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Dispersive Blow-Up II. Schrödinger-Type Equations, Optical and Oceanic Rogue Waves***

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(*Dedicated to Professor Roger Temam with Friendship and Admiration*)

Abstract Addressed here is the occurrence of point singularities which owe to the focusing of short or long waves, a phenomenon labeled dispersive blow-up. The context of this investigation is linear and nonlinear, strongly dispersive equations or systems of equations. The present essay deals with linear and nonlinear Schrödinger equations, a class of fractional order Schrödinger equations and the linearized water wave equations, with and without surface tension.

Commentary about how the results may bear upon the formation of rogue waves in fluid and optical environments is also included.

Keywords Rogue waves, Dispersive blow-up, Nonlinear dispersive equations, Nonlinear Schrödinger equation, Water wave equations, Propagation in optical cables, Weak turbulence models

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1 Introduction

1.1 General setting

The notion of a dispersive singularity has its roots in some remarks in [4], in the context of the linearized Korteweg-de Vries equation

$$u_t + u_x + u_{xxx} = 0 \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty) \tag{1.1}$$

(sometimes referred to as the Airy equation). The gist of the analysis sketched in [4] was that an initial wave profile $u(x,0) = u_0(x)$ which was infinitely smooth, bounded and possessed of finite energy (a square integrable function) could result in a solution $u = u(x,t)$ that blows up in L^∞-norm in finite time. Moreover, the blow-up point (x^*, t^*) could be specified arbitrarily in the upper half plane $\mathbb{R} \times (0,\infty)$.

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This result was elaborated and given mathematical precision in [8] where these solutions were shown to be smooth and bounded at all points $(x,t) \in \mathbb{R} \times (0,\infty)$ except the singular point (x^*,t^*). The term dispersive blow-up was coined to describe this situation, and the theory extended from (1.1) to include the full, nonlinear Korteweg-de Vries equation (KdV-equation henceforth) and its generalizations

$$u_t + u_x + u^p u_x + u_{xxx} = 0 \tag{1.2}$$

for integers $p \geq 1$.

A key observation that allowed extension of the theory to a nonlinear setting was the smoothing property of the double integral term

$$\int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{3}}} \mathrm{Ai}\Big(\frac{x-z+s}{(t-s)^{\frac{1}{3}}}\Big) u^p(y,s) u_y(y,s) \mathrm{d}y \mathrm{d}s$$

in the Duhamel representation of the solution. Here, Ai is the unique, non-trivial, bounded solution of Airy's ordinary differential equation (5.7), normalized so that

$$\int_{\mathbb{R}} \mathrm{Ai}\,(z) \mathrm{d}z = 1.$$

(The last integral is an improper Riemann integral, as are many of the integrals appearing below.) This regularity property was obtained in [8] by developing a suitable theory for the Cauchy problem for equation (1.2) in weighted spaces and in [20] (but only for $p \geq 2$) by using Strichartz-type estimates.

It deserves remark that smooth, bounded initial data that lead to dispersive blow-up can be found having arbitrarily small L^2- and L^∞-norm, thus emphasizing that such singularities are not induced by nonlinear effects. Indeed, dispersive blow-up phenomena are related to the fact that evolution equations like the Airy equation or the linear Schrödinger equation are ill-posed in L^∞, or what is the same, that $\mathrm{e}^{\mathrm{i}\xi^3}$ and $\mathrm{e}^{\mathrm{i}|\xi|^2}$ are not Fourier multipliers in L^∞ (see, e.g., [17]). Actually, dispersive blow-up can be viewed as a striking expression of this ill-posedness.

In the original, physical variables, dispersive blow-up is a focusing phenomenon which is due to both the unbounded domain in which the problem is set and the propensity of the dispersion relation to propagate energy at different speeds. These two aspects allow the possibility that widely separated, small disturbances have the potential of coming together locally in space-time, thereby resulting in a large deviation from the rest position. In this light, the idea has possible relevance for explaining the genesis of rogue waves on the surface of large bodies of water (see, e.g., [15, 18, 19]) and in optical networks (see, e.g., [14, 23]). In a little more detail, one of the proposed routes to oceanic rogue-wave formation is what we here call concurrence. This is exactly the idea that the ambient wave motion in a big body of water possesses a large amount of energy which could, in the right circumstances, temporarily coalesce in space, thereby leading to giant waves. Related remarks apply to the optical situation, where a range of frequencies input at one end of an optical cable over a relatively large fetch of time can coalesce, thereby forming a spike in space-time.

Oceanic rogue waves, or freak waves as they are sometimes termed, occur in both deep and shallow water (see [15, 18, 19, 26]). While the free surface Euler equations could be taken as the overall governing equations in both deep and shallow water, there is much to be learned from approximate models. These differ in deep and shallow water regimes. Our earlier work dealt with the shallow water situation, exemplified by the Korteweg-de Vries equation and Boussinesq-type systems of equations.

In the present script, interest is initially focussed upon the deep water regime for surface water waves. The development begins with the linear Schrödinger equation and the linear theory then informs an analysis of the equation with cubic and other power nonlinearities. The ideas pertaining to the Schrödinger equation are easily generalized to include related models for the propagation of pulses in fiber optics cables, and so to perhaps bear upon rogue wave formation in this context. Guided by the work on the Schrödinger equation, attention is then turned to the linearization about the rest state of the full water wave system. As an offshoot of the framework erected in the analysis of the foregoing situations, results for a class of fractional order Schrödinger equations are also brought forth.

An important point that appears in the analysis is that the dispersive blow-up phenomenon has a certain robustness to it that makes it more likely to be observed in reality (see Remarks 2.3 and 2.5 in Section 2). This robustness was already introduced in our earlier work [8] on KdV-type equations.

1.2 Organization of the paper

The paper proceeds as follows. In Section 2, the dispersive blow-up properties are stated precisely and proved to hold for linear and nonlinear Schrödinger-type equations. The development begins with the linear Schrödinger equation. The linear theory is then used to complete an analysis of dispersive blow-up for nonlinear Schrödinger equations. Potential application of the ideas to explain the formation of rogue waves in optical fibers is then discussed. A similar discussion applies to the hyperbolic Schrödinger equation in the context of deep-water, oceanic rogue waves. In Section 3, attention is turned to the linearized water-wave equations, both with and without surface tension. A central ingredient in the analysis of dispersive blow-up for this system is the precise asymptotic estimates, obtained in [21] and [27], of the Fourier transform of kernels of the form $K_a(|\xi|) = \psi(|\xi|)\mathrm{e}^{\mathrm{i}|\xi|^a}$, $0 < a < 1$, where ψ is a C^∞-function which vanishes in a neighborhood of the origin and is identically one for large values of ξ. The same type of estimates that are effective when used on the water wave equations is applied in Section 4 to prove dispersive blow-up results for fractional order Schrödinger equations of the form

$$\mathrm{i}u_t + (-\Delta)^{\frac{a}{2}} u = 0, \quad 0 < a < 1,$$

which occur as the linearization of weak turbulence models (see [12]). Finally, Section 5 contains various auxiliary remarks, in particular on the distinction between weakly dispersive and strongly dispersive equations or systems. There is also further discussion of rogue waves and of some possible extensions of the present results.

1.3 Notation

Partial differentiation with respect to, say, x or t of a function u is denoted indifferently by $\partial_x u$ or u_x (respectively, $\partial_t u$ or u_t). The standard Lebesgue spaces are denoted by $L^p(\mathbb{R}^d)$ and the norm of a function f defined on $\mathbb{R}^d$ is written as $|f|_p$ ($1 \le p \le \infty$). The Fourier multiplier notation $f(D)u$ is defined by $\mathcal{F}(f(D)u)(\xi) = f(\xi)\widehat{u}(\xi)$, where $\mathcal{F}$ and $\widehat{\cdot}$ both connote the Fourier transform. The standard notation $H^s(\mathbb{R}^d)$, or simply H^s if the underlying domain is clear from the context, is used for the L^2-based Sobolev spaces; their norm is written as $|\cdot|_{H^s}$.

For a Banach space X, $C_b(\mathbb{R}_+; X)$ denotes the space of continuous and bounded functions defined on $[0, +\infty)$ with values in X.

2 Dispersive Blow-Up for the Schrödinger Equation

The body of the paper commences with analysis of Schrödinger equations. The crux of the matter is the linear case, dealt with first. This analysis then informs a companion development for nonlinear problems. Commentary on rogue wave formation appears in Subsection 2.3.

2.1 The linear case

Considered here is the linear Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u = 0 & \text{for } (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \\ u(x,0) = \phi(x) & \text{for } x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

where Δ is the d-dimensional Laplacian. The main result of this subsection is the following theorem.

Theorem 2.1 *Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$ be given. There exist functions ϕ lying in the class $C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the corresponding solution u of* (2.1) *satisfies*

(1) *$u \in C_b(\mathbb{R}_+; L^2(\mathbb{R}^d))$,*

(2) *u is a continuous function of (x,t) on $\mathbb{R}^d \times ((0,+\infty) \setminus \{t^*\})$,*

(3) *$u(\cdot, t^*)$ is a continuous function of x on $\mathbb{R}^d \setminus \{x^*\}$,*

(4)

$$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*) \\ (x,t)\neq(x^*,t^*)}} |u(x,t)| = +\infty.$$

Remark 2.1 In particular, one deduces from Theorem 2.1 that for any fixed $t \in (0, +\infty) \setminus \{t^*\}$, the function $x \mapsto u(x,t)$ is continuous on $\mathbb{R}^d$.

Remark 2.2 A stronger result than that stated in Theorem 2.1 is valid. One can show in fact that $u \in C(\mathbb{R}^d \times (0,\infty) \setminus \{x^*, t^*\})$ (see the earlier paper [8] on KdV-type equations). The somewhat technical proof of this fact will be developed in subsequent work. A similar comment applies to all the results in the present essay.

Remark 2.3 As a technical aside, because of the Sobolev embedding theorem and conservation laws associated to dispersive equations such as (2.1) in d spatial dimensions, $d = 1, 2, \cdots$, solutions which become unbounded in finite time cannot be associated with initial data taken from $H^k(\mathbb{R}^d)$ if $k > \frac{d}{2}$. However, we shall see that our results imply as a corollary that there are H^k-initial data that are everywhere small, but which become as large as we like at a given point (x^*, t^*), $t^* > 0$. Indeed, given $\epsilon > 0$ small and $M > 0$ large, there are elements $u_0 \in H^k(\mathbb{R}^d)$, $k > \frac{d}{2}$, and an $r > 0$ such that for any v_0 in the ball $B_r(u_0)$ in $H^k(\mathbb{R}^d)$, $|v_0|_\infty < \epsilon$, but the solution v of (2.1) associated to v_0 has $|v(x^*, t^*)| \geq M$. In physical terms, this means that the property of large solutions emanating from small data is robust in that it applies to whole neighborhoods of smooth data. This point, which is valid for all the equations considered in this essay, is expounded at more length presently.

Proof of Theorem 2.1 Without loss of generality, take it that $(x^*, t^*) = (0, \frac{1}{4})$. For any $\phi \in L^2(\mathbb{R}^d)$, the unique solution u of (2.1) has the representation

$$u(x,t) = \frac{1}{(4\mathrm{i}\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{\frac{\mathrm{i}|x-y|^2}{4t}} \phi(y)\mathrm{d}y, \tag{2.2}$$

where the integral is taken in the improper Riemann sense. (As mentioned already, many of the integrals appearing below are taken in this sense.) Choose the initial data ϕ in (2.1) to be

$$\phi(y) = \frac{\mathrm{e}^{-\mathrm{i}|y|^2}}{(1+|y|^2)^m}. \tag{2.3}$$

If m is chosen in the range $m > \frac{d}{4}$, then the function ϕ does in fact lie in the space $C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Formally, the solution $u(x,t)$ of (2.1) corresponding to the initial data ϕ in (2.3), when evaluated at $(x,t) = (0, \frac{1}{4})$, has the value

$$C \int_{\mathbb{R}^d} \frac{\mathrm{d}y}{(1+|y|^2)^m} \mathrm{d}y,$$

where C is a non-zero constant of no consequence. The latter integral is divergent provided $m \le \frac{d}{2}$. As we want the solution to become infinite at this point, it seems propitious to presume that

$$\frac{d}{4} < m \le \frac{d}{2}. \tag{2.4}$$

With the restriction (2.4) in force, fix a point $(x,t) \in \mathbb{R}^d \times (0,\infty) \setminus \{(0, \frac{1}{4})\}$. The analysis proceeds in two steps.

