^{-\sqrt{3}\rho L}\,, \tag{2.36}$$

as ρ goes to $+\infty$. Note that the coefficients of the highest order terms in these asymptotic forms are independent of L. An application of Lemmas 2.5 and 2.6 yields a constant C independent of L such that

$$\begin{aligned}\|I_1^+(x,\cdot)\|_{L_2(0,T)} &\le \sum_{j=1}^{3}\int_0^{+\infty}\left|\frac{\Delta^+_{j,1}(\rho,L)}{\Delta^+(\rho,L)}\right|^2\left(e^{\mathrm{Re}\,\lambda_j^+(\rho)L}+1\right)^2\left|\hat{h}_1^+(\rho)3\rho^2\right|^2 d\rho \\ &\le C\sum_{j=1}^{3}\int_0^{+\infty}\left|\hat{h}_1^+(\rho)\rho^2\right|^2 d\rho \le C\int_0^{+\infty}(1+\mu)^{2/3}\left|\int_0^{+\infty}e^{-i\mu\tau}h_1(\tau)d\tau\right| d\mu \\ &\le C\|h_1\|^2_{H^{\frac{1}{3}}(R^+)}\,.\end{aligned} \tag{2.37}$$

The same argument can be used to obtain the inequality

$$\|I_1^-(x,\cdot)\|_{L_2(0,T)} \le C\|h_1\|^2_{H^{\frac{1}{3}}(R^+)}$$

where C is independent of L. Moreover, observe that

$$\partial_x I_1^+(x,t) = \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty} e^{ist}\lambda_j^+(s^{1/3})e^{\lambda_j^+(s^{1/3})x}\frac{\Delta^+_{j,1}(s^{1/3},L)}{\Delta^+(s^{1/3},L)}\hat{h}_1^+(s^{1/3})ds\,.$$

The Plancherel Theorem (with respect to t) implies that for any $x\in(0,L)$,

$$\|\partial_x I_1^+(x,\cdot)\|^2_{L_{2,t}(R^+)} \le \sum_{j=1}^{3}\frac{1}{2\pi}\int_0^{+\infty}\left|\lambda_j^+(s^{1/3})e^{\lambda_j^+(s^{1/3})x}\frac{\Delta^+_{j,1}(s^{1/3},L)}{\Delta^+(s^{1/3},L)}\hat{h}_1^+(s^{1/3})\right|^2 ds\,. \tag{2.38}$$

By Lemmas 2.5 and 2.6 and analysis analogous to that leading to (2.37), there obtains

$$\int_0^L \|\partial_x I_1^+(x,\cdot)\|^2_{L_2(R^+)}dx \le C\sum_{j=1}^3 \int_0^{+\infty} \left|\lambda_j^+(s^{1/3})\frac{\Delta_{j,1}^+(s^{1/3},L)}{\Delta^+(s^{1/3},L)}\left(e^{\mathrm{Re}\,\lambda_j^+(s^{1/3})L}+1\right)\hat{h}_1^+(s^{1/3})\right|^2 ds$$
$$\le C\int_0^{+\infty}(1+\mu)^{2/3}\left|\int_0^{+\infty}e^{-i\mu\tau}h_1(\tau)d\tau\right|d\mu \le C\|h_1\|^2_{H^{\frac{1}{3}}(R^+)}.$$

In addition, (2.38) yields that for $k=0,1$,

$$\sup_{x\in[0,L]}\|\partial_x^k I_1^+(x,\cdot)\|^2_{L_2(R^+)} \le C\sum_{j=1}^3\int_0^{+\infty}\sup_{x\in[0,L]}\left|\left(\lambda_j^+(s^{1/3})\right)^k e^{\lambda_j^+(s^{1/3})x}\frac{\Delta_{j,1}^+(s^{1/3},L)}{\Delta^+(s^{1/3},L)}\hat{h}_1^+(s^{1/3})\right|^2 ds$$
$$\le C\sum_{j=1}^3\int_0^{+\infty}(1+\mu)^{2/3}|\hat{h}_1^+(i\mu)|^2d\mu \le C\|h_1\|^2_{H^{\frac{1}{3}}(R^+)}.$$

The same proof may be carried out for $\partial_t^{\frac{1}{3}}I_1^+(t,x)$. Since λ_i is independent of L for $i=1,2,3$, the constant C can be chosen independently of L.

For arbitrary $h_1\in H^{\frac{1}{3}}(R^+)$, a limiting process together with an application of Fatou's Lemma yields the advertised estimates, thereby completing the proof of the proposition. □

Similar arguments can be used to prove the following propositions; details for estimates of this sort can also be found in [13].

Proposition 2.8 *There exists a constant C independent of L but which may depend on b, such that for all $h_2\in H^{\frac{1}{3}}(R^+)$,*

$$\sup_{0\le t<\infty}\|u_2(\cdot,t)\|_{L_2(0,L)}+\|u_2\|_{L_2(R^+;H^1(0,L))}\le C\|h_2\|_{H^{\frac{1}{3}}(R^+)}$$

and

$$\sup_{x\in[0,L]}\left(\|\partial_x u_2(x,\cdot)\|_{L_2(R^+)}+\|u_2(x,\cdot)\|_{H_t^{\frac{1}{3}}(R^+)}\right)\le C\|h_2\|_{H^{\frac{1}{3}}(R^+)}.$$

Proposition 2.9 *There exists a constant C independent of L, but which may depend on b, such that for all $h_3\in L_2(R^+)$,*

$$\sup_{0\le t<\infty}\|u_3(\cdot,t)\|_{L_2(0,L)}+\|u_3\|_{L_2(R^+;H^1(0,L))}\le C\|h_3\|_{L_2(R^+)}$$

and

$$\sup_{x\in[0,L]}\left(\|\partial_x u_3(x,\cdot)\|_{L_2(R^+)}+\|u_3(x,\cdot)\|_{H_t^{\frac{1}{3}}(R^+)}\right)\le C\|h_3\|_{L_2(R^+)}.$$

Let $\vec{h}(t) = (h_1(t), h_2(t), h_3(t))$ and write the solution of (2.14)-(2.16) symbolically as

$$u(t) = u_1(t) + u_2(t) + u_3(t) = W_b(t)\vec{h}\,.$$

With the notation

$$\|\vec{h}\|^2_{H_s} = \left(\|h_1\|^2_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|^2_{H^{\frac{s+1}{3}}(0,T)} + \|h_3\|^2_{H^{\frac{s}{3}}(0,T)}\right), \tag{2.39}$$

Propositions 2.7-2.9 imply the following.

Proposition 2.10 *For all $\vec{h}(t)$ with $\|\vec{h}(t)\|_{H_0} < \infty$, the solution $u(x,t) = W_b(t)\vec{h}$ satisfies*

$$\sup_{0\le t<\infty} \|u(\cdot,t)\|_{L_2(0,L)} + \|u\|_{L_2(R^+;H^1(0,L))} \le C\|\vec{h}\|_{H_0}$$

and

$$\sup_{x\in[0,L]} \left(\|\partial_x u(x,\cdot)\|_{L_{2,t}(R^+)} + \|u(x,\cdot)\|_{L_{2,t}(R^+)} + \|u(x,\cdot)\|_{H_t^{\frac{1}{3}}(R^+)}\right) \le C\|\vec{h}\|_{H_0}$$

where C is independent of L and $\vec{h}$.

To study $W_L(t)\phi$, consider the pure initial-value problem

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = 0\,, \qquad x \in R, \quad t > 0, \tag{2.40}$$

$$u(x,0) = \phi^*(x), \quad x \in R,$$

and denote its solution by v, so that

$$v(x,t) = W_R(t)\phi^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi^3-(1+3b^2)\xi)t - 3b\xi^2 t} e^{ix\xi} \int_{-\infty}^{\infty} e^{-iy\xi}\phi^*(y)dyd\xi. \tag{2.41}$$

Let a function ϕ be defined on an finite interval $[0,L]$ and let ϕ^* be an extension of ϕ to the whole real line R. The mapping $\phi \mapsto \phi^*$ can be organized so that it defines a bounded linear operator B from $H^s(0,L)$ to $H^s(R)$ with

$$\|\phi^*(x)\|_{H^s(R)} = \|B\phi(x)\|_{H^s(R)} \le C\|\phi(x)\|_{H^s(0,L)}$$

and C independent of L. (Indeed, the extension can be made so that $\text{supp}\{\phi^*\} \subset [-1, L+1]$, say, if we so desire.) Henceforth, $\phi^* = B\phi$ will refer to the result of such an extension operator applied to $\phi \in H^s(0,L)$. Let v defined by (2.41) correspond to $\phi^* = B\phi$ for some $\phi \in H^s(0,L)$. If $\vec{g}(t) = (v(0,t), v(L,t), v_x(L,t))$, then

$$v_{\vec{g}} = v_{\vec{g}}(x,t) = W_b(t)\vec{g} \tag{2.42}$$

is the corresponding solution of the non-homogeneous boundary-value problem (2.14)-(2.16) with boundary condition $\vec{h}(t) = \vec{g}(t)$ for $t \ge 0$. It is clear that for $x \in [0,L]$, the function $v(x,t) - v_{\vec{g}}(x,t)$ solves the IBVP (2.4)-(2.6), and this leads directly to a representation of the semigroup W_L in terms of $W_b(t)$ and $W_R(t)$.

