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PERIODIC TRAVELING–WAVE SOLUTIONS OF NONLINEAR DISPERSIVE EVOLUTION EQUATIONS

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Abstract. For a general class of nonlinear, dispersive wave equations, existence of periodic, traveling-wave solutions is studied. These traveling waveforms are the analog of the classical cnoidal-wave solutions of the Korteweg-de Vries equation. They are determined to be stable to perturbation of the same period. Their large wavelength limit is shown to be solitary waves.

1. **Introduction.**

A general class of nonlinear wave equations of the form

$$u_t - Lu_x + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{1}$$

has been put forward to describe long-crested, long-wavelength disturbances of small amplitude propagating primarily in one direction in a dispersive media (see Benjamin *et al.* [10]). Here, the dependent variable u, which often represents an amplitude or a velocity, is a real-valued function of the two real variables x and t, f is a real-valued function of one real variable, typically a polynomial with $f(0) = f'(0) = 0$, and L is the dispersion operator defined through its Fourier symbol α, say. In practical situations where such models arise, the independent variable x is usually associated with distance measured from some given point in the spatial domain of propagation while t is proportional to elapsed time. The dispersion operator L applied to a function $v = v(x)$ is related to its symbol α via the Fourier transform, *viz.*

$$\widehat{Lv}(\xi) = \alpha(2\pi\xi)\widehat{v}(\xi) \quad \text{where} \quad \widehat{v}(\xi) = \int_{-\infty}^{\infty} v(x)e^{-2\pi i\xi x}\,dx \tag{2}$$

for all wavenumbers ξ. The symbol α is typically a real-valued, even, continuous function vanishing at the origin and becoming unbounded as $\xi \to \pm\infty$. Such equations arise as rudimentary models for wave propagation in many different physical contexts (see *e.g.* [1], [14], [15] for example).

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Already in the 1870's, Boussinesq had found what we now call the cnoidal-wave solutions of the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \tag{3}$$

which is (1) with both f and α positive and purely quadratic. These cnoidal-wave solutions, so named by Korteweg and de Vries [24], can be written explicitly as

$$u(x,t) = a + b\,\mathrm{cn}^2\big(d(x-ct);k\big) \tag{4}$$

in terms of the Jacobi elliptic function $cn(x;k)$ where the elliptic modulus k, the parameters a, b, c, d and the period P of u are related by a system of nonlinear, transcendental equations (see *e.g.* [23]). The cnoidal waves have been the object of both theoretical and laboratory investigations (see [6]), [27], [28]. The theory, much of which makes use of the exact formula (4), reveals that the Korteweg-de Vries cnoidal waves are stable to periodic perturbations of the same period. Moreover, properly scaled, the cnoidal-wave solutions converge to the well known sech^2 solitary-wave solutions of the Korteweg-de Vries equation as the period length $P \to \infty$.

The present study is concerned with the analog of cnoidal-wave solutions of the more general models (1). Under mild regularity assumptions, such solutions have the form

$$u(x,t) = \phi(x-ct) = \sum_{n=-\infty}^{\infty} \phi_n e^{i\frac{n\pi}{l}(x-ct)} \tag{5}$$

where $2l$ is their spatial period and c their velocity of propagation. Of course, $\phi_n = \bar{\phi}_{-n}$ for all $n \in \mathbb{Z}$ since ϕ is real-valued, the overbar connoting complex conjugation. The overall goal of the present work is to bring forward theory in the general setting of (1) corresponding to what is known about the traveling-wave solutions of the Korteweg-de Vries equation (3) itself. Part of the program involves showing that such solutions exist.

In fact, they do not always exist. Substituting the form (5) into (1) reveals that ϕ satisfies the equation

$$(c+L)\phi = f(\phi) + A \quad \text{or} \quad L\phi = f(\phi) - c\phi + A \tag{6}$$

where A is a constant of integration and

$$L\phi(z) = \sum_{n=-\infty}^{\infty} \alpha(2\pi n)\phi_n e^{i\frac{n\pi}{l}z}.$$

Suppose that f is a polynomial of degree $p > 1$. The polynomial $f(z) - cz + A$ associated to f may or may not have real zeroes. In the case wherein $f(z) - cz + A$ has no real zero, there are no periodic traveling-wave solutions. Indeed, in this case, there is a positive number γ, say, such that $f(z) - cz + A$ is either greater than γ or less than $-\gamma$ for all $z \in \mathbb{R}$. If ϕ was a periodic solution of (6) of the form displayed in (5), then after an integration over the period interval $(-l, l)$, it is discerned that

$$0 = \int_{-l}^{l} \big(f(\phi(x)) - c\phi(x) + A\big)\,dx > 2l\gamma$$

or

$$0 = \int_{-l}^{l} \big(f(\phi(x)) - c\phi(x) + A\big)\,dx < -2l\gamma$$

since $\alpha(0) = 0$. Values of A for which $f(z) - ca + A$ has no real zeroes are henceforth excluded from the discussion.

Suppose that z_0 is a real zero of $f(z)-cz+A$. Then $f(z)-cz+A$ can be written in the form $a_1(z-z_0)+\cdots+a_p(z-z_0)^p$ with $a_p \neq 0$. Make the change of variables $v=\phi-z_0$ so that (6) becomes

$$Lv = a_1 v + \tilde{f}(v) \quad \text{or} \quad (-a_1+L)v = \tilde{f}(v)$$

where $\tilde{f}(v) = a_2 v^2 + \cdots + a_p v^p$. Thus, without loss of generality, we may take it that $z_0 = 0 = A$ and so ϕ satisfies

$$(c+L)\phi = f(\phi) \quad \text{or} \quad L\phi = f(\phi) - c\phi. \tag{7}$$

H. Chen [21] showed the existence of cnoidal-wave solutions of (7) in the form (5) for a range of dispersion relations α and nonlinearities f. Her theory included certain Korteweg-de Vries and Benjamin-Ono type equations where $f(u)=u^p$ for $p \geq 2$ and $\alpha(\xi)=\xi^2$ or $\alpha(\xi)=|\xi|$. In that paper, the symbol α is required to be a real, even, nonnegative and continuous function on $\mathbb{R}$ and monotone increasing on $\mathbb{R}^+$ and the polynomial f is presumed to have positive coefficients. The theory would not apply, for instance, to the Benjamin equation where $\alpha(\xi)=\beta\xi^2-\gamma|\xi|$ for positive real numbers β and γ. The present analysis makes allowance for a broader range of dispersion relations α than considered heretofore. We also investigate the large wavelength limit $l \to \infty$ of these periodic traveling-wave solutions, determining in some cases that they converge to solitary-wave solutions of (1). In particular, this result provides an independent proof of the existence of solitary-wave solutions of (1).