Step 1 Assume first that $t = \frac{1}{4}$, but that $x \neq 0$. In this case, the value of u at the point (x,t) is

$$u\Big(x, \frac{1}{4}\Big) = C\mathrm{e}^{\mathrm{i}|x|^2} \int_{\mathbb{R}^d} \mathrm{e}^{-2\mathrm{i}x.y} \frac{\mathrm{d}y}{(1+|y|^2)^m}.$$

The integral in the last formula is essentially the Fourier transform of a Bessel potential, and is in fact equal to

$$C|x|^{m-\frac{d}{2}} K_{\frac{d}{2}-m}(|x|),$$

where C is a non-zero constant and K_ν is the modified Bessel function of order ν (see [1]). Recall that K_ν is even and smooth on $\mathbb{R}^d \setminus \{0\}$, decays exponentially to zero as $|x| \to +\infty$ and has a singularity at $x = 0$ of the form

$$K_0(|x|) \sim -\log|x|, \quad \text{as } x \to 0,$$
$$K_\nu(|x|) \sim \frac{C}{|x|^\nu}, \quad \nu \neq 0, \quad \text{as } x \to 0$$

(see, again, [1]). In any case, the function $x \mapsto C|x|^{m-\frac{d}{2}} K_{\frac{d}{2}-m}(|x|)$ is continuous for $x \neq 0$ and decays rapidly to 0 at infinity, thus verifying part (3) of the theorem with the choice (2.3) for the initial data ϕ.

Step 2 Consider now the case where $t \neq \frac{1}{4}$, $x \in \mathbb{R}^d$. When $d = 1$, write the solution u in the form

$$u(x,t) = \frac{C}{t^{\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\frac{x^2}{4t}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{2\mathrm{i}xy}{4t}} \frac{\mathrm{e}^{\mathrm{i}y^2(-1+\frac{1}{4t})}}{(1+y^2)^m} \mathrm{d}y = \frac{C}{t^{\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\frac{x^2}{4t}} I(x,t).$$

The integral I just introduced may be broken into two parts, i.e.,

$$I(x,t) = \int_{|y|\leq 1} \mathrm{e}^{-\mathrm{i}\alpha xy} \frac{\mathrm{e}^{\mathrm{i}\beta y^2}}{(1+y^2)^m} \mathrm{d}y + \int_{|y|\geq 1} \mathrm{e}^{-\mathrm{i}\alpha xy} \frac{\mathrm{e}^{\mathrm{i}\beta y^2}}{(1+y^2)^m} \mathrm{d}y = I_1(x,t) + I_2(x,t)$$

with $\alpha = \frac{1}{2t}$ and $\beta = \frac{1}{4t} - 1$ both non-zero. By the Riemann-Lebesgue Lemma, I_1 is continuous in x and t and tends to zero as $|x| \to \infty$, though not necessarily uniformly with respect to t.

Integrate I_2 by parts to reach the formula

$$I_2(x,t) = \frac{C}{\beta} \int_{|y|\geq 1} \frac{1}{y} \Big(\frac{-\mathrm{i}\alpha x}{(1+y^2)^m} - \frac{2my}{(1+y^2)^{m+1}} \Big) \mathrm{e}^{-\mathrm{i}\alpha xy} \mathrm{e}^{\mathrm{i}\beta y^2} \mathrm{d}y + \frac{1}{\beta} F(x),$$

where F is a bounded, continuous function. Observe that the integrand in the last integral is an L^1_y-function; hence the Riemann-Lebesgue Lemma again implies that I_2 is continuous in x and t (and grows at most linearly in x at infinity).

When $d \geq 2$, write

$$u(x,t) = \frac{C}{t^{\frac{d}{2}}} \mathrm{e}^{\mathrm{i}\frac{|x|^2}{4t}} I(x,t),$$

as in the one-dimensional case, where

$$\begin{aligned} I(x,t) &= \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\alpha x\cdot y} \frac{\mathrm{e}^{\mathrm{i}\beta|y|^2}}{(1+|y|^2)^m} \mathrm{d}y \\ &= \int_{|y|\leq 1} \mathrm{e}^{-\mathrm{i}\alpha x\cdot y} \frac{\mathrm{e}^{\mathrm{i}\beta|y|^2}}{(1+|y|^2)^m} \mathrm{d}y + \int_{|y|\geq 1} \mathrm{e}^{-\mathrm{i}\alpha x\cdot y} \frac{\mathrm{e}^{\mathrm{i}\beta|y|^2}}{(1+|y|^2)^m} \mathrm{d}y \\ &= I_1(x,t) + I_2(x,t). \end{aligned}$$

As above, the Riemann-Lebesgue Lemma implies that I_1 is continuous and tends to zero as $|x| \to \infty$. Change variables in the integral I_2, setting $y = r\omega$, $\omega \in \mathbb{S}^{d-1}$ and $r \geq 0$, so that

$$I_2(x,t) = \int_{\mathbb{S}^{d-1}} \int_1^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}\alpha r(x.\omega)} \mathrm{e}^{\mathrm{i}\beta r^2}}{(1+r^2)^m} r^{d-1} \mathrm{d}r \mathrm{d}\omega,$$

where $\mathrm{d}\omega$ is the Lebesgue measure on $\mathbb{S}^{d-1}$. The inner integral has the form

$$\frac{C}{\beta} \int_1^{+\infty} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \Big(\frac{\mathrm{e}^{-\mathrm{i}\alpha r(x.\omega)}}{(1+r^2)^m} r^{d-1} \Big) \mathrm{e}^{\mathrm{i}\beta r^2} \mathrm{d}r.$$

Integrating by parts with respect to r yields the expression

$$\frac{1}{\beta} F(x,\omega) + \frac{C}{\beta} \int_1^{\infty} \frac{\mathrm{e}^{\mathrm{i}\beta r^2}}{r} \mathrm{e}^{-\mathrm{i}\alpha r(x\cdot\omega)} \Big[\frac{(n-1)r^{d-2}}{(1+r^2)^m} - \frac{\mathrm{i}\alpha(x\cdot\omega) r^{d-1}}{(1+r^2)^m} - \frac{2mr^d}{(1+r^2)^{m+1}} \Big] \mathrm{d}r$$

for the inner integral, where $F(x,\omega) = C\mathrm{e}^{-\mathrm{i}\alpha(x\cdot\omega)}$ is bounded and continuous in both x and ω. The mildest decay to 0 in the variable r in the integrand is $\mathcal{O}(\frac{1}{r^{2m+d-2}})$ as $r \to +\infty$. The integrand therefore belongs to L^1_r provided that $m > \frac{d-1}{2}$. This in turn is implied by (2.4) when $n = 2$, but (2.4) is henceforth strengthened to

$$\frac{d-1}{2} < m \leq \frac{d}{2} \tag{2.5}$$

in case the dimension $d \geq 3$.

To conclude the proof, observe that the r-integrability is uniform for $\omega \in \mathbb{S}^{d-1},\ t \in (0, \infty)$ and on bounded sets of x, so that Lebesgue's dominated convergence theorem implies that I_2 is a continuous function of x and t.

Remark 2.4 The principle of superposition allows us to organize initial data ϕ lying in $C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution u of (2.1) associated to ϕ is continuous in $\mathbb{R}^d \times (0, \infty) \setminus \{(x_j, t_j)\}_{j=1}^\infty$, but blows up at the isolated, countable set of points $\{(x_j, t_j)\}$ in $\mathbb{R}^d \times (0, \infty)$. Also, as remarked earlier, by considering the initial value $u(x, 0) = \delta\phi(x)$ and taking δ small, but positive, there obtain solutions that blow up dispersively, the norm of whose initial data is arbitrarily small.

Remark 2.5 It might appear from our proof that dispersive blow-up is a special phenomenon, subsisting upon very delicate spatial organization of energy. In fact, the phenomenon is much more robust than one might imagine. In favor of the persistence of the phenomenon, two arguments present themselves. First, suppose u_0 to be initial data that exhibits the dispersive blow-up as in Theorem 2.1. Let v be any $H^k(\mathbb{R}^d)$-function where $k > \frac{d}{2}$. Then it is clear from standard properties of the Schrödinger group that the solution $u(x, t)$ of the initial-value problem (2.1) with initial data $\phi = u_0 + v$ also features dispersive blow-up. Thus, at least at the linear level, dispersive blow-up is stable to smooth perturbations.

Second, let ϕ be initial data as in (2.3) which leads to dispersive blow-up at $(x, t) = (0, \frac{1}{4})$. As mentioned in Remark 2.3, and which is obvious from Sobolev embedding theory, a solution u can not experience dispersive blow-up if it starts out in $H^k(\mathbb{R}^d)$ with $k > \frac{d}{2}$. However, notice that if we define

$$\phi_R(y) = \rho\Big(\frac{|y|}{R}\Big)\phi(y),$$

where ρ is a non-negative, $\mathcal{C}^\infty$-function with compact support that is identically 1 on $[-1, 1]$, and vanishes for $|y| \geq 2$, then $\phi_R \in H^k(\mathbb{R}^d)$ for all k. Let $\delta > 0$ be small and $M > 0$ large be given. Take for initial data in (2.1) $u_0(y) = \delta\phi_R(y)$. Then, clearly $u_0 \in H^k(\mathbb{R}^d)$ and

$$|u_0|_\infty = \mathcal{O}(\delta)$$

as $\delta \to 0$. But if $u = u_{R,\delta}$ is the solution of (2.1) associated to u_0, then

$$u\Big(0, \frac{1}{4}\Big) = \delta \int_{\mathbb{R}^d} \frac{\rho\big(\frac{|y|}{R}\big)}{(1 + |y|^2)^m} \mathrm{d}y \geq \delta \int_{|y| \leq R} \frac{1}{(1 + |y|^2)^m} \mathrm{d}y.$$

The last integral is bounded below by a quantity of the form $\delta\mathcal{O}(R^{d-2m})$ when $m < \frac{d}{2}$ and of the form $\delta\mathcal{O}(\log R)$ when $m = \frac{d}{2}$, as $R \to \infty$. Thus, while smooth data does not lead to blow up, such data can be organized to be as small as we like in $L^\infty(\mathbb{R}^d)$, but to be such that the associated solution of (2.1) can have very large values at a specified, dispersive blow-up point.

Remark 2.6 With some modifications, similar analysis applies to what is sometimes called the "hyperbolic" Schrödinger equation, namely

$$\begin{cases} \mathrm{i}\partial_t u + \partial_{xx} u - \partial_{yy} u = 0, & \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u(x,0) = \phi(x), & \text{for } x \in \mathbb{R}^2. \end{cases} \tag{2.6}$$

This equation is the linearization about the rest state of a model for surface gravity waves on deep water (see [29]).

The fundamental solution of the hyperbolic Schrödinger equation (2.6) is

$$G(x_1, x_2; t) = \frac{1}{4\mathrm{i}\pi t} \mathrm{e}^{\mathrm{i}\frac{x_1^2 - x_2^2}{4t}}.$$

Inspired by the construction of blowing-up solutions appearing in the proof of Theorem 2.1, take

$$\phi(y) = \phi(y_1, y_2) = \frac{\mathrm{e}^{-\mathrm{i}(y_1^2 - y_2^2)}}{(1 + y_1^2)^m (1 + y_2^2)^p}$$

with $\frac{1}{4} < m, p \le \frac{1}{2}$, so that

$$\phi \in C^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2).$$

Fix a value $t > 0$, with $t \ne \frac{1}{4}$. The solution of problem (2.6) with the initial data just indicated is

$$u(x,t) = \frac{C}{t} \mathrm{e}^{\mathrm{i}\frac{x_1^2 - x_2^2}{4t}} I(x,t),$$

where

$$\begin{aligned} I(x,t) &= \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\alpha(x_1 y_1 - x_2 y_2)} \frac{\mathrm{e}^{\mathrm{i}\beta(y_1^2 - y_2^2)}}{(1 + y_1^2)^m (1 + y_2^2)^p} \mathrm{d}y \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha x_1 y_1} \frac{\mathrm{e}^{\mathrm{i}\beta y_1^2}}{(1 + y_1^2)^m} \mathrm{d}y_1 \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\alpha x_2 y_2} \frac{\mathrm{e}^{-\mathrm{i}\beta y_2^2}}{(1 + y_2^2)^p} \mathrm{d}y_2, \end{aligned}$$

where $\alpha = \frac{1}{2t}$ and $\beta = \frac{1}{4t} - 1$ are both non-zero. The issue of dispersive blow-up in this case is now reduced to that of the ordinary linear Schrödinger equation (2.1) in one space dimension, analyzed already in Theorem 2.1.