Proposition 2.11 *For a given s and $\phi \in H^s(0,L)$, if ϕ^* is its extension to R as described above, then $W_L(t)\phi$ may be written in the form*

$$W_L(t)\phi = W_R(t)\phi^* - W_b(t)\vec{g} \tag{2.43}$$

for any x, $t > 0$, where the components of $\vec{g}$ are the traces of $W_R(t)\phi^*$ at 0 and L and of $\partial_x W_R(t)\phi^*$ at L.

Next, attention is turned to the spatial trace of $W_R(t)\phi$.

Proposition 2.12 *Let $s \geq 0$ be given. There exists a constant C depending only on s such that*

$$\sup_{x\in R} \|W_R(t)\phi\|_{H_t^{\frac{s+1}{3}}(R)} \leq C\|\phi\|_{H^s(R)}\,, \tag{2.44}$$

$$\sup_{x\in R} \|\partial_x W_R(t)\phi\|_{H_t^{\frac{s}{3}}(R)} \leq C\|\phi\|_{H^s(R)}\,, \tag{2.45}$$

for any $\phi \in H^s(R)$.

Proof: We only provide the proof of (2.44). The proof of (2.45) is very similar and is therefore omitted. For convenience, we take $b = 1/3$ so that $1 + 3b^2 = 4/3$. This choice simplifies the formulas, but does not affect the result. Let $u(x,t) = W_R(t)\phi$. Then, the change of variables $\xi^3 = \lambda$ reveals that

$$u(x,t) \;=\; \frac{1}{3}\int_{-\infty}^{\infty} e^{ix\lambda^{1/3}} e^{i(\lambda-\rho\lambda^{1/3})t-\lambda^{2/3}t}\lambda^{-2/3}\hat{\phi}(\lambda^{1/3})d\lambda$$

for $t \geq 0$. The proof of the following assertion will be given in the Appendix.

Claim: The linear mapping defined by

$$\mathcal{L}(g)(t) = \int_{-\infty}^{+\infty} e^{i(\lambda-\rho\lambda^{1/3})t-\lambda^{2/3}t} g(\lambda)d\lambda$$

is bounded from $L^2(R)$ to $L^2(R^+)$.

Using this result, it follows that

$$\|u(x,t)\|_{L_{2,t}(R^+)} \leq C\|\lambda^{-2/3}\hat{\phi}(\lambda^{1/3})\|_{L_2(R)} \leq C\|\phi\|_{H^{-1}(R)}$$

for any $x \in R$. Note that

$$u_t(x,t) = \int_{-\infty}^{\infty} e^{ix\xi} e^{i(\xi^3-\rho\xi)t-\xi^2 t}[i(\xi^3-\rho\xi)-\xi^2]\hat{\phi}(\xi)d\xi\,.$$

Applying the aforementioned change of variables and the Claim yields

$$||u(x,\cdot)||_{H^1(R)} \leq C||\phi||_{H^2(R)}$$

for any $x \in R$. The inequality (2.44) then follows by interpolation. □

By Proposition 2.12, it is known that $\vec{g} = (W_R(t)\phi^*|_{x=0}, W_R(t)\phi^*|_{x=L}, \partial_x W_R(t)\phi^*|_{x=L}) \in H_0$ satisfies

$$||\vec{g}||_{H_0} \leq C||\phi||_{L_2(0,L)}$$

if $\phi \in L_2(0,L)$. Therefore, by Proposition 2.10, $u(x,t) = W_b(t)\vec{g}$ satisfies

$$\sup_{x\in[0,L]} \left(||\partial_x u(x,\cdot)||_{L_{2,t}(R^+)} + ||u(x,\cdot)||_{L_{2,t}(R^+)} + ||u(x,\cdot)||_{H_t^{\frac{1}{3}}(R^+)} \right) \leq C||\vec{g}||_{H_0} \leq C||\phi||_{L_2(0,L)} \tag{2.46}$$

where C is independent of L. Combining (2.43) and (2.46) with Proposition 2.1 gives the following result.

Proposition 2.13 *For all $\phi \in L_2(R)$, the solution $u(x,t) = W_L(t)\phi$ of (2.4)-(2.6) satisfies*

$$\sup_{0\leq t<\infty} ||u(\cdot,t)||_{L_2(0,L)} + ||u||_{L_2(R^+;H^1(0,L))} \leq C||\phi||_{L_2(0,L)}$$

and

$$\sup_{x\in[0,L]} \left(||\partial_x u(x,\cdot)||_{L_{2,t}(R^+)} + ||u(x,\cdot)||_{L_{2,t}(R^+)} + ||u(x,\cdot)||_{H_t^{\frac{1}{3}}(R^+)} \right) \leq C||\phi||_{L_2(0,L)}$$

where C is independent of L.

Combining Propositions 2.2 and 2.13 leads to the following conclusion.

Proposition 2.14 *For all $f(x,t) \in L_1([0,T], L_{2,x}(0,L))$, the aforementioned solution $u(x,t) = \int_0^t W_L(t-\tau)f(\cdot,\tau)d\tau$ of (2.8)-(2.10) satisfies*

$$\sup_{0\leq t\leq T} ||u(\cdot,t)||_{L_2(0,L)} + ||u||_{L_2([0,T];H^1(0,L))} \leq C\int_0^T ||f(\cdot,\tau)||_{L_2(0,L)}d\tau$$

and

$$\sup_{x\in[0,L]} \left(||\partial_x u(x,\cdot)||_{L_{2,t}(0,T)} + ||u(x,\cdot)||_{L_{2,t}(0,T)} + ||u(x,\cdot)||_{H_t^{\frac{1}{3}}(0,T)} \right) \leq C\int_0^T ||f(\cdot,\tau)||_{L_2(0,L)}d\tau$$

where C is independent of L and T.

Proof: Observe that

$$u(x,t) = \int_0^t \Big(W_L(t-\tau)f(\cdot,\tau)\Big)d\tau = \int_0^T \xi_{(0,t)}(\tau)\Big(W_L(t-\tau)f(\cdot,\tau)\Big)d\tau$$

where

$$\xi_{(0,t)}(\tau) = \begin{cases} 1 & if\ \tau \in (0,t); \\ 0 & if\ \tau > t. \end{cases}$$

Applying the Minkowski integral inequality gives

$$\begin{aligned} \|u(x,\cdot)\|_{L^2(0,T)} &\le \int_0^T \left(\int_0^T \left|\xi_{(0,t)}(\tau)W_L(t-\tau)f(\cdot,\tau)\right|^2 dt\right)^{1/2} d\tau \\ &= \int_0^T \left(\int_\tau^T |W_L(t-\tau)f(\cdot,\tau)|^2\, dt\right)^{1/2} d\tau. \end{aligned}$$

Thus, invoking Proposition 2.13 yields

$$\begin{aligned} \sup_{x\in(0,L)} \|u(x,\cdot)\|_{L^2(0,T)} &\le \int_0^T \sup_{x\in(0,L)} \left(\int_\tau^T |W_L(t-\tau)f(\cdot,\tau)|^2\, dt\right)^{1/2} d\tau \\ &\le C\int_0^T \|f(\cdot,\tau)\|_{L^2(0,L)} d\tau \end{aligned}$$

where C is independent of L and T. The proof is complete. □

3 Linear estimates for (1.13)

Consider now the linear IBVP

$$w_t + (1+3b^2)w_x - 3bw_{xx} + w_{xxx} = 0\,, \quad x>0\,, \quad t>0\,, \tag{3.1}$$

$$w(0,t) = h(t), \quad w(x,0) = \phi(x)\,, \tag{3.2}$$

on $R^+ = [0,+\infty)$. The estimates for (3.1)-(3.2) are very similar to those for (2.1)-(2.3). Consequently, we only list the properties of the solutions of (3.1)-(3.2) and omit their proofs. Denote the solution of the problem

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = 0\,, \qquad x>0\,, \quad t>0 \tag{3.3}$$

$$u(0,t) = 0\,, \quad t>0, \qquad u(x,0) = \phi(x)\,, \quad x>0 \tag{3.4}$$

by $u(t) = W_\infty(t)\phi$.

Proposition 3.1 *For any $\phi \in L_2(R^+)$, $u(t) = W_\infty(t)\phi$ satisfies*

$$\|u(\cdot,t)\|^2_{L_2(R^+)} + 6b\int_0^t\int_0^\infty u_x^2(x,\tau)dxd\tau + \int_0^t u_x^2(0,\tau)d\tau = \|\phi\|^2_{L_2(R^+)}\,.$$

The solution $u(t) = \int_0^t W_\infty(t-\tau)f(\cdot,\tau)d\tau$ of the inhomogeneous linear problem

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = f(x,t)\,, \qquad x>0\,, \quad t>0\,, \tag{3.5}$$

$$u(0,t) = 0\,, \quad t>0\,, \qquad u(x,0) = 0\,, \quad x>0\,, \tag{3.6}$$

obeys the inequality in the following proposition.