As mentioned already, the cnoidal-wave solutions of the Korteweg–de Vries equation are known to be stable to perturbations of the same period (see Angulo *et al.* [6] and the references contained in this work). The question of stability of the traveling-wave analogues of the KdV cnoidal-wave solutions of the more general equation (1) is also natrually of interest, the more so since many specializations of (1) arise as models of physical phenomena.

Two related notions of stability will enter into the analysis developed here. In the following two definitions, $(X, \|\cdot\|_X)$ is a Banach space of real-valued, periodic functions with period $2l$.

Definition 1.1. (Stability type–I) A non-constant, periodic traveling-wave solution ϕ of (1) in the form (5) is said to be stable in $(X, \|\cdot\|_X)$ if for any $\epsilon > 0$, there is a $\delta > 0$ such that the relation

$$\inf_{\tau \in [-l,l]} \|u_0(\cdot) - \phi(\cdot+\tau)\|_X < \delta$$

implies that

$$\inf_{\tau \in [-l,l]} \|u(\cdot,t) - \phi(\cdot+\tau)\|_X < \epsilon$$

for all $t > 0$, where $u = u(x,t)$ is the solution of (1) with initial value u_0.

Definition 1.2. (Stability type–II) A set S of traveling-wave solutions of (1) is said to be stable in X if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $u_0 \in X$ with

$$\inf_{\phi \in S} \inf_{\tau \in [-l,l]} \|u_0(\cdot) - \phi(\cdot+\tau)\|_X < \delta,$$

the solution $u = u(x,t)$ of (1) with initial data u_0 remains close to S in the sense that

$$\inf_{\phi \in S} \inf_{\tau \in [-l,l]} \|u(\cdot,t) - \phi(\cdot+\tau)\|_X < \epsilon$$

for all $t > 0$.

Remark 1. Type–I stabilty is the original concept put forward by Benjamin [8] in his pioneering work on stabilty of solitary-wave solutions of the KdV and BBM equations. Benjamin called this stability of the shape of the profile. In fact, type–I stablity is just orbital stabilty of the traveling wave in question. Type–II stabilty coincides with type-I stabilty if the set S consists of only spatial translations of a given traveling wave.

Logically prior to the study of stability is theory for the well-posedness of the initial-value problems under consideration. The techniques for determining well-posedness are quite different from those that come to the fore in existence and stability analysis of traveling waves. In the present paper, we will assume the relevant initial-value problem is well posed in relatively smooth, periodic function spaces. The intention here is not to agonize over how large these spaces can be. However, it will certainly be presumed that the space where well-posedness obtains has 'finite energy'. By this, it is simply meant that the solution u

$$u(x,t)=\sum_{n=-\infty}^{\infty}u_n(t)e^{i\frac{n\pi}{l}x},$$

decomposed into its Fourier series, has the property that

$$\sum_{n=-\infty}^{\infty}\left|\alpha\Big(\frac{n\pi}{l}\Big)\right||u_n(t)|^2<\infty$$

for all $t\geq 0$. We hasten to add that it is not necessarily true that a suitable well-posedness theory is valid in the energy space by itself (see, for example, the work of Molinet, Saut and Tzvetkov [26] on Benjamin-Ono-type equations). Equally, it is not always the case that we know how to establish stability in the smaller spaces where well-posedness is easily ascertained (but see [16] for stabilty results in Sobolev classes of higher order than the energy space).

The following three hypotheses about the dispersion α and nonlinearity f are assumed to hold throughout.

(H1) The nonlinearity f is a polynomial of degree $p-1$, with $p\geq 3$, having non-negative coefficients for which $f(0)=f'(0)=0$. We write f in the form $f(z)=3\gamma_3z^2+\cdots p\gamma_pz^{p-1}$ where $\gamma_j\geq 0$ for $j=3,\cdots,p-1$ and $\gamma_p>0$.

Remark 2. In fact, the primitive

$$F(z)=\int_0^z f(x)\,dx=\gamma_3z^3+\cdots+\gamma_pz^p \tag{8}$$

of f will figure prominently in our analysis.

(H2) The symbol α associated with the operator L via (2) is a real, even, continuous function defined on $\mathbb{R}$ with $\alpha(0)=0$. It is presumed to satisfy a growth condition, namely if $p_0=\min\{j:\gamma_j>0\}$ is the lowest-order term appearing in F, then there is an $\tilde{s}\geq\frac{p_0-2}{4}$, such that

$$\lim_{\xi\to 0}|\xi|^{-2\tilde{s}}\alpha(\xi)=\lim_{\xi\to 0}\frac{\alpha(\xi)}{|\xi|^{2\tilde{s}}}=0.$$

(H3) There is an $s>\frac{p-2}{4}$ such that

$$0<\liminf_{\xi\to\infty}\frac{\alpha(\xi)}{|\xi|^{2s}}\leq\limsup_{\xi\to\infty}\frac{\alpha(\xi)}{|\xi|^{2s}}<\infty.$$

A fourth restriction of the dispersion relation α will only be needed in part of Section 5.