2.2 The nonlinear case

Attention is now turned to the nonlinear Cauchy problem

$$\begin{cases} \mathrm{i}u_t + u_{xx} + \epsilon |u|^p u = 0, & x \in \mathbb{R},\ t > 0,\ \epsilon = \pm 1, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.7}$$

The following result is built upon the linear analysis appearing in Theorem 2.1.

Theorem 2.2 *Let $t^* > 0$ and $x^* \in \mathbb{R}$ be given and suppose that $1 \le p < 3$. There is initial data $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ for which the associated solution of (2.7) is such that*

(1) $u \in C_b(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^q(\mathbb{R}_+^*; L^r(\mathbb{R})) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+^*; H^{\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R}))$, *where* $r \in [2, +\infty]$ *and* $\frac{2}{q} = (\frac{1}{2} - \frac{1}{r})$,

(2) *the conclusions* (2)–(4) *of Theorem* 2.1 *hold for* u.

Proof Part (1) of this theorem comprises well-known facts about the one-dimensional initial-value problem (2.7) (see, for instance, [25]).

Again, without loss of generality, we take it that $(x^*, t^*) = (0, \frac{1}{4})$. Let $u_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$ be the initial data constructed in the proof of Theorem 2.1 which leads to dispersive blow-up at the point $(x, t) = (0, \frac{1}{4})$. The Schröthe dinger group $S(t) = \mathrm{e}^{\mathrm{i}t\Delta}$ and the Duhamel's formula can be used to obtain the representation

$$\begin{aligned} u(x,t) &= S(t)u_0 + C\int_0^t \int_{\mathbb{R}} \frac{1}{(t-s)^{\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\frac{(x-y)^2}{4(t-s)}} |u|^p u(y,s)\mathrm{d}y\mathrm{d}s \\ &=: S(t)u_0 + CI(x,t), \end{aligned} \tag{2.8}$$

where C is a non-zero constant. If the integral term $I(x,t)$ in (2.8) is a continuous function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, then the desired results follow from what is already known about $S(t)u_0$ from Theorem 2.1. Continuity of this double integral will follow from Lebesgue's theorem as soon as it is known to be locally bounded as a function of x and t. The provision of such bounds is the next order of business. By the Hölder's inequality, for all $x \in \mathbb{R}$,

$$|I(x,t| \le \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} |u(\cdot,s)|_{p+1}^{p+1} \mathrm{d}s \le \Big(\int_0^t \frac{\mathrm{d}s}{(t-s)^{\frac{\gamma}{2}}}\Big)^{\frac{1}{\gamma}} \Big(\int_0^t |u(\cdot,s)|_{p+1}^{\gamma'(p+1)} \mathrm{d}s\Big)^{\frac{1}{\gamma'}}, \tag{2.9}$$

where $\gamma \in (1,2)$ and $\frac{1}{\gamma'} = 1 - \frac{1}{\gamma}$ will be fixed presently. The standard Strichartz estimates enunciated in part (1) of the theorem assert that for any $T > 0$, $u \in L^q(0,T; L^r(\mathbb{R}))$, where q and r are restricted as already noted. Take $r = p+1$ (implying $p \ge 1$). The corresponding value of q is $q = \frac{4(p+1)}{p-1}$. The condition $\gamma'(p+1) \le q$ entails that $\gamma' \le \frac{4}{p-1}$, which is to say, $\frac{1}{\gamma'} = 1 - \frac{1}{\gamma} \ge \frac{p-1}{4}$ or $\gamma \ge \frac{4}{5-p}$. This choice is compatible with $\gamma < 2$ if and only if $p < 3$. Thus, for $p < 3$, the integral I is a bounded function of x and t in $\mathbb{R} \times [0,T]$ for any $T > 0$. The proof is completed.

2.3 Possible connexion with optical rogue waves

The analysis of optical rogue-wave formation in [14] (see also [13]) is based on the generalized nonlinear Schrödinger equation

$$\frac{\partial A}{\partial t} + \frac{\alpha}{2}A - \sum_{k\ge 2} \frac{\mathrm{i}^{k+1}}{k!}\beta_k \frac{\partial^k A}{\partial z^k} = \mathrm{i}\gamma\Big(1 + \mathrm{i}\tau_{\mathrm{shock}}\frac{\partial}{\partial z}\Big)\Big(A(z,t)\int_{-\infty}^{+\infty} R(z')|A(z',t)|^2 \mathrm{d}z'\Big).$$

It deserves remark that in the application, the variable here denoted by t in fact connotes distance along the fiber, whereas z is in reality the temporal variable. The physical problem is in fact a boundary-value problem, but this is normally converted to an initial-value problem by viewing the independent variables as indicated in the present notation. On the other hand, when interpreting predictions of the model, one must keep in mind that space and time have been traded in the partial differential equation.

In this generalized Schrödinger equation, the dispersion is represented by its Taylor series and the nonlinearity features what is usually called a response function of the form $R(z) = (1 - f_R)\delta + f_R h_R(z)$, where δ is the Dirac mass. Thus the nonlinearity generally includes both instantaneous electronic and delayed Raman contributions.

Sketched here is a proof that dispersive blow-up also occurs in this model, thus providing a rigorous account of a possible explanation of optical rogue wave formation.

Consider first the linear part and for convenience, truncate the Taylor expansion of the dispersion so the linear model becomes

$$\begin{cases} \dfrac{\partial A}{\partial t} + \dfrac{\alpha}{2} A - \displaystyle\sum_{2 \leq k \leq K} \mathrm{i}^{k+1} \gamma_k \frac{\partial^k A}{\partial z^k} = 0, \\ A(x, 0) = A_0(x), \end{cases} \tag{2.10}$$

where $\gamma_K \neq 0$. By changing the independent variable from A to $B = \mathrm{e}^{-\alpha t} A$, one may take it that the damping coefficient α is zero.

Demonstrating dispersive blow-up for the linear equation (2.10) can be reduced (by perturbation arguments very similar to those used below for the linearized water-wave equation) to showing dispersive blow-up for the linear equation with homogeneous dispersion, i.e.,

$$\begin{cases} \dfrac{\partial A}{\partial t} - \mathrm{i}^{K+1} \dfrac{\partial^K A}{\partial z^K} = 0, \\ A(x, 0) = A_0(x), \end{cases} \tag{2.11}$$

where γ_K is set equal to 1 without loss of generality. Equation (2.11) specializes to the linear KdV-equation and the linear Schrödinger equations as particular cases when $K = 3$ and 2, respectively. When $K \geq 4$ one can use [22] or [3] to evaluate the corresponding fundamental solution and then construct suitable smooth initial data (a weighted version of the fundamental solution) which leads to dispersive blow-up.

This linear theory may then be extended to the nonlinear case. When the coefficient $\tau_{\text{shock}} = 0$, the equation is "semilinear" and the result follows by using Strichartz estimates as for the one-dimensional nonlinear Schrödinger equation. (This is especially transparent when the instantaneous electronic contribution vanishes, that is when $f_R = 1$, but it holds without this restriction.)

When $\tau_{\text{shock}} \neq 0$, the nonlinear term involves a derivative with respect to z. Assume now that $K \geq 3$. The crux of the matter is to analyze the double integral term in the Duhamel representation of the solution and to show that it defines a continuous function of space and time. When $K = 3$, we are reduced to the Korteweg-de Vries case which was dealt with already in [8] by using a theory of the Cauchy problem in weighted L^2-spaces. This analysis was also extended in [8] to a class of fifth-order Korteweg-de Vries equations. This extension is easily made for any odd value of K greater than 7. When $K \geq 4$ is even, the equation is of Schrödinger type and the weighted space theory (which is used in a crucial way that the phase velocity of the linear equation has a definite sign) does not appear to work. One has to rely instead on the higher-order smoothing properties of the linear group that appertains to the higher-order dispersion.

3 The Linearized Water-Wave Equations

Dispersive equations like the KdV-equation and the nonlinear Schrödinger equation are typically well-posed in L^2-based Sobolev classes. When such an equation features dispersive blow-up, it is clear that they are not well-posed in L^∞, however. Indeed, as we have seen, there will be initial data that is arbitrarily small in both L^2 and L^∞, but while it remains small in L^2, it becomes unbounded in L^∞ in finite time, indeed in small time if we so desire.

While model equations such as the KdV-equation and the Schrödinger equation exhibit aspects that are indicative of important phenomena actually arising in physical situations, the issue of real oceanic rogue wave formation in either shallow or deep water is most convincingly understood in the context of the full water wave equations. And, while singularity formation in finite time might point to the formation of rogue waves, a lack of well-posedness in L^∞-spaces reveals the same qualitative behavior that we associate with rogue-wave formation.

So, attention is focused on the full surface water wave equations linearized about the rest state (flat free surface and zero velocity) in both one and two horizontal dimensions. The question in front of us is whether or not they are well-posed in L^∞. In both one and two spatial dimensions, $d = 1, 2$, the full linear equations reduce to the single, evolution equation

$$\begin{cases} \eta_{tt} + \omega^2(|D|)\eta = 0, \quad x \in \mathbb{R}^d,\ t \in (0,\infty), \\ \eta(x,0) = \eta_0(x), \\ \eta_t(x,0) = \eta_1(x) \end{cases} \tag{3.1}$$

for the elevation $\eta = \eta(x,y,t)$ (or $\eta(x,t)$ in case the motion does not vary much in the y direction) of the wave (see, e.g., [28]). Here, $\omega(|\mathbf{k}|)$ is the usual linearized dispersion relation

$$\omega^2(|\mathbf{k}|) = g|\mathbf{k}|\tanh(|\mathbf{k}|h_0) \tag{3.2}$$

for water waves, where h_0 is the undisturbed depth, $\mathbf{k} = (k_1, k_2)$ and $|\mathbf{k}| = (k_1^2 + k_2^2)^{\frac{1}{2}}$ when $d = 2$. The phase velocity is therefore

$$\mathbf{c}(\mathbf{k}) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|}\widehat{\mathbf{k}} = g^{\frac{1}{2}}\Big(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|}\Big)^{\frac{1}{2}}\widehat{\mathbf{k}},$$

where $\widehat{\mathbf{k}}$ is the unit vector in the $\mathbf{k}$-direction. For waves of extreme length wherein $|\mathbf{k}| \to 0$, the phase velocity tends to $\sqrt{gh_0}\,\widehat{\mathbf{k}}$. For water waves on an infinite layer (corresponding to $h_0 = +\infty$), the phase velocity is

$$\mathbf{c}(\mathbf{k}) = g^{\frac{1}{2}}\frac{1}{|\mathbf{k}|^{\frac{1}{2}}}\widehat{\mathbf{k}}.$$

Thus, on deep water, plane waves travel faster and faster as the wavelength becomes large, contrary to the cases of the linear KdV or linear Schrödinger equation where large phase velocities occur for short waves (large wavenumbers k).

Considered also will be the case of gravity-capillary waves whose linear dispersion relation is

$$\omega^2(|\mathbf{k}|) = g|\mathbf{k}|\tanh(|\mathbf{k}|h_0)\Big(1 + \frac{T}{\rho g}|\mathbf{k}|^2\Big), \tag{3.3}$$

where ρ is the density and T is the surface tension coefficient. In this case, the phase velocity is

$$\mathbf{c}(\mathbf{k}) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|}\widehat{\mathbf{k}} = g^{\frac{1}{2}}\Big(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|}\Big)^{\frac{1}{2}}\Big(1 + \frac{T}{\rho g}|\mathbf{k}|^2\Big)^{\frac{1}{2}}\widehat{\mathbf{k}},$$

whose modulus tends to infinity as $|\mathbf{k}|$ tends to $+\infty$, that is in the limit of short wavelengths. In the infinite depth case, the phase velocity tends to infinity in the limit of both infinitely long and infinitely short waves.

In the sequel, when the depth is finite, the equations are scaled so that the gravity constant g and the mean depth h_0 are both equal to 1.

The solution of (3.1) with the dispersion law (3.2) is easily computed in Fourier transformed variables to be

$$\widehat{\eta}(\mathbf{k},t) = \widehat{\eta}_0(\mathbf{k})\cos[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}] + \frac{\sin[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}]}{(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}}\widehat{\eta}_1(\mathbf{k}). \tag{3.4}$$

In consequence, the Cauchy problem is clearly well-posed in L^2-based Sobolev classes. More precisely for any $(\eta_0,\eta_1) \in H^k(\mathbb{R}^d)\times H^{k-\frac{1}{2}}(\mathbb{R}^d)$, $k \geq 0$, (3.1) possesses a unique solution $\eta \in C(\mathbb{R}, H^k(\mathbb{R}^d))$.