Proposition 3.2 *For any $f \in L_{1,loc}(R^+; L_{2,x}(R^+))$, the solution $u(t) = \int_0^t W_L(t-\tau)f(\cdot,\tau)d\tau$ of (3.5) and (3.6) satisfies*

$$\sup_{t\in[0,T]}\|u(\cdot,t)\|^2_{L_2(R^+)} + 6b\int_0^T\int_0^\infty u_x^2 dxdt + \int_0^T u_x^2(0,\tau)d\tau \le 8\|f\|^2_{L_1([0,T];L_2(R^+))}$$

for any $T \ge 0$.

Next, consider the boundary-value problem

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = 0\,, \qquad x>0\,, \quad t>0\,, \tag{3.7}$$

$$u(0,t) = h(t)\,, \qquad u(x,0) = 0\,. \tag{3.8}$$

By applying the Laplace transform to (3.7) with respect to t, the solution u may be written as

$$u(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\hat{h}(s)e^{\lambda_1(s)x}ds = W_{qb}h\,, \tag{3.9}$$

where λ_1 satisfies

$$s + (1+3b^2)\lambda - 3b\lambda^2 + \lambda^3 = 0 \tag{3.10}$$

and $\mathrm{Re}\,\lambda_1 < 0$ for $s \neq 0$. If $s = i\rho$, then for ρ small,

$$\lambda_1 \sim -\frac{i\rho}{1+3b^2}\,, \tag{3.11}$$

and for $\rho > 0$ large

$$\lambda_1^+(\rho) \sim -\frac{\sqrt{3}}{2}\rho - \frac{1}{2}\rho i\,. \tag{3.12}$$

The proofs of the following propositions follow by the same arguments as for the analogous results on $[0,L]$, and so are omitted.

Proposition 3.3 *There exists a constant C such that for all $h \in H^{\frac{1}{3}}(R^+)$, $u(x,t) = W_{qb}h$ satisfies*

$$\sup_{0 \le t < +\infty} \|u(\cdot,t)\|_{L_2(R^+)} + \|u\|_{L_2(R^+;H^1(R^+))} \le C\|h\|_{H^{\frac{1}{3}}(R^+)}$$

and

$$\sup_{x \in (R^+)} \left(\|\partial_x u(x,\cdot)\|_{L_2(R^+)} + \|u(x,\cdot)\|_{H_t^{1/3}(R^+)} \right) \le C\|h\|_{H^{\frac{1}{3}}(R^+)}.$$

Using the properties of $W_R(t)\phi$ that follow from the representation (2.41), the same proofs as those appearing in support of Propositions 2.13 and 2.14 yield the following spatial trace result for $W_\infty(t)\phi$.

Proposition 3.4 *For all $\phi \in L_2(R^+)$, the solution $u(x,t) = W_\infty(t)\phi$ of (3.3)-(3.4) satisfies*

$$\sup_{0 \le t < \infty} \|u(\cdot,t)\|_{L_2(R^+)} + \|u\|_{L_2(R^+;H^1(R^+))} \le C\|\phi\|_{L_2(R^+)}$$

and

$$\sup_{x \in [0,\infty)} \left(\|\partial_x u(x,\cdot)\|_{L_{2,t}(R^+)} + \|u(x,\cdot)\|_{L_{2,t}(R^+)} + \|u(x,\cdot)\|_{H_t^{\frac{1}{3}}(R^+)} \right) \le C\|\phi\|_{L_2(R^+)}.$$

Proposition 3.5 *For all $f(x,t) \in L_1([0,T], L_{2,x}(R^+))$, the solution*

$$u(x,t) = \int_0^t W_\infty(t-\tau) f(\cdot,\tau)d\tau$$

of (3.5)-(3.6) satisfies

$$\sup_{0 \le t < T} \|u(\cdot,t)\|_{L_2(R^+)} + \|u\|_{L_2([0,T];H^1(R^+))} \le C \int_0^T \|f(\cdot,\tau)\|_{L_2(R^+)} d\tau$$

and

$$\sup_{x \in (R^+)} \left(\|\partial_x u(x,\cdot)\|_{L_{2,t}([0,T])} + \|u(x,\cdot)\|_{L_{2,t}([0,T])} + \|u(x,\cdot)\|_{H_t^{\frac{1}{3}}([0,T])} \right) \le C \int_0^T \|f(\cdot,\tau)\|_{L_2(R^+)} d\tau$$

where the constants C are independent of T.

4 Local well-posedness

In this section, consideration is given to the full nonlinear IBVP

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} + e^{(b+b^3)t-bx}(u_x - bu)u = f(x,t) \text{ for } (x,t) \in [0,L] \times R^+, \tag{4.1}$$

$$u(0,t) = h_1(t), \qquad u(L,t) = h_2(t), \qquad u_x(L,t) = h_3(t), \qquad t > 0, \tag{4.2}$$

$$u(x,0) = \phi(x), \quad 0 < x < L. \tag{4.3}$$

For any $T > 0$, let $X_{s,L,T}$ be defined to be

$$X_{s,L,T} = L_1([0,T]; H^s(0,L)) \times H^s(0,L) \times H^{\frac{1+s}{3}}(0,T) \times H^{\frac{1+s}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

with its product topology and let $Y_{s,L,T}$ be the collection of

$$v \in C([0,T]; H^s(0,L)) \cap L_2([0,T]; H^{s+1}(0,L))$$

such that $v \in C([0,L]; H^{\frac{1+s}{3}}(0,T))$ and $v_x \in C([0,L]; H^{\frac{s}{3}}(0,T))$. A norm $\|\cdot\|_{Y_{s,L,T}}$ on the space $Y_{s,L,T}$ is the obvious one, namely,

$$\begin{aligned} \|v\|^2_{Y_{s,L,T}} :=& \|v\|^2_{C([0,T];H^s(0,L))} + \|v\|^2_{L_2([0,T];H^{1+s}(0,L))} \\ &+ \|v\|^2_{C([0,L];H^{\frac{1+s}{3}}(0,T))} + \|v_x\|^2_{C((0,L);H^{\frac{s}{3}}(0,T))}. \end{aligned} \tag{4.4}$$

The space $X_{s,L,T}$ was used in [13] while $Y_{s,L,T}$ was not defined before. However, it was shown in [13] that the solutions obtained there have finite norms defined in (4.4). Here, because of the KdV-Burgers nature of (4.1), we can derive the uniform estimates of solutions with respect to L and prove the local well-posedness of the problem in a small time interval which is independent of L. For notational convenience, write $X_{0,L,T} = X_{L,T}$ and $Y_{0,L,T} = Y_{L,T}$. The space $Y_{s,L,T}$ possesses the following helpful property.

Lemma 4.1 *For any $T > 0$, $s \geq 0$ and u, $v \in Y_{s,L,T}$,*

$$\int_0^T \left\| \Big(u(\cdot,t)v(\cdot,t)\Big)_x \right\|_{H^s(0,L)} dt \leq C\left(T^{1/2} + T^{1/3}\right) \|u\|_{Y_{s,L,T}} \|v\|_{Y_{s,L,T}}$$

where the constant C is independent of T and L.

Proof: The proof is exactly the same as the proof of Lemma 3.1 of [13] except that in this earlier analysis, the dependence on the constant C on L was not examined. Thus, it remains to be shown that the constant can be chosen independent of L. Elementary consideration implies that

$$\max_{x\in[0,L]} u^2(x,t) \leq \frac{1}{L}\int_0^L u^2(s,t)ds + 2\|u(\cdot,t)\|_{L_2(0,L)}\|u_x(\cdot,t)\|_{L_2(0,L)}.$$

Thus, if $L \geq 1$, it follows that

$$\max_{x\in[0,L]} |u(x,t)| \leq \|u(\cdot,t)\|_{L_2(0,L)} + \sqrt{2}\|u(\cdot,t)\|^{1/2}_{L_2(0,L)}\|u_x(\cdot,t)\|^{1/2}_{L_2(0,L)},$$

and this is independent of L. The remainder of the proof is the same as that of Lemma 3.1 in [13] and is therefore omitted. □

The next step is to show that the IBVP (4.1)-(4.3) is locally well-posed in the space $X_{L,T}$.