(H4) The number s in (H3) is at least $\frac{1}{2}$. Furthermore, the symbol $\alpha(\xi)$ is smooth on $(0,\infty)$ and there is a number $\theta > 0$ such that for any $\xi > 0$

$$|\alpha'(\xi)| \le \theta\,(1+|\xi|)^{2s-1}. \tag{9}$$

Here is the layout of the paper. Notation and preliminaries are provided in Section 2. In Section 3, for a given $\lambda > 0$, the variational problem

$$\Gamma(\lambda) = \inf\Big\{M_l(u) : \, u(x) = \sum_{n=-\infty}^{\infty} u_n e^{i\frac{n\pi}{l}x}, u_n = \overline{u}_{-n} \in \mathbb{C}, \tag{10}$$
$$\sum_{n=-\infty}^{\infty} \alpha\Big(\frac{n\pi}{l}\Big)|u_n|^2 < \infty,\, E_l(u) = \lambda\Big\}$$

is investigated. The functionals E_l and M_l are given by

$$E_l(u) = \frac{1}{2}\int_{-l}^{l} u^2(x)\,dx = l\sum_{n=-\infty}^{\infty}|u_n|^2,$$
$$M_l(u) = \int_{-l}^{l}\Big(\frac{1}{2}uLu - F(u)\Big)\,dx \tag{11}$$
$$= 2l\sum_{n=-\infty}^{\infty}\frac{1}{2}\alpha\Big(\frac{n\pi}{l}\Big)|u_n|^2 - 2l\sum_{j=3}^{p}\gamma_j \sum_{n_1+\cdots+n_j=0} u_{n_1}\cdots u_{n_j}.$$

where F is the primitive of f as in (8).

Remark 3. It will transire that E_l and M_l are independent of the temporal variable t if $u = u(x,t) = u(x+2l,t)$ solves (1).

The main result in Section 3 is the following, stated somewhat informally for the moment.

Theorem 1.3. *Let $\lambda > 0$ be given. Then for $l > 0$ sufficiently large, (10) has at least one nontrivial minimizer. For each minimizer ϕ, there is a $c > 0$ such that $\phi(x-ct) = \sum_n \phi_n e^{i\frac{n\pi}{l}(x-ct)}$ is an infinitely smooth, traveling-wave solution of (1). The set of all such solutions, denoted by S_λ, is stable (type-II) in the Sobolev class H^s_l (to be defined in Section 2).*

Section 4 provides a brief review of the variational problem

$$\Gamma_\infty(\lambda) = \inf\Big\{M_\infty(\eta) : \, \eta \in H^s(\mathbb{R}),\, E_\infty(\eta) = \lambda\Big\} \tag{12}$$

which has been studied in connection with the existence and stablity theory for solitary-wave solutions of (1). The functionals E_∞ and M_∞ are analogous to E_l and M_l, *viz.*

$$E_\infty(\eta) = \frac{1}{2}\int_{-\infty}^{\infty}\eta^2(x)\,dx \quad \text{and} \quad M_\infty(\eta) = \int_{-\infty}^{\infty}\Big(\frac{1}{2}\eta L\eta - F(\eta)\Big)\,dx. \tag{13}$$

It is straightforward to verify that if u is a suitably smooth solution of (1), then $E_\infty(u)$ and $M_\infty(u)$ are independent of t. Any minimizer ϕ of the variational problem (12) is the shape function of a solitary-wave solution of (1). That is, $u(x,t) = \phi(x-c\,t)$ is a solution of (1) for some speed $c > 0$. As mentioned, results of this nature begin with Benjamin [8]. His work has been continued in a large number of subsequent publications; see for example [2], [3], [11], [12], [17], [18], [29] for some of the earlier extensions of Benjamin's theory.

The variational problem Γ_∞ comes to the fore in Section 5. Using the further regularity of the dispersion symbol α embodied in (H4) and assuming as before that (H1), (H2) and (H3) hold, the following result is established. Again, the statement here is informal.

Theorem 1.4. *For fixed $\lambda > 0$, we have*

$$\lim_{l\to\infty} \Gamma_l(\lambda) = \Gamma_\infty(\lambda). \tag{14}$$

For fixed s, there is a one-parameter family of bounded linear operators $T_l : H_l^s \to H^s(\mathbb{R})$ with the following property. Let $\{l_n\}_{n=1}^\infty$ be an increasing and unbounded sequence of positive real numbers with l_1 sufficiently large and for each l_n, let ϕ_n be one of the minimizers of (10) with $l = l_n$. Then, $\{T_{l_n}\phi_n\}_{n=1}^\infty$ forms a minimizing sequence for the variational problem (12).

Remark 4. The operators T_l are defined in (67)–(68).

It is also shown in Section 5 that in some circumstances, the minimizing sequence $\{T_{l_n}\phi_n\}_{n=1}^\infty$ whose existence is asserted in Theorem 1.4 converges to a solitary-wave solution. The paper closes with an Appendix where certain technical points arising in the main development are settled.

2. **Notation and preliminaries.**

Throughout the paper, a bold-faced letter $\mathbf{u}$ stands for a sequence $\{u_n\}_{n=-\infty}^\infty$. The summation $\sum_{n=-\infty}^\infty$ is abbreviated $\sum_n$, or simply just $\sum$ if the range of summation is clear from the context. Similar liberties are taken with $\sup_{-\infty<n<\infty}$, which is written simply $\sup_n$.

For $1 \le r < \infty$, the sequence space

$$\ell_r = \Big\{\mathbf{u} = \{u_n\} : u_n = \overline{u}_{-n} \in \mathbb{C},\ \sum |u_n|^r < \infty\Big\}$$

is a Banach space equipped with the norm

$$|\mathbf{u}|_r = \Big(\sum |u_n|^r\Big)^{\frac{1}{r}}.$$

For $r = \infty$, the Banach space

$$\ell_\infty = \big\{\mathbf{u} = \{u_n\} : u_n = \overline{u}_{-n} \in \mathbb{C}, \sup_n |u_n| < \infty\big\}$$

is equipped with its usual norm

$$|\mathbf{u}|_\infty = \sup_n |u_n|.$$

Of course, if $1 \le r_1 < r_2 \le \infty$,

$$\ell_{r_1} \subset \ell_{r_2}$$

and for any $\mathbf{u} = \{u_n\} \in \ell_{r_1}$,

$$|\mathbf{u}|_{r_2} \le |\mathbf{u}|_{r_1}.$$

For $\sigma \ge 0$, let

$$\ell_2^\sigma = \Big\{\mathbf{u} \in \ell_2 : \sum (1+|n|)^{2\sigma}|u_n|^2 < \infty\Big\}.$$