To establish ill-posedness in L^∞, it suffices to consider the situation wherein $\widehat{\eta}_1 = 0$. Ill-posedness then amounts to proving that for each $t \neq 0$, the kernel

$$m_t(\mathbf{k}) = \mathrm{e}^{\mathrm{i}t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}}$$

is not a Fourier multiplier in L^∞, which is the same as showing its Fourier transform is not a bounded Borel measure.

Let $t>0$ be fixed and focus on $m_t(\mathbf{k})$. The first point to note is that

$$(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}} = |\mathbf{k}|^{\frac{1}{2}}\Big(1-\frac{2}{1+\mathrm{e}^{2|\mathbf{k}|}}\Big)^{\frac{1}{2}} = |\mathbf{k}|^{\frac{1}{2}} + r(|\mathbf{k}|), \tag{3.5}$$

where $r \in C(\mathbb{R})\cap C^\infty(\mathbb{R}\setminus\{0\})$ and $r(|\mathbf{k}|)$ behaves like $-\frac{|\mathbf{k}|^{\frac{1}{2}}}{1+\mathrm{e}^{2|\mathbf{k}|}}$ as $|\mathbf{k}| \to +\infty$ and like $-|\mathbf{k}|^{\frac{1}{2}}(1-|\mathbf{k}|^{\frac{1}{2}})$ as $|\mathbf{k}| \to 0$. Note that $r \equiv 0$ when the depth h_0 is infinite.

Decompose the kernel m_t as follows:

$$\mathrm{e}^{\mathrm{i}t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}} = \mathrm{e}^{\mathrm{i}r(|\mathbf{k}|)t}\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}} = (1+f_t(|\mathbf{k}|))\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}}, \tag{3.6}$$

where

$$f_t(|\mathbf{k}|) = 2\mathrm{i}\sin\Big(\frac{r(|\mathbf{k}|)t}{2}\Big)\mathrm{e}^{\mathrm{i}\frac{r(|\mathbf{k}|)t}{2}}$$

is continuous, smooth on $\mathbb{R}^d\setminus\{0\}$, and decays exponentially to 0 as $|\mathbf{k}| \to +\infty$, uniformly on bounded temporal sets, since $r(\mathbf{k})$ does so. This decomposition leads to an associated splitting of the Fourier transform $I_t(\mathbf{x})$ of $m_t(\mathbf{k})$, namely

$$I_t(\mathbf{x}) =: \int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{k} + \int_{\mathbb{R}^d} f_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{k} = I_t^1(x) + I_t^2(x). \tag{3.7}$$

Study of $I_t^2(x)$ Because f_t decays rapidly to 0 as $|\mathbf{k}|$ becomes large, the Riemann-Lebesgue Lemma implies that I_t^2 is a bounded, continuous function of $\mathbf{x}$ and thus locally integrable, in both dimensions 1 and 2. In fact, when $d=1$, it is actually a continuous L^1-function. To see this, restrict to $|x| \geq 1$ and integrate by parts to obtain the formula

$$\begin{aligned} I_t^2(x) &= -\frac{1}{\mathrm{i}x}\int_{-\infty}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}k}(f_t(k)\mathrm{e}^{\mathrm{i}t|k|^{\frac{1}{2}}})\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \\ &= -\frac{1}{\mathrm{i}x}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{\mathrm{i}t|k|^{\frac{1}{2}}}\Big[f_t'(k) + \frac{\mathrm{i}t\,\mathrm{sgn}\,k}{2|k|^{\frac{1}{2}}}f_t(k)\Big]\mathrm{d}k. \end{aligned}$$

The term in square brackets decays exponentially to 0 as $|k| \to \infty$ and has a singularity of order $|k|^{-\frac{1}{2}}$ at the origin, coming from $f_t'(k)$ (note that $f_t(k)|k|^{-\frac{1}{2}}$ is bounded at 0). It is therefore the Fourier transform of an L^p-function, where $1 \le p < 2$, and so, by the Riesz-Thorin theorem, must itself be an L^q-function where $2 < q \le +\infty$. Since $\frac{1}{x} \in L^s(|x| \ge 1)$ for any $s > 1$, the Hölder's inequality thus insures that $I_t^2 \in L^1(\mathbb{R})$.

Study of $I_t^1(x)$ The analysis of I_t^1 relies on detailed results about the Fourier transform of the kernel $\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^a}$, for a in the range $0 < a < 1$, where $\psi \in C^\infty(\mathbb{R})$, $0 \le \psi \le 1$, $\psi \equiv 0$ on $[0,1]$, $\psi \equiv 1$ on $[2,+\infty)$. For $0 < a < 1$ and $\mathbf{k} \in \mathbb{R}^d$, let $F_a(x) = \mathcal{F}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^a})(x)$ be the Fourier transform of the kernel. Since $\mathbf{k} \mapsto \psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^a} \in \mathcal{S}'(\mathbb{R}^d)$, $\psi(|\mathbf{k}|)\mathrm{e}^{-\epsilon|\mathbf{k}|}\mathrm{e}^{\mathrm{i}|\mathbf{k}|^a}$ converges to $\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^a}$ as $\epsilon \to 0$, at least in the sense of distributions. It follows that $\mathcal{F}(\psi(|\mathbf{k}|)\mathrm{e}^{-\epsilon|\mathbf{k}|}\mathrm{e}^{\mathrm{i}|\mathbf{k}|a}) \to \mathcal{F}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|a})$ as $\epsilon \to 0$.

For the readers' convenience, we recall the following general result of Wainger [27, Theorem 9] and Miyachi [21, Proposition 5.1].

Theorem 3.1 (see [21, 27]) *Let $0 < a < 1$, $b \in \mathbb{R}$ and define $F_{a,b}^\epsilon(\mathbf{x}) =: \mathcal{F}(\psi(|\mathbf{k}|)|\mathbf{k}|^{-b} \exp(-\epsilon|\mathbf{k}| + i|\mathbf{k}|^a))(\mathbf{x})$ for $\epsilon > 0$ and $\mathbf{x} \in \mathbb{R}^d$. The following is true of the function $F_{a,b}^\epsilon$.*

(i) *$F_{a,b}^\epsilon(\mathbf{x})$ depends only on $|\mathbf{x}|$.*

(ii) *$F_{a,b}(\mathbf{x}) = \lim_{\epsilon \to 0^+} F_{a,b}^\epsilon(\mathbf{x})$ exists pointwise for $\mathbf{x} \neq 0$ and $F_{a,b}$ is smooth on $\mathbb{R}^d \setminus \{0\}$.*

(iii) *For all $N \in \mathbb{N}$, and $\mu \in \mathbb{N}^d$, $|(\frac{\partial}{\partial x})^\mu F_{a,b}(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-N})$ as $|\mathbf{x}| \to +\infty$.*

(iv) *If $b > d(1 - \frac{a}{2})$, $F_{a,b}$ is continuous on $\mathbb{R}^d$.*

(v) *If $b \le d(1 - \frac{a}{2})$, then for any $m_0 \in \mathbb{N}$, the function $F_{a,b}$ has the asymptotic expansion*

$$F_{a,b}(\mathbf{x}) \sim \frac{1}{|\mathbf{x}|^{\frac{1}{1-a}(d-b-\frac{ad}{2})}} \exp\Big(\frac{\mathrm{i}\xi_a}{|\mathbf{x}|^{\frac{a}{1-a}}}\Big) \sum_{m=0}^{m_0} \alpha_m |\mathbf{x}|^{\frac{ma}{1-a}} + o(|\mathbf{x}|)^{\frac{(m_0+1)a}{(1-a)}} + g(\mathbf{x}), \tag{3.8}$$

as $\mathbf{x} \to 0$, where $\xi_a \in \mathbb{R}$, $\xi_a \neq 0$, and g is a continuous function.

(vi) *When $a > 1$ and $b \in \mathbb{R}^d$, $F_{a,b}$ is smooth on $\mathbb{R}^d$ and has the asymptotic expansion*

$$F_{a,b}(\mathbf{x}) \sim C(a,b,d)|\mathbf{x}|^{\frac{b-d+\frac{da}{2}}{1-a}} \exp\{\mathrm{i}B(a)|\mathbf{x}|^{-\frac{a}{1-a}}\} + o(|\mathbf{x}|^{\frac{b-d+\frac{da}{2}}{1-a}}), \tag{3.9}$$

as $|\mathbf{x}| \to +\infty$, where $C(a,b,d)$ and $B(a)$ are explicit positive constants.

(vii) *For any $b \in \mathbb{R}$, $F_{1,b}$ is smooth on $\mathbb{R}^d \setminus \{|x| = 1\}$ and for every $\mu \in \mathbb{N}^d$ and $N \in \mathbb{N}$,*

$$\Big|\Big(\frac{\partial}{\partial x}\Big)^\mu F_{1,b}(x)\Big| = \mathcal{O}(|x|^{-N}), \quad \text{as } |x| \to +\infty.$$

If $b < \frac{d+1}{2}$, then

$$F_{1,b}(x) = C(b)\Big(1 - |x| + \mathrm{i}0\Big)^{b - \frac{d-1}{2}}, \quad \text{as } |x| \to 1.$$

Remark 3.1 The previous theorem will be used with $b = 0$. For notational convenience, set $F_{a,0} = F_a$. If $a = \frac{1}{2}$, the first three terms in the asymptotic expansion of F_a are, respectively, $\alpha_0|x|^{\frac{-3d}{2}}\exp\{\theta\}$, $\alpha_1|x|^{\frac{-3d+2}{2}}\exp\{\theta\}$ and $\alpha_2|x|^{\frac{-3d+4}{2}}\exp\{\theta\}$, where $\theta = \mathrm{i}\xi_a|x|^{\frac{-a}{1-a}}$. Notice that $F_a \notin L^1_{\mathrm{loc}}(\mathbb{R}^d)$, but that it is defined as a distribution since, because of its oscillatory nature, it is locally integrable around 0 in the sense of generalized Riemann integration. For example, when $d = 1$, one has

$$F_{\frac{1}{2}}(x) = \frac{1}{|x|^{\frac{3}{2}}} \exp\Big\{\mathrm{i}\frac{\xi}{|x|}\Big\} + G(x),$$

where $\xi \neq 0$ and $G \in L^1(\mathbb{R})$. For any $A > 0$, integration by parts reveals that

$$\int_{-A}^{A} |x|^{-\frac{3}{2}} \exp\Big\{\mathrm{i}\frac{\xi}{|x|}\Big\}\mathrm{d}x = 2\int_0^A \frac{\mathrm{e}^{\mathrm{i}\frac{\xi}{r}}}{r^{\frac{3}{2}}}\mathrm{d}r = 2\frac{\mathrm{i}}{\xi}\sqrt{A}\,\mathrm{e}^{\mathrm{i}\frac{\xi}{A}} - \frac{\mathrm{i}}{\xi}\int_0^A \frac{1}{\sqrt{r}}\mathrm{e}^{\mathrm{i}\frac{\xi}{r}}\mathrm{d}r.$$

The last integral exists in the $L^1(0, A)$-sense. Thus, $F_{\frac{1}{2}}$ can be defined as a distribution by writing it as $F_{\frac{1}{2}} = G + \mathfrak{F}$, where $G \in L^1_{\mathrm{loc}}(\mathbb{R})$, and for any test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$,

$$\langle \mathfrak{F}, \phi\rangle = -\frac{\mathrm{i}}{2\xi}\int_0^{+\infty} \frac{\mathrm{e}^{\mathrm{i}\frac{\xi}{x}}}{\sqrt{x}}[\phi(x) - \phi(-x)]\mathrm{d}x - \frac{\mathrm{i}}{\xi}\int_0^{+\infty} \mathrm{e}^{\mathrm{i}\frac{\xi}{x}}\sqrt{x}\,[\phi'(x) - \phi'(-x)]\mathrm{d}x.$$

Returning to the study of $I_t^1(x)$, notice first that for dimensions $d = 1, 2$,

$$I_t^1(x) = \frac{1}{t^{2d}}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\frac{x}{t^2}}\mathrm{d}\mathbf{k} = \frac{1}{t^{2d}}J\Big(\frac{x}{t^2}\Big),$$

where

$$J(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k}.$$

Introducing a truncation function ψ as above which is zero near the origin and one near infinity, the integral J can be broken down as

$$J(x) = \int_{\mathbb{R}^d} \psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} + \int_{\mathbb{R}^d}(1 - \psi(|\mathbf{k}|))\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} =: J_1(x) + J_2(x). \tag{3.10}$$

Arguing as in the analysis of I_t^2, one checks that in dimension $d = 1$, the continuous function J_2 lies in $L^1(\mathbb{R})$. In dimension $d = 2$, J_2 is a bounded continuous function of x.