Proposition 4.2 *Let $T > 0$ be given. For any $(f, \phi, \vec{h}) \in X_{L,T}$ with $\vec{h} = (h_1, h_2, h_3)$, there is a $T^* \in (0, T]$ depending only on $\|(f, \phi, \vec{h})\|_{X_{L,T}}$ such that the IBVP (4.1)-(4.3) admits a unique solution $u \in Y_{L,T^*}$. Moreover, for any $T' < T^*$, there is a neighborhood U of $(f, \phi, \vec{h})$ such that the IBVP (4.1)-(4.3) admits a unique solution in the space $Y_{L,T'}$ for any $(g, \psi, \vec{h}_1) \in U$ and the corresponding solution map from U to $Y_{L,T'}$ is Lipschitz continuous.*

Proof: Write the IBVP (4.1)-(4.3) in its integral equation form, *viz.*

$$u(t) = W_L(t)\phi + W_b(t)\vec{h} + \int_0^t W_L(t-\tau)\Big\{f(\cdot,\tau) - e^{(b+b^3)\tau - bx}(u_x - bu)u\Big\}d\tau\,, \tag{4.5}$$

where the operator $W_b(t)$ is as defined in Proposition 2.10 and the spatial variable is suppressed throughout. For given $(f, \phi, \vec{h}) \in X_{L,T}$, let $r > 0$ and $\theta > 0$ be constants to be determined and let

$$S_{\theta,r} = \{v \in Y_{L,\theta} : \|v\|_{Y_{L,\theta}} \leq r\}.$$

The set $S_{\theta,r}$ is a closed, bounded subset of the space $Y_{L,\theta}$ and is therefore a complete metric space in the topology induced by that of $Y_{L,\theta}$. Define a map Γ on $S_{\theta,r}$ by

$$\Gamma(v) = W_L(t)\phi + W_b(t)\vec{h} + \int_0^t W_L(t-\tau)\{f - e^{(b+b^3)\tau - bx}(v_x - bv)v\}d\tau\,.$$

For any $v \in S_{\theta,r}$,

$$\begin{aligned}\|\Gamma(v)\|_{Y_{L,\theta}} &\leq C_0(\|\phi\|_{L_2(0,T)} + \|\vec{h}\|_{H_0}) \\ &\quad + C_1\int_0^\theta \Big(\|f(\cdot,\tau)\|_{L_2(0,L)} + e^{(b+b^3)\theta}(\|vv_x(\cdot,\tau)\|_{L_2(0,L)} + \|bv^2(\cdot,\tau)\|_{L_2(0,L)})\Big)\,d\tau \\ &\leq (C_0 + C_1)\|(f,\phi,\vec{h})\|_{X_{L,T}} + C_2 e^{(b+b^3)\theta}\left(\theta^{1/2} + \theta^{1/3}\right)\|v\|^2_{Y_{L,\theta}}\end{aligned}$$

where C_0, C_1 and C_2 are constants independent of L and T. As the norm on $Y_{L,\theta}$ has three parts, this amounts to three inequalities, all of which follow immediately from the linear estimates in Section 2 and in Lemma 4.1. Choosing $r > 0$ and $\theta > 0$ so that

$$\begin{cases} r = 2(C_0 + C_1)\|(f,\phi,\vec{h})\|_{X_{L,T}}, \\ C_2 e^{(b+b^3)\theta}\left(\theta^{1/2} + \theta^{1/3}\right) r \leq \frac{1}{4}, \end{cases} \tag{4.6}$$

or, what is the same,

$$e^{(b+b^3)\theta}\left(\theta^{1/2}+\theta^{1/3}\right) \le \frac{1}{8C_2(C_0+C_1)\|(f,\phi,\vec{h})\|_{X_{L,T}}}, \tag{4.7}$$

then

$$\|\Gamma(v)\|_{Y_{L,\theta}} \le r$$

for any $v \in S_{\theta,r}$. Thus, with such a choice of r and θ, Γ maps $S_{\theta,r}$ into $S_{\theta,r}$. The same type of estimates allow one to deduce that for r and θ chosen as in (4.6) and (4.7),

$$\|\Gamma(v_1)-\Gamma(v_2)\|_{Y_{L,\theta}} \le \frac{1}{2}\|v_1-v_2\|_{Y_{L,\theta}}$$

for any $v_1, v_2 \in S_{\theta,r}$. In other words, the map Γ is a contraction mapping of $S_{r,\theta}$. Its fixed point $u=\Gamma(u)$ is the unique solution of the integral equation (4.5) in $S_{\theta,r}$. The Lipschitz continuity follows directly from the contraction mapping principle. The local well-posedness of (4.1)-(4.3) is thereby established. □

Next, attention is given to the local well-posedness of the associated quarter-plane problem, which is the IBVP

$$u_t+(1+3b^2)u_x - 3bu_{xx} + u_{xxx} + e^{(b+b^3)t-bx}(u_x-bu)u = f(x,t)\,, \quad x>0\,, \quad t>0\,, \tag{4.8}$$

$$u(0,t)=h(t)\,, \quad t>0\,, \qquad u(x,0)=\phi(x)\,, \quad x>0\,. \tag{4.9}$$

For any $T>0$, let $X_{s,\infty,T}$ denote the space

$$X_{s,\infty,T} = L_1([0,T];H^s(0,\infty)) \times H^s(0,\infty) \times H^{\frac{1+s}{3}}(0,T)$$

with its product topology and let $Y_{s,\infty,T}$ be the collection of

$$v \in C([0,T];H^s(0,\infty)) \cap L_2([0,T];H^{1+s}(0,\infty))$$

with $v \in C([0,\infty);H^{\frac{1+s}{3}}(0,T))$ and $v_x \in C([0,\infty);H^{\frac{s}{3}}(0,T))$. A norm $\|\cdot\|_{Y_{s,\infty,T}}$ on the space $Y_{s,\infty,T}$ is defined by

$$\begin{aligned}\|v\|^2_{Y_{s,\infty,T}} :=& \|v\|^2_{C([0,T];H^s(R^+)} + \|v\|^2_{L_2([0,T];H^{1+s}(R^+))} \\ &+ \|v\|^2_{C(R^+;H^{\frac{1+s}{3}}(0,T))} + \|v_x\|^2_{C(R^+;H^{\frac{s}{3}}(0,T))}\,.\end{aligned} \tag{4.10}$$

These spaces have not been defined previously for the solutions of KdV equation and can only be introduced for solutions of KdV equation with exponential decay at positive infinity. Again, denote $X_{0,\infty,T}=X_{\infty,T}$ and $Y_{0,\infty,T}=Y_{\infty,T}$. In this context, Lemma 4.1 also holds, thereby yielding the following proposition about local well-posedness, whose proof is omitted.

Proposition 4.3 *Let $T > 0$ be given. For any $(f, \phi, h) \in X_{\infty,T}$, there is a $T^* \in (0, T]$ depending only on $\|(f, \phi, h)\|_{X_{\infty,T}}$ such that the IBVP (4.8)-(4.9) admits a unique solution $u \in Y_{\infty,T^*}$. Moreover, for any $T' < T^*$, there is a neighborhood U of (f, ϕ, h) such that the IBVP (4.8)-(4.9) admits a unique solution in the space $Y_{\infty,T'}$ for any $(g, \psi, h_1) \in U$ and the corresponding solution map from U to $Y_{\infty,T'}$ is Lipschitz continuous.*

Next, consider the forced linear problem

$$u_t + (1 + 3b^2)u_x - 3bu_{xx} + u_{xxx} = f(x, t)\,, \qquad x \in [0, L]\,, \quad t > 0\,, \tag{4.11}$$

$$u(0, t) = h_1(t)\,, \quad u(L, t) = h_2(t)\,, \quad u_x(L, t) = h_3(t)\,, \qquad t > 0\,, \tag{4.12}$$

$$u(x, 0) = \phi(x)\,, \qquad x \in [0, L]\,. \tag{4.13}$$

If $(f, \phi, \vec{h}) \in X_{L,T}$, then the linear estimates derived in Section 2 imply the corresponding solution u of (4.11)-(4.13) belongs to the space $Y_{L,T}$ and satisfies

$$\|u\|_{Y_{L,T}} \leq C\|(f, \phi, \vec{h})\|_{X_{L,T}} \tag{4.14}$$

for a constant C independent of L. The next lemma gives an estimate on solutions of (4.11)-(4.13) in the space $Y_{s,L,T}$ with s in the range $0 \leq s \leq 3$.

Lemma 4.4 *For given $T > 0$ and s in the range $[0, 3]$, let there be given $(f, \phi, \vec{h}) \in X_{s,L,T}$ satisfying the compatibility conditions*

$$\begin{cases} \phi(0) = h_1(0), \quad \phi(L) = h_2(0)\,, & \text{if } \tfrac{1}{2} < s \leq \tfrac{3}{2}, \text{ or} \\ \phi(0) = h_1(0), \quad \phi(L) = h_2(0), \quad \phi'(L) = h_3(0)\,, & \text{if } \tfrac{3}{2} < s \leq 3. \end{cases} \tag{4.15}$$

Then (4.11)-(4.13) admits a unique solution $u \in Y_{s,L,T}$ and

$$\|u\|_{Y_{s,L,T}} \leq C\|(f, \phi, \vec{h})\|_{X_{s,L,T}} \tag{4.16}$$

for a constant $C > 0$ independent of $f, \phi, \vec{h}$ and L.