The norm on this Hilbert space is written $\|\cdot\|_\sigma$ and is defined by

$$\|\mathbf{u}\|_\sigma = \Big(\sum \big(1+|n|\big)^{2\sigma}|u_n|^2\Big)^{\frac{1}{2}}.$$

When $\sigma = 0$, $\ell_2^\sigma = \ell_2$, $\|\mathbf{u}\|_0 = |\mathbf{u}|_2$, and this norm is written unadorned as $\|\mathbf{u}\|$. For $l > 0$, it will be useful to have an equivalent norm on ℓ_2^σ, namely

$$\|\mathbf{u}\|_{\sigma,l} = \Big(\sum \Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{2\sigma} \big|u_n\big|^2\Big)^{\frac{1}{2}}. \tag{15}$$

Clearly, $\|\mathbf{u}\|_{\sigma,\pi} = \|\mathbf{u}\|_\sigma$ and $\|\mathbf{u}\|_{0,l} = \|\mathbf{u}\|$.

For $\sigma \geq 0$, the function space

$$H_l^\sigma = \Big\{u(x) = \sum u_n e^{i\frac{n\pi}{l}x} : \mathbf{u} = \{u_n\} \in \ell_2^\sigma\Big\}$$

with norm

$$\|u\|_{\sigma,l} = \Big(2l \sum \Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{2\sigma} |u_n|^2\Big)^{\frac{1}{2}} = \sqrt{2l}\|\mathbf{u}\|_{\sigma,l} \tag{16}$$

is a Hilbert space of real-valued, periodic functions of period $2l$. We can identify H_l^σ with the sequence space ℓ_2^σ and will do so when it is convenient.

For $1 \leq q \leq \infty$, $L_q = L_q(\mathbb{R})$ is the standard Lebesgue space with its usual norm denoted by $|u|_q$. For any $\sigma \geq 0$,

$$H^\sigma = H^\sigma(\mathbb{R}) = \Big\{u \in L_2 : \int_{-\infty}^{\infty} (1 + |2\pi\xi|)^{2\sigma} |\widehat{u}(\xi)|^2 \, d\xi < \infty\Big\}$$

is the usual, L_2–based Sobolev space with norm

$$\|u\|_\sigma = \Big(\int_{-\infty}^{\infty} (1 + |2\pi\xi|)^{2\sigma} |\widehat{u}(\xi)|^2 \, d\xi\Big)^{\frac{1}{2}}.$$

Here, $\widehat{u}$ is the Fourier transform of the function u, *viz.*

$$\widehat{u}(\xi) = \int_{-\infty}^{\infty} u(x) e^{-2\pi i x \xi} \, dx.$$

If $\sigma = 0$, $H^0 = L_2$, so $\|u\|_0 = |u|_2$, both of which are usually written unadorned as $\|u\|$.

If u and v are two Lebesgue measurable functions defined on $\mathbb{R}$, the notation $u*v$ stands for the convolution of u and v, that is

$$(u * v)(x) = \int_{-\infty}^{\infty} u(x - y) v(y) \, dy.$$

In the case $u = v$, $u * u$ is denoted by $*_2 u$ and inductively $*_j u = (*_{j-1} u) * u$ for $j \geq 3$.

The same symbol $*$ is used to represent discrete convolution. If $\mathbf{u} = \{u_n\}$ and $\mathbf{v} = \{v_n\}$ are two sequences, then $\mathbf{w} = \{w_n\} = \mathbf{u} * \mathbf{v}$ is defined by

$$w_n = \sum_{k=-\infty}^{\infty} u_{n-k} v_k$$

for $n \in \mathbb{Z}$. It is convenient to write $\mathbf{u} * \mathbf{u}$ as $*_2 \mathbf{u}$ and for $j \geq 3$, $*_j \mathbf{u} = (*_{j-1} \mathbf{u}) * \mathbf{u}$. The n^{th} element $(*_j \mathbf{u})_n$ of $*_j \mathbf{u}$ is

$$(*_j \mathbf{u})_n = \sum_{k_1 + \cdots + k_j = n} u_{k_1} \cdots u_{k_j}.$$

By Young's inequality for convolutions, if $\mathbf{u}_1 \in \ell_{r_1}, \cdots, \mathbf{u}_N \in \ell_{r_N}$ where $1 \leq r_1 \leq \cdots \leq r_N \leq \infty$ and $\frac{1}{r_1} + \cdots + \frac{1}{r_N} = N - 1$, then

$$\mathbf{w} = \{w_n\} = \mathbf{u}_1 * \cdots * \mathbf{u}_N \in \ell_\infty$$

and

$$|\mathbf{w}|_\infty \le |\mathbf{u_1}|_{r_1} \cdots |\mathbf{u_N}|_{r_N}.$$

We will frequently identify a periodic function $u(x) = \sum u_n e^{i\frac{n\pi}{l}x}$ with its Fourier coefficients $\mathbf{u} = \{u_n\}$.

Remark 5. If $\mathbf{u} \in \ell_{\frac{j}{j-1}}$, then $\int_{-1}^{l} u^j(x)\,dx = 2l(*_j\mathbf{u})_0 = 2l\sum_{k_1+\cdots+k_j=0} u_{k_1}\cdots u_{k_j}$ and the integral is bounded by $2l|\mathbf{u}|^j_{\frac{j}{j-1}}$.

The following proposition is standard and will find frequent use.