But, Theorem 3.1 implies that $(1 - \psi(|\mathbf{k}|))\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}$ is not an $L^\infty(\mathbb{R}^d)$-multiplier. These considerations allow the following conclusion.

Proposition 3.1 *In both one and two horizontal spatial dimension, $d = 1, 2$, the linearized water-wave problem* (3.1) *is ill-posed in $L^\infty(\mathbb{R}^d)$.*

Proof Take $\eta_1 \equiv 0$ in (3.1) and an appropriate choice of η_0 (see the proof of Theorem 3.2 for more details).

This proposition is reinforced by the following, more specific, dispersive blow-up result.

Theorem 3.2 *Let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$ $(d = 1, 2)$ be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d \setminus \{0\}) \cap C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of* (3.1) *with $\eta_1 \equiv 0$ is such that*

(i) *η is a continuous function of x and t on $\mathbb{R} \times ((0, +\infty) \setminus \{t^*\})$,*

(ii) *$\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R} \setminus \{x*\}$,*

(iii) $\displaystyle\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |\eta(x, t)| = +\infty.$

Proof One may assume that $(x^*, t^*) = (0, 1)$. Again, take $\widehat{\eta}_1 = 0$ in (3.1) so that the corresponding solution is

$$\eta(\cdot, t) = \eta_0 \star \mathcal{F}^{-1}(\exp\{\mathrm{i}[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}]\}).$$

Using the notation introduced in our earlier ruminations, we write

$$\begin{aligned}\eta(\cdot,t) &= \eta_0 \star \mathcal{F}^{-1}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}} + (1-\psi(|\mathbf{k}|))\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}} + f_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}})\\ &= \eta_0 \star \mathcal{F}^{-1}(f_1(t,\mathbf{k}) + f_2(t,\mathbf{k}) + f_3(t,\mathbf{k}))\\ &= N_1(\cdot,t) + N_2(\cdot,t) + N_3(\cdot,t).\end{aligned}$$

In spatial dimension $d=1$, it has already been shown that for any fixed $t \in (0,+\infty)$, $\mathcal{F}^{-1}f_2(\cdot,t)$ and $\mathcal{F}^{-1}f_3(\cdot,t)$ are integrable functions of x, and, as is easily confirmed, uniformly so on compact subsets of $t \in (0,+\infty)$. Thus, the functions N_2 and N_3 are continuous on $\mathbb{R}\times(0,+\infty)$, for any $\eta_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. In dimension $d=2$, for a fixed $t \in (0,+\infty)$, $\mathcal{F}^{-1}f_2(\cdot,t)$ and $\mathcal{F}^{-1}f_3(\cdot,t)$ are bounded continuous functions of x, and uniformly so on compact subsets of $t \in (0,+\infty)$.

Choose the initial value η_0 to be $\eta_0(x) = |x|^\lambda \overline{K}(x)$ for $x \in \mathbb{R}$, where $\frac{3d}{2} \le \lambda \le 2d$ and

$$K = \mathcal{F}^{-1}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}).$$

In the notation arising in Theorem 3.1, this amounts to taking $b=0$ and setting $K(x) = \overline{F}_{\frac{1}{2}}(x)$, the overbar connoting complex conjugation. Using the results of Theorem 3.1 along with the choice of λ, it is easily seen that $\eta_0 \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\}) \cap L^1(\mathbb{R}^d)$. In particular, η_0 is an L^2-function.

Note that although $\eta_0 \in L^1(\mathbb{R}^d)$, for $t \neq 0$, the solution $\eta(\cdot,t)$ does not necessarily belong to $L^\infty(\mathbb{R}^d)$ since $\mathcal{F}^{-1}(\exp\{\mathrm{i}[t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}]\})$ is not an L^∞-function. This is in strong contrast with what obtains for the linear KdV-equation (1.1) or the linear Schrödinger equation (2.1). However, the preceding commentary does show that $N_2(\cdot,t)$ and $N_3(\cdot,t)$ are convolutions of an L^1-function with a bounded, continuous function of x. Hence, they are themselves bounded and continuous in x, and uniformly so on compact temporal subsets.

Theorem 3.1 applied to N_1 implies that as $(x,t)\to(0,1)$, the solution $\eta(x,t)$ tends to

$$C_1 + C_2\int_{\mathbb{R}^d} |K(y)|^2|y|^\lambda \mathrm{d}y = +\infty,$$

since $\lambda \le 2d$.

It is now demonstrated that η is continuous on $\mathbb{R}^d \times (0,+\infty) \setminus \{(0,1)\}$, which is to say, everywhere except at the dispersive blow-up point. Since N_2 and N_3 are already known to be continuous in x and t, it remains to consider $N_1(\cdot,t) = \eta_0 \star \mathcal{F}^{-1}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}})$.

We first show that $N_1(\cdot,1)$ is a continuous function of x on $\mathbb{R}^d \setminus \{0\}$. According to the definition of η_0, we get

$$N_1(x,1) = \int_{\mathbb{R}^d} |x-y|^\lambda \overline{F}_{\frac{1}{2}}(x-y)F_{\frac{1}{2}}(y)\mathrm{d}y. \tag{3.11}$$

Let $\delta > 0$ be fixed and suppose that $|x| > \delta$. Decompose the last integral in the form

$$\begin{aligned}N_1(x,1) &= \int_{B_{\frac{\delta}{2}}(0)} |x-y|^\lambda \overline{F}_{\frac{1}{2}}(x-y)F_{\frac{1}{2}}(y)\mathrm{d}y + \int_{B_{\frac{\delta}{2}}(x)} |x-y|^\lambda \overline{F}_{\frac{1}{2}}(x-y)F_{\frac{1}{2}}(y)\mathrm{d}y\\ &\quad + \int_{\mathbb{R}^d\setminus B_{\frac{\delta}{2}}(0)\cup B_{\frac{\delta}{2}}(x)} |x-y|^\lambda \overline{F}_{\frac{1}{2}}(x-y)F_{\frac{1}{2}}(y)\mathrm{d}y\\ &= N_1^1(x,1) + N_1^2(x,1) + N_1^3(x,1).\end{aligned}$$

By Theorem 3.1, $N_1^3(\cdot,1)$ is a continuous function of x. By our choice of λ, $N_1^2(\cdot,1)$ is a continuous function of x. The treatment of $N_1^1(\cdot,1)$ is a bit more delicate and makes use of the oscillatory nature of the integrand. By Theorem 3.1, $F_{\frac{1}{2}}(x)$ behaves like

$$\Big[\frac{\alpha_1}{|x|^{\frac{3d}{2}}}+\frac{\alpha_2}{|x|^{\frac{3d}{2}-1}}+\frac{\alpha_3}{|x|^{\frac{3d}{2}-2}}+g(x)\Big]\mathrm{e}^{\mathrm{i}C_3|x|^{-1}} \tag{3.12}$$

for x near 0, where g is continuous.

When $d=1$, only the first term in (3.12) gives trouble as regards the continuity of $N_1^1(\cdot,1)$. Integration by parts reveals immediately that the integral

$$\int_{B_{\frac{\delta}{2}}(0)}|x-y|^{\lambda}\overline{F}_{\frac{1}{2}}(x-y)\frac{\mathrm{e}^{\mathrm{i}C_3|y|^{-1}}}{|y|^{\frac{3}{2}}}\mathrm{d}y$$

defines a continuous function of x.

When $d=2$, the first two terms in (3.12) both lead to situations that are possibly singular. We are therefore lead to consider the two integrals

$$\int_{B_{\frac{\delta}{2}}(0)}|x-y|^{\lambda}\overline{F}_{\frac{1}{2}}(x-y)\frac{\mathrm{e}^{\mathrm{i}C_3|y|^{-1}}}{|y|^{3}}\mathrm{d}y \tag{3.13}$$

and

$$\int_{B_{\frac{\delta}{2}}(0)}|x-y|^{\lambda}\overline{F}_{\frac{1}{2}}(x-y)\frac{\mathrm{e}^{\mathrm{i}C_3|y|^{-1}}}{|y|^{2}}\mathrm{d}y. \tag{3.14}$$

Straightforward integration by parts shows that both these integrals define continuous functions of x.

Attention is now turned to the region $D_1=\{(x,t);\,x\in\mathbb{R}^d,\ t>0,\ t\neq 1\}$. It will be shown that N_1 is continuous throughout this domain. A first observation is

$$\mathcal{F}^{-1}(\psi(|\mathbf{k}|\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}})(x)=\frac{1}{t^{\frac{d}{2}}}\mathcal{F}^{-1}\Big(\psi\Big(\frac{|\mathbf{k}|}{t^{\frac{1}{2}}}\Big)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}\Big)\Big(\frac{x}{t^{\frac{1}{2}}}\Big). \tag{3.15}$$

Notice also that

$$\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}-\psi\Big(\frac{|\mathbf{k}|}{t^{\frac{1}{2}}}\Big)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}=\widetilde{\psi}_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}},$$

where $\widetilde{\psi}_t$ is smooth, compactly supported and vanishes in a neighborhood of 0. Thus, the inverse Fourier transform of $\widetilde{\psi}_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{1}{2}}}$ is smooth and decays rapidly to 0 as $|x|\to\infty$; it is certainly bounded and continuous on D_1. We may therefore write

$$N_1(\cdot,t)=\frac{1}{t^{\frac{d}{2}}}\eta_0\star F_{\frac{1}{2}}\Big(\frac{\cdot}{t^{\frac{1}{2}}}\Big)+G(\cdot,t)=:\widetilde{N}_1(\cdot,t)+G(\cdot,t), \tag{3.16}$$

where G is continuous in x and t. Split $\widetilde{N}_1(x,t)$ as follows:

$$\begin{aligned}\widetilde{N}_1(x,t)&=\frac{1}{t^{\frac{d}{2}}}\int_{\mathbb{R}^d}\eta_0(x-y)F_{\frac{1}{2}}\Big(\frac{y}{t^{\frac{1}{2}}}\Big)\mathrm{d}y\\&=\frac{1}{t^{\frac{d}{2}}}\Big(\int_{|y|\le 1}\eta_0(x-y)F_{\frac{1}{2}}\Big(\frac{y}{t^{\frac{1}{2}}}\Big)\mathrm{d}y+\int_{|y|\ge 1}\eta_0(x-y)F_{\frac{1}{2}}\Big(\frac{y}{t^{\frac{1}{2}}}\Big)\mathrm{d}y\Big)\\&=\widetilde{N}_1^1(x,t)+\widetilde{N}_1^2(x,t).\end{aligned}$$

Since $\eta_0(x-y)$ behaves like $C|x-y|^{-\frac{3d}{2}+\lambda}$ when y is close to x, the choice of λ and the properties of $F_{\frac{1}{2}}$ imply that $\widetilde{N}_1^2$ is continuous in x and t.

The choice $\eta_0 = |x|^\lambda \overline{K}(x)$ entails

$$\widetilde{N}_1(x,t) = \frac{1}{t^{\frac{d}{2}}} \int_{|y|\le 1} |x-y|^\lambda \overline{F}_{\frac{1}{2}}(x-y) F_{\frac{1}{2}}\Big(\frac{y}{t^{\frac{1}{2}}}\Big) \mathrm{d}y. \tag{3.17}$$

When $x \neq 0$, the singularity at $y = 0$ can be dealt with as in the preceding analysis of N_1^1. Attention is thus restricted to $x = 0$ and the aim is to prove that the integral

$$\widetilde{N}_1(0,t) = \frac{1}{t^{\frac{d}{2}}} \int_{|y|\le 1} |y|^\lambda \overline{F}_{\frac{1}{2}}(y) F_{\frac{1}{2}}\Big(\frac{y}{t^{\frac{1}{2}}}\Big) \mathrm{d}y, \tag{3.18}$$

taken in the sense of generalized Riemann integration, is finite when $t \neq 1$. According to Theorem 3.1, the singular part of the integral defining $\widetilde{N}_1(0,t)$ is

$$\Gamma(t) = t^{\frac{d}{4}} \int_{|y|\le 1} |y|^{\lambda - 3d} \mathrm{e}^{\mathrm{i}\frac{C_3}{|y|}(t^{\frac{1}{2}}-1)} \mathrm{d}y.$$

This integral is finite, as seen by integration by parts, provided $\lambda > 2d-1$, which is compatible with the restriction

$$\frac{3d}{2} \le \lambda \le 2d$$

on λ. The proof is completed.