Proof: The proof is provided for $s = 3$. The result for other values of s can be established by interpolation using (4.14). For the solution u of (4.11) with $f = 0$, let $v = u_t$. Then the function v is a solution of the linear problem

$$v_t + (1 + 3b^2)v_x - 3bv_{xx} + v_{xxx} = 0\,, \qquad x \in [0, L]\,, \quad t > 0\,, \tag{4.17}$$

$$v(0, t) = h_{1t}(t)\,, \qquad v(L, t) = h_{2t}(t)\,, \qquad v_x(L, t) = h_{3t}(t)\,, \qquad t > 0\,, \tag{4.18}$$

$$v(x, 0) = -(1 + 3b^2)\phi_x + 3b\phi_{xx} - \phi_{xxx} = \phi_1(x)\,, \quad x \in [0, L]\,. \tag{4.19}$$

Applying (4.14) to v in (4.18)-(4.19) yields that

$$\|v\|_{Y_{L,T}} \leq C\|(0, \phi_1, \vec{h}_t)\|_{X_{L,T}}.$$

Define the function u by

$$u(x,t) = \int_0^t v(x,\tau)d\tau + \phi(x).$$

Then $u(x,0) = \phi(x)$ and

$$u(0,t) = \int_0^t v(0,\tau)d\tau + \phi(0) = \int_0^t h_1'(\tau)d\tau + \phi(0) = h_1(t) - h_1(0) + \phi(0) = h_1(t).$$

Similarly, $u(L,t) = h_2(t)$ and $u_x(L,t) = h_3(t)$. Furthermore, it is easily verified that $u(x,t)$ satisfies (4.11) with $f = 0$. Thus, u solves the IBVP (4.11)-(4.13) with $f = 0$. Since

$$u_{xxx} = -v - (1 + 3b^2)u_x + 3bu_{xx},$$

it follows that $u \in Y_{3,L,T}$ and satisfies (4.11) with $s = 3$. Therefore, for $0 \leq s \leq 3$, it follows that

$$\|W_L(t)\phi\|_{Y_{s,L,T}} \leq C\|(0, \phi, 0)\|_{X_{s,L,T}}.$$

Next, consider the solution of (4.11)-(4.13) with $\vec{h} = \phi = 0$. The corresponding solution is

$$u(x,t) = \int_0^t W_L(t-\tau)f(\cdot,\tau)d\tau.$$

By a proof similar to that of Proposition 2.14, there obtains the inequalities

$$\begin{aligned}\|u(x,t)\|_{Y_{s,L,T}} \leq & C\int_0^T \|\xi_{(0,t)}(\tau)W_L(t-\tau)f(\cdot,\tau)\|_{Y_{s,L,T}}d\tau \\ \leq & C\int_0^T \|(0, f(\cdot,\tau), 0)\|_{X_{s,L,T}}d\tau \leq C\|(f,0,0)\|_{X_{s,L,T}}.\end{aligned}$$

The proof of the Lemma is complete. □

Theorem 4.5 *Let $T > 0$ and let s in the range $0 \leq s \leq 3$ be given. Suppose that $(f, \phi, \vec{h}) \in X_{s,L,T}$ satisfies the compatibility conditions appearing in Lemma 4.4. Then, there exists a $T^* \in (0,T]$ depending only on $\|(f,\phi,\vec{h})\|_{X_{s,L,T}}$ such that (4.1)-(4.3) admit a unique solution $u \in Y_{s,L,T^*}$. Moreover, for any $T' < T^*$, there is a neighborhood U of $(f,\phi,\vec{h})$ such that the IBVP (4.1)-(4.3) admits a unique solution in the space $Y_{s,L,T'}$ for any $(g,\psi,\vec{h}_1) \in U$ and the corresponding solution map is Lipschitz continuous.*

Proof : For given $(f, \phi, \vec{h}) \in X_{s,L,T}$ satisfying the compatibility conditions in Lemma 4.4, let $r > 0$ and $\theta > 0$ be given and let $S_{\theta,r}$ be the collection of functions $v \in Y_{s,L,\theta}$ such that $\|v\|_{Y_{s,L,\theta}} \leq r$. Then, arguing as in the proof of Proposition 4.2 and using Lemmas 4.1 and 4.4, one shows using a suitable choice of r and θ, that the contraction mapping principle applies and gives a solution of (4.1)-(4.3) in $Y_{s,L,T'}$. The Lipschitz continuity follows directly from the contraction mapping principle. The proof is complete. □

Similar properties hold for the problem posed on R^+, *viz.*

$$u_t + (1+3b^2)u_x - 3bu_{xx} + u_{xxx} = f(x,t)\,, \quad x > 0\,, \quad t > 0\,, \tag{4.20}$$
$$u(0,t) = h(t)\,, \quad t > 0\,, \quad u(x,0) = \phi(x)\,, \quad x > 0\,. \tag{4.21}$$

For $(f, \phi, h) \in X_{\infty,T}$, the corresponding solution u of (4.20)-(4.21) belongs to the space $Y_{\infty,T}$ and satisfies

$$\|u\|_{Y_{\infty,T}} \leq C\|(f,\phi,h)\|_{X_{\infty,T}} \tag{4.22}$$

for some constant C.

Lemma 4.6 *For given $T > 0$ and s in the range $[0,3]$, let there be given $(f, \phi, h) \in X_{s,\infty,T}$ satisfying the compatibility conditions*

$$\phi(0) = h(0), \quad \textit{if} \quad \tfrac{1}{2} < s \leq 3\,. \tag{4.23}$$

Then (4.20)-(4.21) admits a unique solution $u \in Y_{s,\infty,T}$ and

$$\|u\|_{Y_{s,\infty,T}} \leq C\|(f,\phi,h)\|_{X_{s,\infty,T}} \tag{4.24}$$

for some constant $C > 0$ independent of f, ϕ, h.

Theorem 4.7 *Let there be given $T > 0$ and s with $0 \leq s \leq 3$. Suppose that $(f, \phi, h) \in X_{s,\infty,T}$ satisfies the compatibility conditions stated in Lemma 4.6. Then there exists a $T^* \in (0,T]$ depending only on $\|(f,\phi,h)\|_{X_{s,\infty,T}}$ such that (4.8)-(4.9) admits a unique solution $u \in Y_{s,\infty,T^*}$. Moreover, for any $T' < T^*$, there is a neighborhood U of (f, ϕ, h) such that the IBVP (4.8)-(4.9) admits a unique solution in the space $Y_{s,L,T'}$ for any $(g, \psi, h_1) \in U$ and the corresponding solution map is Lipschitz continuous.*

5 Global well-posedness

The results presented in Theorems 4.5 and 4.7 are local in the sense that the time interval $(0, T^*)$ on which the solution exists depends on $\|(f,\phi,\vec{h})\|_{X_{s,L,T}}$ or $\|(f,\phi,h)\|_{X_{s,\infty,T}}$. In general, the larger $\|(f,\phi,\vec{h})\|_{X_{s,L,T}}$ or $\|(f,\phi,h)\|_{X_{s,\infty,T}}$, the smaller will be T^*. However, if $T^* = T$ no matter what the size of $\|(f,\phi,\vec{h})\|_{X_{s,L,T}}$ or $\|(f,\phi,h)\|_{X_{s,\infty,T}}$, the IBVP (4.1)-(4.3) or (4.8)-(4.9) is said to be globally well-posed. In this section we study global well-posedness of these problems. To simplify the exposition, we only consider the IBVP's

$$u_t+(1+3b^2)u_x - 3bu_{xx} + u_{xxx} + e^{(b+b^3)t-bx}(u_x - bu)u = 0\,, \quad x \in [0,L]\,, \quad t>0\,, \tag{5.1}$$

$$u(0,t) = h(t)\,, \qquad u(L,t) = u_x(L,t) = 0\,, \quad t > 0\,, \tag{5.2}$$

$$u(x,0) = \phi(x)\,, \qquad x \in [0,L]\,, \tag{5.3}$$

and

$$u_t+(1+3b^2)u_x - 3bu_{xx} + u_{xxx} + e^{(b+b^3)t-bx}(u_x - bu)u = 0\,, \quad x>0\,, \quad t>0\,, \tag{5.4}$$

$$u(0,t) = h(t)\,, \quad t>0\,, \qquad u(x,0) = \phi(x)\,, \quad x > 0\,. \tag{5.5}$$

First, to study the global well-posedness, it is helpful to introduce some additional Banach spaces. For s in the interval $0 \le s \le 3$, let

$$Z_{s,L,T} = H^s(0,L) \times H^{\frac{1+s+\epsilon}{3}}(0,T)\,,$$

and

$$Z_{s,\infty,T} = H^s(0,\infty) \times H^{\frac{1+s+\epsilon}{3}}(0,T)\,,$$

where ϵ is any fixed small positive constant. The compatibility conditions

$$\begin{cases} \phi(0) = h(0), \quad \phi(L) = 0 & \text{if } \frac{1}{2} < s \le \frac{3}{2}, \text{ or} \\ \phi(0) = h(0), \quad \phi(L) = 0, \quad \phi'(L) = 0 & \text{if } \frac{3}{2} < s \le 3, \end{cases} \tag{5.6}$$

are imposed for $(\phi, h) \in Z_{s,L,T}$, whilst it is insisted that

$$\phi(0) = h(0), \quad \text{if} \quad \tfrac{1}{2} < s \le 3, \tag{5.7}$$

for $(\phi, h) \in Z_{s,\infty,T}$.