Proposition 1. *Let $r \in [1,2]$. Then for any $\sigma > \frac{1}{r} - \frac{1}{2}$,*

$$\ell_2^\sigma \subset \ell_r. \tag{17}$$

In more detail,

(a) if $\mathbf{u} \in \ell_2^\sigma$, then $\mathbf{u} \in \ell_r$ and

$$|\mathbf{u}|_r \le \beta\,\|\mathbf{u}\|_{\sigma,l} \tag{18}$$

where $\beta = \beta(l, r, \sigma) > 0$ is given explicitly by

$$\begin{cases} \beta = 1 & \text{if} \quad r = 2, \\ \beta^{2r/(2-r)} = \sum \left(1 + \left|\frac{n\pi}{l}\right|\right)^{-2\sigma r/(2-r)} & \text{if} \quad 1 \le r < 2. \end{cases}$$

(b) For any $r_1 \in (\frac{2-r}{2\sigma}, r]$, the bound on the right-hand side of (18) can be improved as follows;

$$|\mathbf{u}|_r \le \beta\,\|\mathbf{u}\|^{1-\frac{r_1}{r}}\,\|\mathbf{u}\|^{\frac{r_1}{r}}_{\sigma,l} \tag{19}$$

where $\beta = \beta(l, r, r_1, \sigma) > 0$ is given by

$$\begin{cases} \beta = 1 & \text{if} \quad r = 2 \\ \beta^{2r/(2-r)} = \sum \left(1 + \left|\frac{n\pi}{l}\right|\right)^{-2\sigma r_1/(2-r)} & \text{if} \quad 1 \le r < 2. \end{cases}$$

(c) The imbedding (17) is compact. Namely, if $\{\mathbf{u}^{(k)}\}_k$ is a bounded sequence in ℓ_2^σ, then there is a subsequence $\{\mathbf{u}^{(k_j)}\}_j$ and there is $\mathbf{u} = (u_n) \in \ell_2^\sigma$ such that

$$\lim_{j\to\infty} u_n^{(k_j)} = u_n$$

for all $n \in \mathbb{Z}$ and

$$\lim_{j\to\infty} |\mathbf{u}^{(k_j)} - \mathbf{u}|_r = 0.$$

The continuous versions of (a) and (b) of Proposition 1 will also be useful.

Proposition 2. *If $\sigma > 0$ and $2 \le q \le \infty$ are restricted by the relation $\sigma > \frac{1}{2} - \frac{1}{q}$, then*

$$H^\sigma \hookrightarrow L_q \tag{20}$$

and this embedding is a bounded linear operator. More precisely,

(a) if $u \in H^\sigma$, then $u \in L_q$ and

$$|u|_q \le |\widehat{u}|_{\frac{q}{q-1}} \le \beta\,\|u\|_\sigma \tag{21}$$

where $\beta = \beta(q, \sigma) > 0$ is given by

$$\begin{cases} \beta = 1 & \text{if} \quad r = 2, \\ \beta^{2q/(q-2)} = \displaystyle\int_{-\infty}^{\infty} \big(1 + |2\pi\xi|\big)^{-2\sigma q/(q-2)} \, d\xi & \text{if } 1 \le r < 2. \end{cases}$$

(b) Let $r_1 \in (\frac{q-2}{2\sigma(q-1)}, \frac{q}{q-1}]$. Then, the bound on the right-hand side of (21) can be improved as follows;

$$|u|_q \le \beta \, \|u\|^{1-\frac{r_1(q-1)}{q}} \, \|u\|_\sigma^{\frac{r_1(q-1)}{q}}, \tag{22}$$

where $\beta = \beta(q, r_1, \sigma) > 0$ is given by

$$\begin{cases} \beta = 1 & \text{if} \quad r = 2, \\ \beta^{2q/(q-2)} = \displaystyle\int_{-\infty}^{\infty} \big(1 + |2\pi\xi|\big)^{-2\sigma r_1(q-1)/(q-2)} \, d\xi & \text{if} \quad 1 \le r < 2. \end{cases}$$

3. **The periodic variational problem.** Hypotheses (H1)-(H3) specified in Section 1 are assumed to be valid. Throughout, the parameter $\lambda > 0$ is fixed, but arbitrary.

Let u be a periodic function of period $2l$ with Fourier coefficients $\mathbf{u} = \{u_n\}$. In terms of the Fourier coefficients $\mathbf{u}$, the functionals $E(u) = E_l(u)$ and $M(u) = M_l(u)$ take the form

$$\begin{aligned} E(u) &= E(\mathbf{u}) = l \sum |u_n|^2 = l\|\mathbf{u}\|^2, \\ M(u) &= M(\mathbf{u}) = l \sum \alpha\Big(\frac{n\pi}{l}\Big)|u_n|^2 - 2l \sum_{j=3}^{p} \gamma_j (*_j \mathbf{u})_0 \end{aligned} \tag{23}$$

where, as noted in Section 2, $(*_j\mathbf{u})_0 = \sum_{k_1+\cdots k_j=0} u_{k_1} \cdots u_{k_j}$. Hypothesis (H3) asserts that $\alpha(\xi)$ is of order $|\xi|^{2s}$ as $|\xi| \to +\infty$. Hence, the variational problem (10) can be written as

$$\Gamma(\lambda) = \inf \Big\{ M(\mathbf{u}) : \, \mathbf{u} = \{u_n\} \in \ell_2^s, \, E(\mathbf{u}) = \lambda \Big\}. \tag{24}$$

Since $\gamma_j \ge 0$ for $3 \le j \le p$ and $(*_j\mathbf{u})_0 = \sum_{k_1+\cdots k_j=0} u_{k_1} \cdots u_{k_j} \le \sum_{k_1+\cdots k_j=0} |u_{k_1}| \cdots |u_{k_j}|$, solutions of the minimization problem (10) can be assumed to have non-negative Fourier coefficients $\{u_n\}$. Thus, (24) has the same solutions as does the more restricted variational problem

$$\Gamma(\lambda) = \inf \Big\{ M(\mathbf{u}) : \, \mathbf{u} = \{u_n\} \in \ell_2^s, \, u_n = u_{-n} \ge 0, \, \|\mathbf{u}\|^2 = \frac{\lambda}{l} \Big\}. \tag{25}$$

Proposition 3. *The mapping $M : \ell_2^s \mapsto \mathbb{R}$ is continuous.*

This follows directly from Proposition 2.1 since $s > \frac{p-2}{4}$ implies that the imbeddings $\ell_2^s \hookrightarrow \ell_{\frac{j}{j-1}}$ are continuous for $j \in [3, p]$.