Remark 3.2 As noted previously, the phase velocity $g^{\frac{1}{2}}(\frac{\tanh(|\mathbf{k}|h_0)}{|\mathbf{k}|})^{\frac{1}{2}}\widehat{\mathbf{k}}$ is a bounded function of $\mathbf{k}$. This is contrary to the case of the linear KdV-equation (Airy-equation) and the linear Schrödinger equation, where both the phase velocity and the group velocity become unbounded in the short wave limit. It is also unlike the case of linear gravity waves on the surface of an infinite layer of fluid, where the phase velocity is unbounded in the long wave limit. The dispersive blow-up phenomenon observed here is thus not linked to the unboundedness of the phase velocity. Further comments on this point appear in Section 5.

Consider now the case of the linearized gravity-capillary waves where the dispersion relation is given in formula (3.3). Again taking all the physical constants equal to 1 to simplify the discussion, the solution of (3.1) becomes

$$\begin{aligned}\widehat{\eta}(\mathbf{k},t) = {}& \widehat{\eta}_0(\mathbf{k}) \cos[t(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}] \\ &+ \widehat{\eta}_1(\mathbf{k}) \frac{\sin[t(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}]}{(|\mathbf{k}| \tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}.\end{aligned} \tag{3.19}$$

From this formula, it is readily discerned that for $(\eta_0,\eta_1) \in H^k(\mathbb{R}^d) \times H^{k-\frac{3}{2}}(\mathbb{R}^d)$, $k \in \mathbb{N}$, (3.1) has a unique solution $\eta \in C_b(\mathbb{R}; H^k(\mathbb{R}^d)) \cap L^2_{\mathrm{loc}}(\mathbb{R}; H^{k+\frac{1}{4}}_{\mathrm{loc}}(\mathbb{R}^d))$; the local smoothing property is a consequence of the general theory supplied in [11], for example.

The next theorem is the analogue of Theorem 3.2 in the capillary-gravity-wave context. It implies in particular that the Cauchy problem (3.1)–(3.3) is ill-posed in L^∞.

Theorem 3.3 *For $d = 1$ or 2, let $(x^*, t^*) \in \mathbb{R}^d \times (0, +\infty)$ be given. There exists $\eta_0 \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that the solution $\eta \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ of (3.1)–(3.3) with $\eta_1 \equiv 0$ satisfies*

(i) *η is a continuous function of x and t on $\mathbb{R} \times ((0, +\infty) \setminus \{t^*\})$),*
(ii) *$\eta(\cdot, t^*)$ is continuous in x on $\mathbb{R} \setminus \{x*\}$,*
(iii)

$$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |\eta(x,t)| = +\infty.$$

Proof One may assume that $(x^*, t^*) = (0, 1)$. Again, take $\widehat{\eta}_1 = 0$ in (3.1) and search for an η_0 that leads to dispersive blow-up. Define η_0 by

$$\eta_0(x) = \frac{\overline{F}_{\frac{3}{2}}(x)}{(1 + |x|^2)^m},$$

where $\frac{3d}{4} < m \leq d$ and $F_{\frac{3}{2}} = F_{\frac{3}{2},0}$, defined in Theorem 3.1, is smooth on $\mathbb{R}^d$ and is asymptotic to $|x|^{\frac{d}{2}}\mathrm{e}^{\mathrm{i}C|x|^3}$ as $|x| \to \infty$ for a suitable non-zero constant C. It is straightforwardly verified that $\eta_0 \in C^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. As in the gravity-wave case, the fact that $\eta_0 \in L^1(\mathbb{R}^d)$ does not imply that $\eta(\cdot, t) \in L^\infty(\mathbb{R}^d)$, $t \neq 0$, since $F_{\frac{3}{2}} \notin L^1(\mathbb{R}^d)$.

In place of (3.5), we have the decomposition

$$\begin{aligned}(|\mathbf{k}| \tanh |\mathbf{k}|)^{\frac{1}{2}}(1 + |\mathbf{k}|^2)^{\frac{1}{2}} &= |\mathbf{k}|^{\frac{1}{2}}(1 + |\mathbf{k}|^2)^{\frac{1}{2}} + (1 + |\mathbf{k}|^2)^{\frac{1}{2}} r(|\mathbf{k}|)\\ &= |\mathbf{k}|^{\frac{1}{2}}(1 + |\mathbf{k}|^2)^{\frac{1}{2}} + s(|\mathbf{k}|),\end{aligned} \tag{3.20}$$

where the remainder function r is as introduced in (3.5). The function s is continuous, smooth on $\mathbb{R}^d \setminus \{0\}$, and decays exponentially to 0 as $|\mathbf{k}| \to +\infty$. So, instead of (3.6), there appears

$$\mathrm{e}^{\mathrm{i}t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}} = \mathrm{e}^{\mathrm{i}ts(|\mathbf{k}|)}\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}} = (1 + g_t(|\mathbf{k}|))\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}, \tag{3.21}$$

where

$$g_t(|\mathbf{k}|) = 2\mathrm{i}\mathrm{e}^{\frac{\mathrm{i}}{2}ts(|\mathbf{k}|)} \sin\Big(\frac{ts(|\mathbf{k}|)}{2}\Big)$$

is continuous, smooth on $\mathbb{R}^d \setminus \{0\}$, and decays exponentially to 0 as $|\mathbf{k}| \to +\infty$.

As for the situation without surface tension, split the crucial, oscillatory integral according to the decomposition (3.21), i.e.,

$$\begin{aligned}J_t(x) &= \int_{R^d} \mathrm{e}^{\mathrm{i}t(|\mathbf{k}|\tanh|\mathbf{k}|)^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}\mathrm{d}\mathbf{k}\\ &= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} + \int_{\mathbb{R}^d} g_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{1}{2}}(1+|\mathbf{k}|^2)^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k}\\ &= J_t^1(x) + J_t^2(x).\end{aligned} \tag{3.22}$$

One proves just as in the gravity-wave case that J_t^2 is bounded, continuous, and tends to 0 as its argument becomes unbounded (and that it lies in L^1 when $d = 1$). As will be seen presently, the treatment of J_t^1 follows a line of argument more akin to that used in the analysis of the linear KdV-equation and the linear Schrödinger equation, a consequence of what we will, in Section 5, term the "strongly dispersive" character of its dispersion.

Study of $J_t^1(x)$ Define the function $\alpha = \alpha(|\mathbf{k}|)$ by

$$|\mathbf{k}|^{\frac{1}{2}}(1 + |\mathbf{k}|^2)^{\frac{1}{2}} = |\mathbf{k}|^{\frac{3}{2}} + \alpha(|\mathbf{k}|),$$

so that

$$\alpha(|\mathbf{k}|) = \frac{|\mathbf{k}|^{\frac{1}{2}}}{(1+|\mathbf{k}|^2)^{\frac{1}{2}} + |\mathbf{k}|}. \tag{3.23}$$

In terms of α, the integral $J_t^1(x)$ is written as

$$\begin{aligned} J_t^1(x) &= \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x} \mathrm{d}\mathbf{k} + 2\mathrm{i} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}\frac{\alpha(\mathbf{k})t}{2}} \sin\Big(\frac{\alpha(\mathbf{k})t}{2}\Big) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x} \mathrm{d}\mathbf{k} \\ &= J_t^{1,1}(x) + J_t^{1,2}(x). \end{aligned}$$

Theorem 3.1, part (vi) with $a = \frac{3}{2}$ and $b = 0$ implies that, due to its growth at ∞, $J_t^{1,1}$ is not a bounded measure. More precisely, write

$$\begin{aligned} J_t^{1,1}(x) &= \int_{\mathbb{R}^d} (1-\psi(|\mathbf{k}|)) \mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x} \mathrm{d}\mathbf{k} + \int_{\mathbb{R}^d} \psi(|\mathbf{k}|) \mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x} \mathrm{d}\mathbf{k} \\ &= K_t^1(x) + K_t^2(x), \end{aligned} \tag{3.24}$$

where $\psi \in C^\infty(\mathbb{R}),\ 0 \le \psi \le 1,\ \psi \equiv 0$ on $[0,1],\ \psi \equiv 1$ on $[2, +\infty)$ is as introduced in the statement of Theorem 3.1, the Wainger-Miyachi results.

The first integral in (3.27) is, by the Riemann-Lebesgue lemma, continuous in x and tends to 0 at infinity. Moreover, as is easily checked via two integrations by parts, the function $x \mapsto |x|^2 K_t^1(x)$ is a bounded function of x (and thus an L^1-function when $d = 1$). The modulus of K_t^2 grows like $|x|^{\frac{d}{2}}$ as $|x|$ tends to infinity.

The preceding calculations allow us to express the solution $\eta(\cdot, t)$ of the linear, capillary-gravity wave initial-value problem (3.1)–(3.3) in the form

$$\eta(\cdot, t) = \eta_0 \star (K_t^1 + K_t^2 + J_t^{1,2} + J_t^2) = N_t^1 + N_t^2 + N_t^3 + N_t^4. \tag{3.25}$$

Consider first the "singular" term N_t^2. By Theorem 3.1, $N_t^2(x,t)$ is continuous away from the point $(0,1)$, but tends to

$$\int_{\mathbb{R}^d} \frac{|F_{\frac{3}{2}}(y)|^2}{(1+|y|^2)^m} \mathrm{d}y = +\infty$$

as $(x,t) \to (0,1)$ because of the restriction imposed on the parameter m.

In consequence of this remark, if it is known that the integrals $N_t^{\mathrm{i}},\ i = 1,3,4$ all define continuous functions of (x,t) on $\mathbb{R}^d \times \mathbb{R}^+$, the result will be established. Of course, $N_t^1 = \eta_0 \star K_t^1$ is continuous since it is a convolution of an L^1-function with a bounded, continuous function. The following result provides the desired information about N_t^3.

Lemma 3.1 *$J_t^{1,2}$ is a continuous function of $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$ and is a bounded function of x, uniformly on compact subsets of $t \in (0,\infty)$.*

Proof By its definition,

$$J_t^{1,2}(x) = 2\mathrm{i} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}} \mathrm{e}^{\mathrm{i}\frac{\alpha(\mathbf{k})t}{2}} \sin\Big(\frac{\alpha(\mathbf{k})t}{2}\Big) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x} \mathrm{d}\mathbf{k},$$

where α is defined in (3.23). A straightforward Taylor expansion shows that

$$\mathrm{e}^{\mathrm{i}\frac{\alpha(\mathbf{k})t}{2}} \sin\Big(\frac{\alpha(\mathbf{k})t}{2}\Big) = \frac{\alpha(|\mathbf{k}|)t}{2} + \mathrm{i}\frac{\alpha^2(|\mathbf{k}|)t^2}{4} + r(|\mathbf{k}|, t),$$

where $r(|\mathbf{k}|,t) = t^3\mathcal{O}(\alpha^3(|\mathbf{k}|))$ as $\alpha(|\mathbf{k}|)\to 0$, t bounded. Since

$$\alpha(|\mathbf{k}|) = \frac{1}{2|\mathbf{k}|^{\frac{1}{2}}} + \mathcal{O}\Big(\frac{1}{|\mathbf{k}|^{\frac{5}{2}}}\Big),$$

it follows that, uniformly for bounded values of t,

$$\mathrm{e}^{\mathrm{i}\frac{\alpha(\mathbf{k})t}{2}} \sin\Big(\frac{\alpha(\mathbf{k})t}{2}\Big) = \frac{t}{4|\mathbf{k}|^{\frac{1}{2}}} + \frac{t^2}{16|\mathbf{k}|} + t^3\mathcal{O}\Big(\frac{1}{|\mathbf{k}|^{\frac{3}{2}}}\Big) \tag{3.26}$$

as $|\mathbf{k}|\to\infty$.