Theorem 5.1 *Let $T > 0$ and s be such that $0 \le s \le 3$. For any $(\phi,\, h) \in Z_{s,L,T}$ satisfying (5.6), the IBVP (5.1)-(5.3) admits a unique solution $u \in Y_{s,L,T}$. Moreover, the corresponding solution map is locally Lipschitz continuous.*

Proof: In the context of an established local well-posedness result, it suffices to prove the following global *a priori* H^s-estimate for smooth solutions of the IBVP (5.1)-(5.3).

Lemma 5.2 *For given $T > 0$, there exists a continuous and non-decreasing function $\gamma : R^+ \to R^+$, independent of L such that for any smooth solution u of (5.1)-(5.3),*

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_{H^s(0,L)} \le \gamma(\|(\phi,h)\|_{Z_{s,L,T}}) .$$

Proof of Lemma: For a smooth solution u of the IBVP (5.1)-(5.3), write $u = w + v$, where v solves

$$\begin{aligned} &v_t + (1+3b^2)v_x - 3bv_{xx} + v_{xxx} = 0, \qquad x \in [0,L], \quad t > 0, \\ &v(t,0) = h(t), \qquad v(t,L) = 0, \qquad v_x(t,L) = 0, \quad t > 0, \\ &v(0,x) = h(0)e^{-x}, \quad x \in [0,L], \end{aligned}$$

and w solves

$$w_t + (1+3b^2)w_x - 3bw_{xx} + w_{xxx} + e^{(b+b^3)t}((e^{-bx}(v+w))_x(v+w)) = 0,$$
$$\text{for} \quad x \in [0,L], \ t > 0, \tag{5.8}$$
$$w(0,t) = 0, \quad w(L,t) = 0, \quad u_x(L,t) = 0, \quad t > 0, \tag{5.9}$$
$$w(x,0) = \phi(x) - h(0)e^{-x}, \quad x \in [0,L], \tag{5.10}$$

if $h(0)$ is well-defined (otherwise, set $h(0) = 0$). By Lemma 4.4, it transpires that

$$\|v\|_{Y_{s,T,L}} \le C\|h\|_{H^{\frac{1+s}{3}}(0,T)}. \tag{5.11}$$

Multiply both sides of the equation in (5.8) by w and integrate over $[0,L]$ with respect to x. Integration by parts leads to

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L_2(0,L)} + 6b\int_0^L w_x^2(\cdot,t)dx \le C\left|\int_0^L e^{-bx}(v(\cdot,t)+w(\cdot,t))(v(\cdot,t)w(\cdot,t)+w^2(\cdot,t))_x dx\right|$$

$$\begin{aligned} \le C\int_0^L e^{-bx}\big(&|v^2(\cdot,t)w(\cdot,t)| + |v(\cdot,t)v_x(\cdot,t)w(\cdot,t)| \\ &+ |w^2(\cdot,t)|(|v_x(\cdot,t)| + |v(\cdot,t)| + |v(\cdot,t)+w(\cdot,t)|)\big)dx. \end{aligned} \tag{5.12}$$

As was shown in [14], for any $T > 0$ and $s \ge -\frac{3}{2}$, there is a constant C such that

$$\|v\|_{L_2((0,T);H^{s+\frac{3}{2}}(0,L))} \le C\|h\|_{H^{\frac{1+s}{3}}(0,L)} \tag{5.13}$$

for any $h \in H^{\frac{1+s}{3}}(0,L)$. Of course, $\tilde{u}(x,t) = e^{-bx}(v(x,t)+w(x,t))$ is a solution of the original IBVP (1.7)-(1.9) for the KdV equation with $g(t) = e^{(b+b^3)t}h(t)$ and $u_0(x) = e^{-bx}\phi(x)$. In consequence of the results in [13, 14], it is the case that

$$\|\tilde{u}(x,t)\|_{Y_{s,L,T}} \le \alpha(\|(u_0,g)\|_{Z_{s,L,T}}) \tag{5.14}$$

where $\alpha : R^+ \to R^+$ is a nondecreasing continuous function. Also observe that the right-hand side of (5.12) is less than or equal to the quantity

$$C\Big(\sup_{x\in[0,L]} |v(x,t)|\, \|v(\cdot,t)\|_{L_2(0,L)} \|w(\cdot,t)\|_{L_2(0,L)} + \sup_{x\in[0,L]} |v(x,t)|\, \|v_x(\cdot,t)\|_{L_2(0,L)} \|w(\cdot,t)\|_{L_2(0,L)}$$
$$+ \sup_{x\in[0,L]} |w(x,t)|\, \|w(\cdot,t)\|_{L_2(0,L)} \|v_x(\cdot,t)\|_{L_2(0,L)} + \sup_{x\in[0,L]} |v(x,t)|\, \|w(\cdot,t)\|^2_{L_2(0,L)}$$
$$+ \sup_{x\in[0,L]} |w(x,t)|\, \|\tilde{u}(\cdot,t)\|_{L_2(0,L)} \|w(\cdot,t)\|_{L_2(0,L)}\Big)$$

$$\le C\Big(\|v(\cdot,t)\|^2_{H^1(0,L)} (\|w(\cdot,t)\|^2_{L_2(0,L)} + \|w(\cdot,t)\|_{L_2(0,L)})$$
$$+ (\|w(\cdot,t)\|_{L_2(0,L)} + \|w_x(\cdot,t)\|_{L_2(0,L)}) \|w(\cdot,t)\|_{L_2(0,L)} (\|v_x(\cdot,t)\|_{L_2(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)})\Big)$$

$$\le C\Big(\|v(\cdot,t)\|^2_{H^1(0,L)} (\|w(\cdot,t)\|^2_{L_2(0,L)} + \|w(\cdot,t)\|_{L_2(0,L)})$$
$$+ \|w(\cdot,t)\|^2_{L_2(0,L)} (\|v_x(\cdot,t)\|_{L_2(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)}) + (b/C) \|w_x(\cdot,t)\|_{L_2(0,L)}$$
$$+ C_1 \|w(\cdot,t)\|^2_{L_2(0,L)} (\|v_x(\cdot,t)\|_{L_2(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)})^2\Big).$$

Thus, because of (5.12)-(5.13) and the preceding inequality,

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L_2[0,L]} + 5b\int_0^L w_x^2(\cdot,t)dx \le C(\|v(\cdot,t)\|^2_{H^1(0,L)} \|w(\cdot,t)\|_{L_2(0,L)}$$
$$+ \|w(\cdot,t)\|^2_{L_2(0,L)} (\|v(\cdot,t)\|_{H^1(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)})(\|v(\cdot,t)\|_{H^1(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)} + 1)).$$

In consequence, one has that

$$\frac{d}{dt}\|w(\cdot,t)\|_{L_2(0,L)} \le C(\|v(\cdot,t)\|^2_{H^1(0,L)}$$
$$+ \|w(\cdot,t)\|_{L_2(0,L)} (\|v(\cdot,t)\|_{H^1(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)})(\|v(\cdot,t)\|_{H^1(0,L)} + \|\tilde{u}(\cdot,t)\|_{L_2(0,L)} + 1))$$

for any $t \ge 0$. The estimate in the Lemma with $s = 0$ then follows by applying the Gronwall Lemma and (5.13)-(5.14). The proofs for $s = 3$ and then for $0 < s < 3$ follow in the same way as do the analogous inequalities in [13]. □

In a similar manner, one establishes the following theorem for the quarter-plane case.

Theorem 5.2 *Let $T > 0$ and let s be so that $0 \le s \le 3$. For any $(\phi, h) \in Z_{s,\infty,T}$ satisfying (5.7), the IBVP (5.4)-(5.5) admits a unique solution $u \in Y_{s,\infty,T}$. Moreover, the corresponding solution map for the IBVP (5.4)-(5.5) is Lipschitz continuous.*

6 Comparison

In this section, the solutions of the KdV equation in a quarter plane and on a finite spatial domain are directly compared.