Notice that the constraint set in (25) includes a unique trivial point $\mathbf{u} = \{u_n\}$ with $u_0 = \sqrt{\frac{\lambda}{l}}$ and $u_n = 0$ for $n \neq 0$. This corresponds to the non-zero constant function $u(x) \equiv \sqrt{\frac{\lambda}{l}}$. The following lemma rules out the possibility that this trivial point is a minimizer of (25), at least for larger values of the half-period l.

Lemma 3.1. *If $l > 0$ is large enough that*

$$l^{\frac{p_0-2}{2}}\alpha\Big(\frac{\pi}{l}\Big) < 4(\sqrt{2}-1)\gamma_{p_0}\lambda^{\frac{p_0-2}{2}} \tag{26}$$

where p_0 is as in (H2), the smallest index j such that $\gamma_j > 0$, then,

$$-\infty < \Gamma(\lambda) < -2lF\Big(\sqrt{\frac{\lambda}{l}}\Big).$$

Proof. Let $\mathbf{u} = \{u_n\}$ where $u_0 = \sqrt{\frac{\lambda}{2l}}$, $u_1 = u_{-1} = \sqrt{\frac{\lambda}{4l}}$ and $u_n = 0$ for $n \neq 0, \pm 1$. A calculation shows that $\|\mathbf{u}\|^2 = \frac{\lambda}{l}$. For any $j \geq 3$,

$$\begin{aligned}(*_j\mathbf{u})_0 &= \sum_{k_1+k_2+\cdots+k_j=0} u_{k_1}\cdots u_{k_j}\\ &= \sum_{k=0}^{\lfloor j/2\rfloor}\binom{j}{j-2k}\binom{2k}{k}\binom{k}{k}u_0^{j-2k}u_1^k u_{-1}^k\\ &= \sum_{k=0}^{\lfloor j/2\rfloor}\binom{j}{j-2k}\binom{2k}{k}u_0^{j-2k}u_1^{2k}\\ &= \sum_{k=0}^{\lfloor j/2\rfloor}\binom{j}{j-2k}\binom{2k}{k}\Big(\frac{1}{2}\Big)^{\frac{2k+j}{2}}\Big(\frac{\lambda}{l}\Big)^{\frac{j}{2}}\\ &= \sigma_j\Big(\frac{\lambda}{l}\Big)^{\frac{j}{2}} \geq \sqrt{2}\Big(\frac{\lambda}{l}\Big)^{\frac{j}{2}},\end{aligned} \tag{27}$$

where, for $x \in \mathbb{R}$ the notation $\lfloor x\rfloor$ represents the largest integer which is less than or equal to x. The inequality $\sigma_j = \sum_{k=0}^{\lfloor j/2\rfloor}\binom{j}{j-2k}\binom{2k}{k}\big(\frac{1}{2}\big)^{\frac{2k+j}{2}} \geq \sqrt{2}$ is established in Lemma 6.1 in the Appendix. In consequence of this latter result and (27),

$$\sum_{j=3}^{p}\gamma_j(*_j\mathbf{u})_0 \geq \sum_{j=3}^{p}\sigma_j\gamma_j\Big(\frac{\lambda}{l}\Big)^{\frac{j}{2}} \geq \sqrt{2}F\Big(\sqrt{\frac{\lambda}{l}}\Big).$$

It thus transpires that

$$\Gamma(\lambda) \leq l\sum\alpha\Big(\frac{n\pi}{l}\Big)u_n^2 - 2l\sum_{j=3}^{p}\gamma_j(*_j\mathbf{u})_0 \leq \alpha\Big(\frac{\pi}{l}\Big)\frac{\lambda}{2} - 2l\sqrt{2}F\Big(\sqrt{\frac{\lambda}{l}}\Big), \tag{28}$$

whence

$$\begin{aligned}\Gamma(\lambda) + 2lF\Big(\sqrt{\frac{\lambda}{l}}\Big) &\leq \alpha\Big(\frac{\pi}{l}\Big)\frac{\lambda}{2} - 2l(\sqrt{2}-1)F\Big(\sqrt{\frac{\lambda}{l}}\Big)\\ &\leq \alpha\Big(\frac{\pi}{l}\Big)\frac{\lambda}{2} - 2l(\sqrt{2}-1)\gamma_{p_0}\Big(\frac{\lambda}{l}\Big)^{\frac{p_0}{2}}.\end{aligned} \tag{29}$$

By (26), the right-hand side of (29) is strictly less than zero, so

$$\Gamma(\lambda) < -2lF\Big(\sqrt{\frac{\lambda}{l}}\Big).$$

Attention is now turned to showing that $\Gamma(\lambda) > -\infty$. Let $\mathbf{u}$ lie in the constraint set of (25). Because $s > \frac{p-2}{4} \geq \frac{j-2}{4}$ for all $j \in [3, p]$, Proposition 1 implies that

$$(*_j\mathbf{u})_0 = |(*_j\mathbf{u})_0| \leq \Big\{\sum|u_n|^{\frac{j}{j-1}}\Big\}^{j-1} = |\mathbf{u}|^j_{\frac{j}{j-1}} \leq \beta_j\|\mathbf{u}\|^{j-(j-1)r_j}\|\mathbf{u}\|_{s,l}^{(j-1)r_j} \tag{30}$$

for any r_j lying in the interval $\left(\frac{j-2}{2s(j-1)}, \frac{2}{j-1}\right) \subset \left(\frac{j-2}{2s(j-1)}, \frac{j}{j-1}\right]$, where

$$\beta_j = \left\{ \sum \left(1 + \left|\frac{n\pi}{l}\right|\right)^{-\frac{2sr_j(j-1)}{j-2}} \right\}^{\frac{j-2}{2}}. \tag{31}$$