With ψ denoting the same cut-off function introduced earlier, one decomposes $J_t^{1,2}$ in a form that is by now familiar, namely,

$$\begin{aligned} J_t^{1,2}(x) &= 2\mathrm{i}\int_{\mathbb{R}^d}\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}}\mathrm{e}^{\mathrm{i}\frac{\alpha(\mathbf{k})t}{2}}\sin\Big(\frac{\alpha(\mathbf{k})t}{2}\Big)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} \\ &\quad + 2\mathrm{i}\int_{\mathbb{R}^d}(1-\psi(|\mathbf{k}|))\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}}\mathrm{e}^{\mathrm{i}\frac{\alpha(\mathbf{k})t}{2}}\Big(\sin\frac{\alpha(\mathbf{k})t}{2}\Big)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} \\ &= M_t^1(x) + M_t^2(x). \end{aligned}$$

The Riemann-Lebesgue Lemma indicates that the second integral M_t^2 is a bounded, continuous function of x, the spatial bound being uniform on compact temporal subsets of $(0,\infty)$.

Using (3.26), the integral M_t^1 may be expressed as

$$\begin{aligned} M_t^1(x) &= \frac{\mathrm{i}t}{2}\int_{\mathbb{R}^d}\psi(|\mathbf{k}|)\frac{\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}}}{|\mathbf{k}|^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\mathbf{k}.x}\mathrm{d}\mathbf{k} + \frac{\mathrm{i}t^2}{16}\int_{\mathbb{R}^d}\psi(|\mathbf{k}|)\frac{\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}}}{|\mathbf{k}|}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} \\ &\quad + 2\mathrm{i}t^3\int_{\mathbb{R}^d}\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}}\mathcal{O}\Big(\frac{1}{|\mathbf{k}|^{\frac{3}{2}}}\Big)\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot x}\mathrm{d}\mathbf{k} \\ &= M_t^{1,1}(x) + M_t^{1,2}(x) + M_t^{1,3}(x). \end{aligned}$$

The integral $M_t^{1,1}$ is essentially $F_{\frac{3}{2},\frac{1}{2}}$. Thus, by Theorem 3.1, it behaves like $|x|^{\frac{d}{2}-1}$ as $|x|\to\infty$. Hence, in one and two spatial dimensions, it comprises a bounded, continuous function of x. Similarly, the integral $M_t^{1,2}$ corresponds to $F_{\frac{3}{2},1}$, and so it behaves like $|x|^{-2+\frac{d}{2}}$ as $|x|\to\infty$. It is also a bounded, continuous function of x. The last term $M_t^{1,3}$ can be treated in a similar way, thereby completing the proof of Lemma 3.1.

Proof of Theorem 3.3 (Continuation) Since J_t^2 is a bounded, continuous function, N_t^4 is likewise a bounded, continuous function of x, and uniformly so on compact temporal subsets of $(0,\infty)$.

To complete the proof of Theorem 3.3, it remains to prove that N_t^2 is continuous on $\mathbb{R}^d\times(0,+\infty)\setminus\{(0,1)\}$. It is first established that $N_1^2(x_0)$ is continuous at any point $x_0\in\mathbb{R}^d\setminus\{0\}$. To this end, consideration is given to the integral

$$I(x) = \int_{\mathbb{R}^d}\frac{\overline{F}_{\frac{3}{2}}(x-y)}{(1+|x-y|^2)^m}F_{\frac{3}{2}}(y)\mathrm{d}y$$

for, say, $x\in B_\delta(x_0)$, where δ is small enough that this ball does not include $x=0$. In view of Theorem 3.1(vi), it suffices to show that the integral

$$\int_{|y|\geq 1}\frac{|x-y|^{\frac{d}{2}}|y|^{\frac{d}{2}}}{(1+|x-y|^2)^m}\mathrm{e}^{\mathrm{i}C(|y|^3-|x-y|^3)}\mathrm{d}y$$

is finite in the sense of generalized Riemann integration. (Note that if $x = 0$, the oscillation disappears and the following considerations do not apply.) Consider first the one-dimensional case $d = 1$. Plainly, it suffices to show that the integral over $\{y : |y| \geq |x|\}$ is finite. The integrals where $y \geq |x| > 0$ and $y \leq -|x| < 0$ are of a similar nature, so we consider only the integral

$$J(x) = \mathrm{e}^{-\mathrm{i}x^3} \int_{y\geq|x|} \frac{(y-x)^{\frac{1}{2}} y^{\frac{1}{2}}}{(1+(x-y)^2)^m} \mathrm{e}^{3\mathrm{i}x^2 y} \mathrm{e}^{-3\mathrm{i}xy^2} \mathrm{d}y.$$

Integrating by parts gives

$$J(x) = -\mathrm{i}\frac{\mathrm{e}^{-\mathrm{i}x^3}}{6x} \int_{y\geq|x|} \frac{\mathrm{d}}{\mathrm{d}y}\Big[\frac{(y-x)^{\frac{1}{2}} y^{\frac{1}{2}}}{y^{\frac{1}{2}}(1+(x-y)^2)^m} \mathrm{e}^{3\mathrm{i}x^2 y}\Big] \mathrm{e}^{-3\mathrm{i}xy^2} \mathrm{d}y.$$

The worst term from the point of view of the present considerations is

$$\frac{1}{3} x \mathrm{e}^{\mathrm{i}x^3} \int_{y\geq|x|} \Big[\frac{(y-x)^{\frac{1}{2}}}{y(1+(x-y)^2)^m} \mathrm{e}^{3\mathrm{i}x^2 y}\Big] \mathrm{e}^{-3\mathrm{i}xy^2} \mathrm{d}y,$$

which is a convergent integral since the restriction on m implies that $m > \frac{1}{2}$.

To treat the case $d = 2$, use polar coordinates, writing $y = r\mathrm{e}^{\mathrm{i}\theta}$, and integrate twice by parts the integral with respect to r on the region $\{r > |x|\}$. The computation is analogous to that made already for the nonlinear Schrödinger equation in the proof of Theorem 2.1, and so is omitted.

Finally, it is established that $N_t^2 = \eta_0 \star \mathcal{F}^{-1}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}})$ is a continuous function of (x,t) in the domain $D_1 = \{(x,t) : x \in \mathbb{R}^d,\ t > 0,\ t \neq 1\}$. A simple rescaling reveals that

$$\mathcal{F}^{-1}(\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}t|\mathbf{k}|^{\frac{3}{2}}})(x) = \frac{1}{t^{\frac{2d}{3}}} \mathcal{F}^{-1}\Big(\psi\Big(\frac{|\mathbf{k}|}{t^{\frac{2}{3}}}\Big)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{3}{2}}}\Big)\Big(\frac{x}{t^{\frac{2}{3}}}\Big). \tag{3.27}$$

As for the gravity-wave case, one observes that

$$\psi(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{3}{2}}} - \psi\Big(\frac{|\mathbf{k}|}{t^{\frac{2}{3}}}\Big)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{3}{2}}} = \widetilde{\psi}_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{3}{2}}},$$

where $\widetilde{\psi}_t$ is smooth, compactly supported and vanishes in a neighborhood of 0. Hence the function $\widetilde{\psi}_t(|\mathbf{k}|)\mathrm{e}^{\mathrm{i}|\mathbf{k}|^{\frac{3}{2}}}$ is likewise smooth and compactly supported, so its inverse Fourier transform is, as a function of x, smooth, decays rapidly to 0 at $\pm\infty$ and is jointly continuous in x and t. It follows that N_t^2 can be written in the form

$$N_t^2(\cdot) = \frac{1}{t^{\frac{2d}{3}}} \eta_0 \star F_{\frac{3}{2}}\Big(\frac{\cdot}{t^{\frac{2}{3}}}\Big) + \widetilde{G}(\cdot,t) =: \widetilde{N}_t^2(\cdot) + \widetilde{G}(\cdot,t), \tag{3.28}$$

where $\widetilde{G}$ is continuous in x and t.

Split $\widetilde{N}_t^2$ in two as follows:

$$\widetilde{N}_t^2(x) = \frac{1}{t^{\frac{2d}{3}}}\Big(\int_{|y|\leq 1} + \int_{|y|\geq 1}\Big)\eta_0(x-y) F_{\frac{3}{2}}\Big(\frac{y}{t^{\frac{2}{3}}}\Big)\mathrm{d}y = \widetilde{M_t^1}(x) + \widetilde{M_t^2}(x). \tag{3.29}$$

Since $F_{\frac{3}{2}}$ is a smooth function, $\widetilde{M_t^1}(\cdot)$ is continuous in x and t. By definition of η_0,

$$\widetilde{M_t^2}(x) = \frac{1}{t^{\frac{2d}{3}}} \int_{|y|\geq 1} \frac{\overline{F}_{\frac{3}{2}}(x-y)}{(1+|x-y|^2)} F_{\frac{3}{2}}\Big(\frac{y}{t^{\frac{2}{3}}}\Big)\mathrm{d}y.$$

Again applying Theorem 3.1(vi) reduces the issue of continuity to checking whether or not the improper Riemann integral

$$\frac{1}{t^{\frac{d}{3}}} \int_{|y| \geq |x|} \frac{|x-y|^{\frac{d}{2}} |y|^{\frac{d}{2}}}{(1+|x-y|^2)^m} \mathrm{e}^{\mathrm{i}C[\frac{|y|^3}{t^2} - |x-y|^3]} \mathrm{d}y \tag{3.30}$$

is finite. A straightforward integration by parts comes to our rescue, showing that the integral is indeed finite. The continuity then follows from standard arguments and the proof of Theorem 3.3 is completed.

Remark 3.3 It is worth remarking that the dispersive blow-up for the full linear gravity-wave problem may bear upon one of the suggested routes to the formation of rogue waves. Many papers in the fluid mechanics and oceanographic literature have recently been devoted to various aspects of oceanic rogue waves (see, e.g., [18, 19] and the references therein). One of the puzzling issues that confront scientists interested in rogue waves is their genesis. While there are several suggestions as to how rogue waves form, a universally accepted explanation is lacking. One of the suggested routes to rogue wave formation is, roughly speaking, that small amounts of energy from disparate parts of the ocean might occasionally come together in space-time and, at least temporarily, result in very large waves. This is precisely what we have demonstrated is possible in establishing dispersive blow-up (dispersive focusing) in the surface water-wave environment. More precisely, following on the commentary in Remarks 2.3 and 2.5 in Section 2, there are open sets $\mathcal{U}$ in $H^k(\mathbb{R}^d)$, $k \geq 3$, such that if initial data u_0 is taken from $\mathcal{U}$, then $|u_0|_\infty \leq \epsilon$, but the solution u of (3.1) with u_0 as initial data has the property that $|u(\cdot, t^*)|_\infty \geq M$, where the positive values of ϵ, M and t^* are specified.

4 Fractional Schrödinger Type Equations

The methods used and the results obtained in the previous section can be extended to a class of fractional-order Schrödinger equations of the type

$$\begin{cases} \mathrm{i}u_t + (-\Delta)^{\frac{a}{2}} u = 0, \quad 0 < a < 1, \\ u(\cdot, t) = u_0(\cdot) \end{cases} \tag{4.1}$$

for $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$. These equations, mentioned already in (2.4), occur as the linearization of some weak turbulence models (see [12]). In Fourier-transformed variables, the solution of (4.1) is

$$\widehat{u}(\xi, t) = \widehat{u}_0(\xi) \mathrm{e}^{\mathrm{i}t|\xi|^a},$$

and so the solution semi-group $S(t)$ is in fact a unitary group on L^2 defined by $S(t)\phi = \mathcal{F}^{-1}(\mathrm{e}^{\mathrm{i}t|\xi|^a}\widehat{\phi})$. Observe that the case $a = \frac{1}{2}$ is essentially the linearized water-wave system on an infinitely deep layer with surface tension neglected. Theorem 3.1 will be used to establish the following dispersive blow-up result for the initial-value problem (4.1).

Theorem 4.1 *Let* $0 < a < 1$. *Then* (2.4) *is ill-posed in* $L^\infty(\mathbb{R}^d)$ *and displays the dispersive-blow-up phenomenon. More precisely, for any fixed* $(x^*, t^*) \in \mathbb{R}^d \times (0, \infty)$, *there exists* $u_0 \in C^\infty(\mathbb{R}^d \backslash \{0\}) \cap C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ *such that the corresponding solution* $u \in C_b(\mathbb{R}; L^2(\mathbb{R}^d))$ *of* (2.4) *is such that*

(i) u *is a continuous function of* (x,t) *on* $\mathbb{R}^d \times ((0, +\infty) \setminus \{t^*\})$,

(ii) $u(\cdot,t^*)$ *is a continuous function of* x *on* $\mathbb{R}^d\setminus\{x^*\}$, *and*

(iii)

$$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}}|u(x,t)|=+\infty.$$

Proof The proof is very similar to that of Theorem 3.2 and so is omitted.