Let $u_L(x,t)$ be the solution of

$$u_t + u_x + uu_x + u_{xxx} = 0\,, \qquad x \in [0,L]\,, \quad t > 0\,, \tag{6.1}$$

$$u(0,t) = g(t)\,, \quad u(L,t) = u_x(L,t) = 0\,, \tag{6.2}$$

$$u(x,0) = u_0(x)\,, \tag{6.3}$$

and $u_\infty(x,t)$ be the solution of

$$u_t + u_x + uu_x + u_{xxx} = 0\,, \qquad x \in [0,\infty)\,, \quad t > 0\,, \tag{6.4}$$

$$u(0,t) = g(t)\,, \qquad u(x,0) = u_0(x)\,, \tag{6.5}$$

where u_0 and g satisfy

$$\|g(t)\|_{H^{\frac{1+s}{3}}(0,T)} + \|e^{2bx}u_0(x)\|_{H^s(R^+)} < \infty \tag{6.6}$$

or

$$\|g(t)\|_{H^{\frac{1+s+\epsilon}{3}}(0,T)} + \|e^{2bx}u_0(x)\|_{H^s(R^+)} < \infty \tag{6.7}$$

for some s with $0 \le s \le 3$, where it is assumed that the compatibility conditions (5.6) or (5.7) hold. By the previously developed existence theorems using the weight function e^{-2bx} for the x-variable (i.e. $b \to 2b$ in the transformation and theorems), the solution $u_\infty = e^{(2b+8b^3)t-2bx}w_\infty$ and $u_L = e^{(2b+8b^3)t-2bx}w_L$ exist and satisfy $w_\infty \in Y_{s,\infty,T}$ and $w_L \in Y_{s,L,T}$. In particular, for all $L \ge 1$,

$$e^{2bL}\|u_\infty(L,\cdot)\|_{H^{\frac{1+s}{3}}(0,T))} + e^{2bL}\|u_{\infty,x}(L,\cdot)\|_{H^{\frac{s}{3}}(0,T))} \le C \tag{6.8}$$

where C is independent of L, but may depend on $T, g(t)$ and $u_0(x)$.

Let $z(x,t) = u_\infty(x,t) - u_L(x,t)$ for $x \in [0,L]$. Then, $w(x,t)$ satisfies

$$z_t + z_x + z_{xxx} + \frac{1}{2}((u_\infty + u_L)z)_x = 0 \quad x \in [0,L]\,, \quad t \in [0,T]\,, \tag{6.9}$$

$$z(0,t) = 0\,, \quad z(L,t) = u_\infty(L,t)\,, \quad z_x(L,t) = u_{\infty,x}(L,t)\,, \tag{6.10}$$

$$z(x,0) = 0\,. \tag{6.11}$$

Defining v by $v(x,t) = e^{-(b+b^3)t+bx}z(x,t)$, it follows that v satisfies the IBVP

$$v_t+(1+3b^2)v_x - 3bv_{xx} + v_{xxx} = \frac{1}{2}\big(-u_{\infty,x} - u_{L,x} + bu_\infty + bu_L\big)v - \frac{1}{2}(u_\infty + u_L)v_x\,, \quad x \in [0,L]\,, \quad t \in [0,T]\,, \tag{6.12}$$

$$v(0,t) = 0\,, \quad v(L,t) = e^{-(b+b^3)t}(e^{bL}u_\infty(L,t)) = h_2(t)\,, \tag{6.13}$$

$$v_x(L,t) = e^{-(b+b^3)L}(e^{bL}u_{\infty,x}(L,t)) = h_3(t)\,, \qquad v(x,0) = 0\,. \tag{6.14}$$

Here, we note that from (6.7)-(6.8), $z(x,t)$ decays like e^{-2bx} at infinity and the transformation from the KdV type equation for z to the KdV-Burgers type equation for v needs a factor e^{-bx}. Thus, v still has the decay rate of e^{-bx}, which is the reason for the weight function e^{-2bx} imposed on $u_0(x)$ in (6.6)-(6.7). If $\vec{h}(t) = (0, h_2(t), h_3(t))$ and $v_1(x,t) = W_b(t)\vec{h}$ defined in (2.42), then by Lemma 4.4 and (6.8), it is deduced that

$$\|v_1(x,t)\|_{Y_{s,L,T}} \le C\|\vec{h}\|_{H_s(0,T)} \le Ce^{-bL} \tag{6.15}$$

where the space H_s is defined in (2.39) and C is independent of L. On the other hand, the function $\tilde{v}(x,t) = v(x,t) - v_1(x,t)$ satisfies the IBVP

$$\tilde{v}_t+(1+3b^2)\tilde{v}_x - 3b\tilde{v}_{xx} + \tilde{v}_{xxx} = \frac{1}{2}\big(-u_{\infty,x} - u_{L,x} + bu_\infty + bu_L\big)(\tilde{v} + v_1) \tag{6.16}$$

$$-\frac{1}{2}(u_\infty + u_L)(\tilde{v} + v_1)_x\,, \quad x \in [0,L]\,, \quad t \in [0,T]\,, \tag{6.17}$$

$$\tilde{v}(0,t) = 0\,, \quad \tilde{v}(L,t) = \tilde{v}_x(L,t) = 0\,, \qquad \tilde{v}(x,0) = 0\,. \tag{6.18}$$

Lemmas 4.1 and 4.4, applied as in the proof of Theorem 4.5, show that there is a $T^* > 0$ such that the solution $\tilde{v}(x,t)$ of (6.16)-(6.18) exists in Y_{s,L,T^*} and obeys the inequality

$$\begin{aligned}\|\tilde{v}\|_{Y_{s,L,T^*}} &\le C\big(\|(u_\infty + u_L)v_{1,x}\|_{L^1((0,T);H^s(0,L))} + \|(|u_{\infty,x} + u_{L,x}| + |u_\infty + u_L|)v_1\|_{L^1((0,T);H^s(0,L))}\big) \\ &\le C(\|u_L\|_{Y_{s,L,T}} + \|u_\infty\|_{Y_{s,\infty,T}})\|v_1\|_{Y_{s,L,T}} \le Ce^{-bL}\end{aligned} \tag{6.19}$$

where C is independent of L. To establish that (6.19) holds for t in the interval $[0,T]$, it is only necessary to obtain a global estimate of $\tilde{v}$. This is accomplished by a standard energy-type argument as follows. Multiply both sides of (6.16) by $\tilde{v}$ and integrate the result from 0 to L. Integration by parts yields

$$\frac{d}{dt}\|\tilde{v}(\cdot,t)\|^2_{L_2[0,L]} + 6b\int_0^L \tilde{v}_x^2(\cdot,t)dx \le C\Big(\int_0^L (|u_{\infty,x}+u_{L,x}| + |u_\infty + u_L|)\tilde{v}^2 dx$$
$$+\int_0^L ((|u_{\infty,x}+u_{L,x}| + |u_\infty+u_L|)v_1\tilde{v} + (\,|u_\infty+u_L|)v_{1,x}\tilde{v})dx\Big)$$

$$\le C(\sup_{x\in[0,L]}|\tilde{v}|\|u_\infty+u_L\|_{H^1(0,L)}\|\tilde{v}\|_{L_2(0,L)} + \sup_{x\in[0,L]}|v_1|\|u_\infty+u_L\|_{H^1(0,L)}\|\tilde{v}\|_{L_2(0,L)}$$
$$+\sup_{x\in[0,L]}|u_\infty+u_L|\|v_{1,x}\|_{L_2(0,L)}\|\tilde{v}\|_{L_2(0,L)}\Big)$$

$$\le C\Big((1+\|u_\infty+u_L\|_{H^1(0,L)})\|u_\infty+u_L\|_{H^1(0,L)}\|\tilde{v}\|^2_{L_2(0,L)} + \sup_{x\in[0,L]}|v_1|\|u_\infty+u_L\|_{H^1(0,L)}\|\tilde{v}\|_{L_2(0,L)}$$
$$+\sup_{x\in[0,L]}|u_\infty+u_L|\|v_{1,x}\|_{L_2(0,L)}\|\tilde{v}\|_{L_2(0,L)}\Big) + b\|\tilde{v}_x\|^2_{L_2(0,L)},$$

where the inequalities

$$\sup_{x\in[0,L]}|\tilde{v}| \le C(\|\tilde{v}(\cdot,t)\|_{L_2(0,L)} + \|\tilde{v}_x(\cdot,t)\|_{L_2(0,L)})$$

and

$$\|\tilde{v}_x(\cdot,t)\|_{L_2(0,L)}\|u_\infty+u_L\|_{H^1(0,L)}\|\tilde{v}\|_{L_2(0,L)} \le \delta\|\tilde{v}_x(\cdot,t)\|^2_{L_2(0,L)} + (4\delta)^{-1}\|u_\infty+u_L\|^2_{H^1(0,L)}\|\tilde{v}\|^2_{L_2(0,L)}$$

have been used with C independent of L and $\delta > 0$ small. Then, the Gronwall Lemma and (6.15) give

$$\|\tilde{v}(\cdot,t)\|_{L_2[0,L]} \le C_0\|v_1\|_{L_2((0,T);H^1(0,L))}\|u_\infty+u_L\|_{L_2((0,T);H^1(0,L))}$$
$$\times\exp(C(\|u_\infty+u_L\|^2_{L_2((0,T);H^1(0,L))} + \|u_\infty+u_L\|_{L_1((0,T);H^1(0,L))})) \le Ce^{-bL},$$

where C is independent of L, but may depend upon T. The estimates of $\tilde{v}$ for $s = 1,2,3$ can be obtained similarly, while those for non-integer s can be derived using interpolation. (Here, note the equation for $\tilde{v}$ is linear). Finally, for $t \in [0,T]$,

$$\|u_\infty(\cdot,t) - u_L(\cdot,t)\|_{H^s(0,L)} = \|z(\cdot,t)\|_{H^s(0,L)} \le C\|v(\cdot,t)\|_{H^s(0,L)}$$
$$\le C\|v_1(\cdot,t)+\tilde{v}(\cdot,t)\|_{H^s(0,L)} \le Ce^{-bL},$$

where C is independent of L but may depend upon T and b. Thus, the proof of Theorem 1.1 is completed.