Choose r_j so that $r_j(j-1) < 2$ for all j with $3 \le j \le p$. Hypotheses (H2) and (H3) imply that there are two positive constants C_1 and C_2 such that

$$C_1 + \alpha(\xi) \ge 2C_2(1 + |\xi|)^{2s} \tag{32}$$

for every $\xi \in \mathbb{R}$. In consequence, it must be the case that

$$\begin{aligned} M(\mathbf{u}) &\ge 2lC_2 \|\mathbf{u}\|_{s,l}^2 - C_1 l \|\mathbf{u}\|^2 - 2l \sum_{j=3}^{p} \gamma_j \beta_j \|\mathbf{u}\|^{j-(j-1)r_j} \|\mathbf{u}\|_{s,l}^{(j-1)r_j} \\ &= 2lC_2 \|\mathbf{u}\|_{s,l}^2 - C_1\lambda - 2l \sum_{j=3}^{p} \gamma_j \beta_j \left(\frac{\lambda}{l}\right)^{\frac{j}{2} - \frac{(j-1)r_j}{2}} \|\mathbf{u}\|_{s,l}^{(j-1)r_j}. \end{aligned} \tag{33}$$

Consider the polynomial

$$P(z) = 2lC_2 z^2 - C_1\lambda - 2l \sum_{j=3}^{p} \gamma_j \beta_j \left(\frac{\lambda}{l}\right)^{\frac{j}{2} - \frac{(j-1)r_j}{2}} z^{(j-1)r_j}$$

for non-negative values of z. Since $r_j(j-1) < 2$ for all j, $\lim_{z\to+\infty} P(z) = +\infty$ and it follows that $\min_{z\ge0} P(z)$ exists and is finite. It is then readily deduced that

$$M(\mathbf{u}) \ge P(\|\mathbf{u}\|_{s,l}) \ge \min\{P(z) : z \ge 0\} > -\infty.$$

In consequence,

$$\Gamma(\lambda) = \inf \left\{ M(\mathbf{u}) :\, \mathbf{u} \in \ell_2^s,\, \|\mathbf{u}\|^2 = \frac{\lambda}{l} \right\} > -\infty$$

and the proof is complete. □

Lemma 3.2. *Every minimizing sequence $\{\mathbf{u}^{(k)}\}_k$ of (25) is bounded in ℓ_2^s under the norm $\|\cdot\|_{s,l}$. Moreover, the bound is independent of the value of $l > 1$.*

Proof. Thanks to (33), one has

$$M(\mathbf{u}^{(k)}) \ge 2lC_2\|\mathbf{u}^{(k)}\|_{s,l}^2 - C_1\lambda - 2l \sum_{j=3}^{p} \gamma_j \beta_j \left(\frac{\lambda}{l}\right)^{\frac{j}{2} - \frac{(j-1)r_j}{2}} \|\mathbf{u}^{(k)}\|_{s,l}^{(j-1)r_j} \tag{34}$$

for every k, where C_1, C_2 and the $r_j's$ are positive, l-indepenent constants with $0 < r_j < \frac{2}{j-1}$ and the corresponding values of the β_j are given in (31). Since $\Gamma(\lambda) < 0$, $M(\mathbf{u}^{(k)}) < 0$ for k sufficiently large. Denote by z_k the quantity

$$z_k = \sqrt{2l}\|\mathbf{u}^{(k)}\|_{s,l} = \left\{ 2l \sum \left(1 + \left|\frac{n\pi}{l}\right|\right)^{2s} |u_n^{(k)}|^2 \right\}^{\frac{1}{2}}.$$

Then inequality (34) combined with the fact that $M(\mathbf{u}^{(k)}) < 0$ for large enough values of k leads to the conclusion

$$C_2 z_k^2 < C_1\lambda + \sum_{j=3}^{p} \gamma_j (2\lambda)^{\frac{j}{2} - \frac{(j-1)r_j}{2}} (2l)^{-\frac{j-2}{2}} \beta_j\, z_k^{(j-1)r_j} = C_1\lambda + \sum_{j=3}^{p} C_j\, z_k^{(j-1)r_j}$$

where $C_j = \gamma_j(2\lambda)^{\frac{j}{2}-\frac{(j-1)r_j}{2}}(2l)^{-\frac{j-2}{2}}\beta_j$ for $j = 3,\cdots,p$. Notice that β_j defined in (31) has the property that

$$\lim_{l\to\infty}(2l)^{-\frac{j-2}{2}}\beta_j = \lim_{l\to\infty}\left\{\frac{1}{2l}\sum\left(1+\left|\frac{n\pi}{l}\right|\right)^{-\frac{2sr_j(j-1)}{j-2}}\right\}^{\frac{j-2}{2}}$$
$$= \left\{\int_{-\infty}^{\infty}(1+|2\pi\xi|)^{-\frac{2sr_j(j-1)}{j-2}}\,d\xi\right\}^{\frac{j-2}{2}}.$$

In particular, the $C_j's, j = 3,\cdots,p$, are bounded, independently of large values of $l \geq 1$. Hence, the values of z_k must be bounded, independently of large values of l.

The lemma is established. □

Lemma 3.3. *For every minimizing sequence $\{\mathbf{u}^{(k)}\}_k$ of (25), there is a convergent subsequence $\{\mathbf{u}^{(k_q)}\}_q$ of $\{\mathbf{u}^{(k)}\}_k$. That is, there is $\mathbf{\Phi} = \{\phi_n\} \in \ell_2^s$ such that*

$$\lim_{q\to\infty}\|\mathbf{u}^{(k_q)} - \mathbf{\Phi}\|_{s,l} = 0 \tag{35}$$

and $\mathbf{\Phi}$ is a minimizer of (25).