Remark 4.1 The class in (4.1) with $a>1$ includes the Schrödinger equation ($a=2$). These may also be treated by appropriate use of Theorem 3.1(vi).

5 Comments and Extensions

5.1 Strongly- versus weakly-dispersive equations: short-wave and long-wave dispersive focussing

In connection with the mathematical analysis of various Boussinesq-type systems for weakly nonlinear, long surface waves, it was observed in [6] that the dispersive blow-up phenomenon does not occur for "weakly dispersive" equations or systems of equations. A paradigm problem of weakly-dispersive type is the generalized BBM-equation

$$u_t+u_x+u^pu_x-u_{xxt}=0, \tag{5.1}$$

where p is a positive integer. Indeed, one of the motivations for introducing the BBM-equation as a water wave model in [4] was its appropriate behavior with regard to the propagation of short waves. When more general dispersion relations are involved in the wave propagation problem at hand, the general class of "regularized" dispersive equations comes to the fore (see, e.g., [5]). These models have the form

$$u_t+u_x+u^pu_x+Lu_t=0, \tag{5.2}$$

where the dispersion operator L is defined as a Fourier multiplier operator by

$$\widehat{Lu}(\xi)=q(\xi)\widehat{u}(\xi) \tag{5.3}$$

and q is a non-negative-valued function.

One can show that for a large class of symbols q (those which generate bounded phase velocities that decay to zero rapidly enough as $|k|\to\infty$), the linearized version

$$\begin{cases}u_t+u_x+Lu_t=0,\\u(\cdot,0)=u_0(\cdot)\end{cases} \tag{5.4}$$

of (5.2) is well-posed in L^∞ (see [6]). In the commentary to follow, attention is restricted to pure power symbols of the form $p(\xi)=|\xi|^\kappa$ whose associated dispersion operator L defined as in (5.3) is denoted by L_κ.

The results in [6] concern the slightly modified version

$$\begin{cases}(I-\partial_x^2)^{\frac{\kappa}{2}}u_t+u_x=0,\\u(\cdot,0)=u_0(\cdot)\end{cases} \tag{5.5}$$

of (5.4). Notice that when $\kappa = 2$, which is the case of the original BBM-equation, the two formulations coincide.

Stated here is a result established in [6, Proposition 4.5] for (5.5), but put into the context of (5.4).

Proposition 5.1 *The initial-value problem* (5.4) *is well-posed in* $L^\infty(\mathbb{R})$ *if and only if* $\kappa > 1$.

Remark 5.1 The phase velocities for (5.4) and (5.5) are

$$c(k) = \frac{1}{1+|k|^\kappa} \quad \text{and} \quad c(k) = \frac{1}{(1+|k|^2)^{\frac{\kappa}{2}}}.$$

In both cases, when $\kappa > 1$, the phase velocity decays rapidly enough to 0 as $k \to \infty$ that $c(k)$ is an L^1-function. On the contrary, for the full linearized gravity- or gravity-capillary-wave problems, the phase velocity behaves like $|k|^{-\frac{1}{2}}$ at infinity, which is not a fast enough decay to prevent dispersion from leading to focussing singularities.

Remark 5.2 Equations (5.4) and (5.5) when $\kappa = 1$ are closely related to the linearization of the Boussinesq systems in [6] which have a phase velocity behaving like $\frac{1}{|k|}$ as $|k| \to \infty$. The so-called "BBM-BBM" systems in [6], which do not feature dispersive blow-up, correspond to $\kappa = 2$.

Proof The solution of (5.4) is

$$u(\cdot,t) = \mathcal{F}^{-1}[\mathrm{e}^{\mathrm{i}t\frac{|k|}{1+|k|^\kappa}}] \star u_0.$$

Observe that

$$\mathrm{e}^{\mathrm{i}t\frac{|k|}{1+|k|^\kappa}} = \mathrm{e}^{\mathrm{i}t|k|(1+|k|^2)^{-\frac{\kappa}{2}}}[(1+h_t(|k|)],$$

where $h_t(|k|)$ is a smooth function of t and $|k|$, and is $\mathcal{O}(|k|^{-2\kappa+1})$ as $|k| \to +\infty$, uniformly on compact subsets of t's in $(0,+\infty)$. Recourse to Theorem 3.1 comes to our aid. The argument is, by now, familiar and we skip over the details.

This last result is now extended to the nonlinear case. Using the Duhamel formula, the solution of (5.2) has the representation

$$u(\cdot,t) = S_\kappa(t)u_0 - \int_0^t S_\kappa(t-s)[u^p u_x]\mathrm{d}s, \tag{5.6}$$

where $S_\kappa(t) = \mathrm{e}^{\mathrm{i}t\partial_x(I+L_\kappa)^{-1}}$ is the associated linear group.

Proposition 5.2 *Let* $\kappa > 1$ *and* $u_0 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. *There exists* $T > 0$ *and a unique solution* $u \in C([0,T]; L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}))$ *of* (5.2) *with initial data* u_0.

Proof This follows from a straightforward iterative argument based on the Duhamel formulation (5.6) and will be omitted (see, e.g., the proof of Theorem 2.1 in [7]).

Remark 5.3 It was shown in [9] that the solution of (5.2) in the case $\kappa = 2$ and $p = 1$ (the original BBM equation) is globally well-posed for data residing only in $L^2(\mathbb{R})$. The same well-posedness result is true of the initial-value problem (5.4) for any $\kappa > \frac{3}{2}$ (see [8]).

When $\kappa \le 1$, both (5.2) and its linearization display the dispersive blow-up phenomenon.

Proposition 5.3 *Let $\kappa \leq 1$ and let $(x^*, t^*) \in \mathbb{R} \times (0, +\infty)$ be given. There exists $\phi \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ such that the corresponding solution u (respectively, v) of* (5.2) (*resp.* (5.4)) *have the properties*

(i) *u, v are continuous in x and t on $\mathbb{R} \times ((0,\infty) \setminus \{t^*\})$,*

(ii) *$(u, v)(\cdot, t^*)$ are continuous on $\mathbb{R} \setminus \{x^*\}$,*

(iii)

$$\lim_{\substack{(x,t)\in\mathbb{R}^d\times(0,+\infty)\to(x^*,t^*)\\(x,t)\neq(x^*,t^*)}} |u, v(x,t)| = +\infty.$$

Proof The details are very similar to those appearing previously in this essay, and hence they are omitted.

5.2 Miscelleanous remarks

(1) Ghidaglia and Jaffard [16] have constructed an explicit, infinite energy solution of (1.1) displaying dispersive blow-up. They took

$$u(x,t) = \frac{1}{t^{\frac{1}{6}}}\mathrm{Ai}^2\Big(\frac{x}{t^{\frac{1}{3}}}\Big),$$

where Ai is the usual Airy-function. The function u is clearly a C^∞-function on the half-space $D = \{(x,t);\ x \in \mathbb{R},\ t > 0\}$. A simple computation using the classical ODE

$$\mathrm{Ai}''(z) - z\mathrm{Ai}(z) = 0 \tag{5.7}$$

defining Ai shows that u solves (1.1) in D. However, one checks that for any fixed $t > 0$, $u(\cdot, t) \in L^p(\mathbb{R})$ for any $p > 2$, and

$$\|u(\cdot,t)\|_p^p = \frac{C}{t^{\frac{p}{6}-\frac{1}{3}}}\int_{\mathbb{R}} \mathrm{Ai}^{2p}(x)\mathrm{d}x.$$

Thus, for any $p > 2$, the L^p-norms of $u(\cdot, t)$ blow up at $t = 0$ like $\frac{C}{t^{\frac{1}{6}-\frac{1}{3p}}}$. In particular, the sup-norm blows up like $\frac{C}{t^{\frac{1}{6}}}$.

(2) We have so far not been able to extend the dispersive blow-up result from the linear to the nonlinear Euler equations for the free-surface, water-wave problem. An intermediate step in this direction would be to extend the results of Section 3 to the case of non constant coefficients. Especially telling would be an extension to the linearization of the water-wave system about a non-horizontal free surface.

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References

[1] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, Dover, New York, 1964.

[2] Akhmediev, N., Soto-Crespo, J. M. and Ankiewicz, A., How to excite a rogue wave, *Phy. Rev. A*, **80**, 2009, 043818, 1–17.

[3] Ben-Artzi, M. and Saut, J.-C., Uniform decay estimates for a class of oscillatory integrals and applications, *Diff. Int. Eqs.*, **12**, 1999, 137–145.

[4] Benjamin, T. B. Bona, J. L. and Mahony, J. J., Model equations for long waves in nonlinear, dispersive media, *Philos. Trans. Royal Soc. London, Ser. A*, **272**, 1972, 47–78.

[5] Bona, J. L. and Chen, H. Q., Well-posedness for regularized dispersive wave equations, *Disc. Cont. Dyn. Systems A*, **23**, 2009, 1253–1275.

[6] Bona, J. L., Chen, M. and Saut, J.-C., Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I : Derivation and linear theory, *J. Nonlinear Sci.*, **12**, 2002, 283–318.

[7] Bona, J. L., Chen, M. and Saut, J.-C., Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory, *Nonlinearity*, **17**, 2004, 925–952.

[8] Bona, J. L. and Saut, J.-C., Dispersive blow-up of solutions of generalized Korteweg-de Vries equations, *J. Diff. Eqs.*, **103**, 1993, 3–57.

[9] Bona, J. L. and Tzvetkov, N., Sharp well-posedness results for the BBM equation, *Disc. Cont. Dyn. Systems A*, **23**, 2009, 1241–1252.

[10] Brenner, P., The Cauchy problem for systems in l_p and $l_{p,\alpha}$, *Ark. Mat.*, **11**, 1973, 75–101.

[11] Constantin, P. and Saut, J.-C., Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.*, **1**, 1988, 413–439.

[12] Dias, F., Guyenne, P., Pushkarev, A. N., et al., Wave turbulence in one-dimensional models, *Physica D*, **152–153**, 2001, 573–619.

[13] Dudley, J., Supercontinuum generation in photonic crystal fiber, *Rev. Mod. Phys.*, **78**, 2006, 1135–1184.

[14] Dudley, J., Genty, G. and Eggleton, B. J., Harnessing and control of optical rogue waves in supercontinum generation, *Optics Express*, **16**, 2008, 3644–3651.

[15] Dysthe, H., Krogstad, E. and Müller, P., Oceanic Rogue Waves, *Ann. Rev. Fluid Mech.*, **40**, 2008, 287–310.

[16] Ghidaglia, J.-M. and Jaffard, S., Personal communication.

[17] Hörmander, L., Estimates for translation invariant operators in L^p, *Acta Math.*, **104**, 1960, 93–140.

[18] Kharif, C. and Pelinovsky, E., Physical mechanism of rogue wave phenomenon, *European J. of Mechanics, B/Fluids*, **22**, 2003, 603–634.

[19] Kharif, C., Pelinovsky, E. and Slunyaev, A., Rogue Waves in the Ocean, Springer-Verlag, Berlin-Heidelberg, 2009.

[20] Linares, F. and Scialom, M., On the smoothing properties of solutions to the modified Korteweg-de Vries equation, *J. Diff. Eqs.*, **106**, 1993, 141–154.

[21] Miyachi, A., On some singular Fourier multipliers, *J. Fac. Sci. Tokyo Section I A*, **28**, 1981, 267–315.

[22] Sidi, A., Sulem, C. and Sulem, P. L., On the long time behavior of a generalized KdV equation, *Acta Applicandae Mathematicae*, **7**, 1986, 35–47.

[23] Solli, D. R., Ropers, C., Koonath, P. and Jalali, B., Optical rogue waves, *Nature*, **450**, 2007, 1054–1057.

[24] Stenflo, L. and Marklund, M., Rogue waves in the atmosphere, 2009. arXiv:0911.1654v1

[25] Tsutsumi, Y., L^2 solutions for the nonlinear Schrödinger equation and nonlinear groups, *Funkcial. Ekvac.*, **30**, 1987, 115–125.

[26] Voronovich, V. V., Shrira, V. I. and Thomas, G., Can bottom friction suppress "freak wave" formation, *J. Fluid Mech.*, **604**, 2008, 263–296.

[27] Wainger, S., Special trigonometric series in k-dimensions, *Mem. American Math. Soc.*, **59**, 1965.

[28] Whitham, G. B., Linear and Nonlinear Waves, John Wiley & Sons, New York, 1974.

[29] Zakharov, V. E., Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.*, **9**, 1968, 86–94.