Appendix

Here, a demonstration is offered of the point left open in the proof of Proposition 2.12.

Proof of the Claim: Let $F_c[\mathcal{L}(g)](k)$ denote the Fourier cosine transform of $\mathcal{L}g$, *viz.*

$$F_c[\mathcal{L}(g)](k) = \int_0^\infty \cos kt \int_{-\infty}^\infty e^{i(\lambda-\rho\lambda^{1/3})t-\lambda^{2/3}t} g(\lambda)d\lambda dt.$$

Since

$$\|\mathcal{L}(g)(t)\|_{L^2_t(R+)} \leq C\|F_c[\mathcal{L}(g)](k)\|_{L^2_k(R)},$$

it is sufficient to estimate appropriately the quantities

$$\int_0^\infty \int_{-\infty}^\infty e^{\pm ikt+i(\lambda-\rho\lambda^{1/3})t-\lambda^{2/3}t} g(\lambda)d\lambda dt = -\int_{-\infty}^\infty \frac{g(\lambda)}{i(\pm k+\lambda-\rho\lambda^{1/3})-\lambda^{2/3}} d\lambda.$$

Direct consideration is given only to

$$[I(g)](k) = \int_{-\infty}^\infty \frac{g(\lambda)}{i(\lambda-\rho\lambda^{1/3}-k)-\lambda^{2/3}} d\lambda$$

as the other case follows by making the change of variables $k \to -k$ in the relevant integral. Break $[I(g)](k)$ into real and imaginary parts thusly;

$$[I(g)](k) = -[I_1(g)](k) - i[I_2(g)](k)$$

with

$$[I_1(g)](k) = \int_{-\infty}^\infty \frac{\lambda^{2/3}}{\lambda^{4/3}+(\lambda-\rho\lambda^{1/3}-k)^2} g(\lambda)d\lambda$$

and

$$[I_2(g)](k) = \int_{-\infty}^\infty \frac{\lambda-\rho\lambda^{1/3}-k}{\lambda^{4/3}+(\lambda-\rho\lambda^{1/3}-k)^2} g(\lambda)d\lambda.$$

To show that $\|I_1(g)\|_{L_2(R)} \leq C\|g\|_{L_2(R)}$, it suffices to show that

$$\text{ess-sup}_\lambda \int_{-\infty}^\infty |K(\lambda,k)|\, dk < C, \qquad \text{ess-sup}_k \int_{-\infty}^\infty |K(\lambda,k)|\, d\lambda < C$$

where

$$K(\lambda,k) = \frac{\lambda^{2/3}}{\lambda^{4/3}+(\lambda-\rho\lambda^{1/3}-k)^2}.$$

References

[1] G.B. Airy, *Tides and waves*, Art. no. 192 in Encyclopaedia Metropolitana, London (1845).

[2] T.B. Benjamin, *Lectures on nonlinear wave motion*, In Lect. Appl. Math. Vol. **15**, (ed. A. Newell) American. Math. Soc.: Providence, RI (1974), 3–47.

[3] J.L. Bona and P.J. Bryant, *A mathematical model for long waves generated by wavemakers in nonlinear dispersive systems,* Proc. Cambridge Philos. Soc. **73** (1973) 391-405.

[4] J.L. Bona and H. Chen, *Comparison of model equations for small-amplitude long waves.* Nonlinear Analysis TMA Series B, **38** (1999) 625-647.

[5] J.L. Bona, H. Chen, S.M. Sun and B.-Y. Zhang, *Comparison of quarter-plane and two-point boundary value problems: the BBM-equation,* Discrete Contin. Dyn. Syst. **13** (2005), 921–940.

[6] J.L. Bona and M. Chen, *A Boussinesq system for the two-way propagation of nonlinear dispersive waves,* Physica D **116** (1998) 191–224.

[7] J.L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory,* J. Nonlinear Sci. **12** (2002) 283–318.

[8] J.L. Bona, M. Chen and J.-C. Saut, *Boussinesq equations and other systems for small amplitude long waves in nonlinear dispersive media. II: Nonlinear theory,* Nonlinearity **17** (2004) 925–952.

[9] J.L. Bona and V. Dougalis, *An initial and boundary value problem for a model equation for propagation of long waves,* J. Math. Anal. Appl. **75** (1980) 503–522.

[10] J.L. Bona, W.G. Pritchard and L.R. Scott, *An evaluation of a model equation for water waves,* Philos. Trans. Royal. Soc. London Ser. A **302** (1981) 457–510.

[11] J.L. Bona, W.G. Pritchard and L.R. Scott, *Numerical schemes for a model for nonlinear dispersive waves,* J. Comput. Phys. **60** (1985) 167–186.

[12] J.L. Bona, S.M. Sun and B.-Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane,* Trans. American Math. Soc. **354** (2002), 427–490

[13] J.L. Bona, S.M. Sun and B.-Y. Zhang, *A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain*, Comm. Partial Differential Eq. **28** (2003), 1391–1436.

[14] J.L. Bona, S.M. Sun and B.-Y. Zhang, *Boundary smoothing properties of the Korteweg-de Vries equation in a quarter plane and applications*, Dynamics Partial Differential Eq. **3** (2006), 1–70.

[15] J.L. Bona and V. Varlamov, *Wave generation by a moving boundary*, Contemp. Math. **371** (2005) 41-71.

[16] J. Boussinesq, *Théorie de l'intumescence liquide appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire*, C. R. Acad. Sci. Paris (1871) 72–73.

[17] J. Boussinesq, *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenue dans ce canal des vitesses sensiblement pareilles de la surface au fond*, J. Math. Pures Appliq. Ser. II **17** (1872) 55-108.

[18] J. Boussinesq, *Essai sur la théorie des eaux courantes*, Mémoires l'Académie des Sciences de Paris, T. 23 et 24, (1877).

[19] B.A. Bubnov, *Generalized boundary value problems for the Korteweg-de Vries equation in bounded domain,* Diff. Eq. **15** (1979) 17-21.

[20] T. Colin and J.-M. Ghidaglia, *An initial-boundary-value problem for the Korteweg-de Vries equation posed on a finite interval,* Adv. Diff. Eq. **6** (2001) 1463–1492.

[21] T. Colin and M. Gisclon, *An initial-boundary-value problem that approximates the quarter-plane problem for the Korteweg-de Vries equation*, Nonlinear Anal. TMA, Ser. A **46** (2001) 869–892.

[22] J. E. Colliander and C. E. Kenig, The generalized Korteweg-de Vries equation on the half line, *Comm. Partial Differential Eq.*, **27** (2002), 2187–2266.

[23] A. V. Faminskii, Mixed problems for the Korteweg-de Vries equation, *Sbornik: Mathematics* **190** (1999), 903–935.

[24] A. V. Faminskii, An initial boundary-value problem in a half-strip for the Korteweg-de Vries equation in fractional-order Sobolev spaces, *Comm. Partial Differential Equations* **29** (2004), 1653–1695.

[25] B.-Y. Guo and J. Shen, *On spectral approximations using modified Legendre rational functions: application to the Korteweg-de Vries equation on the half line*, Indiana Univ. Math. J. **50** (2001) 181–204.

[26] J. Hammack, *A note on tsunamis: their generation and propagation in an ocean of uniform depth,* J. Fluid Mech. **60** (1973) 97–133.

[27] J. Hammack and H. Segur, *The Korteweg-de Vries equation and water waves, Part 2. Comparison with experiments,* J. Fluid Mech. **65** (1974) 237–246.

[28] J. Holmer, *Uniform estimates for the Zakharov system and the initial-boundary value problem for the Korteweg-de Vries and nonlinear Schrdinger equations*, Ph.D. Thesis, University of Chicago, 2004.

[29] J. Holmer, The initial-boundary value problem for the Korteweg-de Vries equation, *Comm. Partial Differential Equations* **31** (2006), 1151–1190.

[30] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Philosophical Magazine, 5th series, **36** (1895) 422–443.

[31] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Vol. 1, Springer-Verlag, Heidelberg, (1972).

[32] R. Miura, *The Korteweg-de Vries equation: a survey of results,* SIAM Review 18 (1976) 412–459.

[33] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences **44**, Springer-Verlag, New York, (1983).

[34] G.G. Stokes, *On the theory of oscillatory waves,* Trans. Cambridge Philos. Soc. 8 (1847) 441–455.

[35] G. B. Whitham, *Linear and Nonlinear Waves*, John Wiley, New York, (1974).