Proof. By Lemma 3.2, the minimizing sequence $\{\mathbf{u}^{(k)}\}_k$ is bounded in ℓ_2^s. Proposition 1 implies that the imbeddings $\ell_2^s \hookrightarrow \ell_2$ and $\ell_2^s \hookrightarrow \ell_{\frac{j}{j-1}}$ are compact, for all $j \in [3,p]$, since $s > \frac{p-2}{4} > \frac{j-1}{j} - \frac{1}{2}$. Hence, there is a subsequence, denoted by $\{\mathbf{u}^{(k_q)}\}_q$, and there is a $\mathbf{\Phi} = \{\phi_n\} \in \ell_2^s$ such that $\{\mathbf{u}^{(k_q)}\}_q$ converges to $\mathbf{\Phi}$ weakly in ℓ_2^s and strongly in ℓ_2 and $\ell_{\frac{j}{j-1}}$. More precisely, for every $n = 0, \pm1, \cdots$,

$$\lim_{q\to\infty} u_n^{(k_q)} = \lim_{q\to\infty} u_{-n}^{(k_q)} = \phi_n = \phi_{-n} \geq 0,$$
$$\lim_{q\to\infty}\|\mathbf{u}^{(k_q)} - \mathbf{\Phi}\| = 0, \tag{36}$$

and

$$\lim_{q\to\infty}|\mathbf{u}^{(k_q)} - \mathbf{\Phi}|_{\frac{j}{j-1}} = 0. \tag{37}$$

The limiting behavior in (36) entails that

$$\|\mathbf{\Phi}\|^2 = \lim_{q\to\infty}\|\mathbf{u}^{(k_q)}\|^2 = \frac{\lambda}{l}. \tag{38}$$

Hence, $\mathbf{\Phi}$ lies in the constraint set of (25). It remains to show that $\mathbf{\Phi}$ is a minimizer of (25). This is a consequence of the next two points.

Claim 1. *For any $j \in [3,p]$,*

$$(*_j\mathbf{\Phi})_0 = \lim_{q\to\infty}(*_j\mathbf{u}^{(k_q)})_0. \tag{39}$$

This follows from

$$\begin{aligned}
&\left|(*_j\mathbf{\Phi})_0 - (*_j\mathbf{u}^{(k_q)})_0\right| \\
=&\left|\Big((\mathbf{\Phi} - \mathbf{u}^{(k_q)}) * \big(*_{j-1}\mathbf{\Phi} + *_{j-2}\mathbf{\Phi} * \mathbf{u}^{(k_q)} + \cdots + *_{j-1}\mathbf{u}^{(k_q)}\big)\Big)_0\right| \\
\leq&|\mathbf{\Phi} - \mathbf{u}^{(k_q)}|_{\frac{j}{j-1}}\Big(|\mathbf{\Phi}|^{j-1}_{\frac{j}{j-1}} + \cdots |\mathbf{u}^{(k_q)}|^{j-1}_{\ell_{\frac{j}{j-1}}}\Big) \\
\to&0 \quad \text{as} \quad q\to\infty.
\end{aligned}$$

Claim 2.

$$\sum\alpha\Big(\frac{n\pi}{l}\Big)|\phi_n|^2 \leq \liminf_{q\to\infty}\sum\alpha\Big(\frac{n\pi}{l}\Big)|u_n^{(k_q)}|^2. \tag{40}$$

This assertion follows from the weak convergence of the subsequence to Φ, or directly as follows. From (H3), we know there is $C>0$ such that $C+\alpha\left(\frac{n\pi}{l}\right)>0$ for every n. Fatou's lemma thus implies that

$$\sum\left(C+\alpha\Big(\frac{n\pi}{l}\Big)\right)|\phi_n|^2 \le \liminf_{q\to\infty}\sum\left(C+\alpha\Big(\frac{n\pi}{l}\Big)\right)|u_n^{(k_q)}|^2.$$

This inequality combined with (36) establishes (40).

With these two points in hand, observe that

$$\begin{aligned} M(\mathbf{\Phi}) =& l\sum\alpha\Big(\frac{n\pi}{l}\Big)|\phi_n|^2 - 2l\sum_{j=3}^{p}\gamma_j(*_j\mathbf{\Phi})_0 \\ \le& \liminf_{q\to\infty}\Big(l\sum\alpha\Big(\frac{n\pi}{l}\Big)|u_n^{(k_q)}|^2\Big) - 2l\sum_{j=3}^{p}\gamma_j\lim_{q\to\infty}(*_j\mathbf{u}^{(k_q)})_0 \\ =& \liminf_{q\to\infty}\Big(l\sum\alpha\Big(\frac{n\pi}{l}\Big)|u_n^{(k_q)}|^2 - 2l\sum_{j=3}^{p}\gamma_j(*_j\mathbf{u}^{(k_q)})_0\Big) \\ =& \liminf_{q\to\infty} M(\mathbf{u}^{(k_q)}) \\ =& \Gamma(\lambda). \end{aligned} \tag{41}$$

Thus, $\mathbf{\Phi}$ is a minimizer of (10) and $M(\mathbf{\Phi})=\Gamma(\lambda)$. Consequently, it must be the case that

$$\sum\alpha\Big(\frac{n\pi}{l}\Big)|\phi_n|^2 = \lim_{q\to\infty}\sum\alpha\Big(\frac{n\pi}{l}\Big)|u_n^{(k_q)}|^2. \tag{42}$$

Thus the subsequence $\{\mathbf{u}^{(k_q)}\}_q$ converges weakly to Φ and the norms of this sequence converge to the norm of Φ in ℓ_2^s. As ℓ_2^s is a Hilbert space, this implies strong convergence of the sequence to Φ.

The proof of the lemma is complete. □

The set of minimizers of (25) is denoted by S_λ^+. Thus,

$$S_\lambda^+ = \Big\{\mathbf{\Phi}=\{\phi_n\}\in\ell_2^s : \phi_n=\phi_{-n}\ge 0,\ E(\mathbf{\Phi})=l\|\mathbf{\Phi}\|=\lambda,\ M(\mathbf{\Phi})=\Gamma(\lambda)\Big\}. \tag{43}$$

Remark 6. If $\mathbf{\Phi}\in S_\lambda^+$, then the periodic function $\phi(x)=\sum\phi_n e^{i\frac{n\pi}{l}x}$ is an even function and it attains its maximum value at $x=0,\pm 2l,\pm 4l,\cdots$.

Theorem 3.4. *For every $\mathbf{\Phi}=\{\phi_n\}\in S_\lambda^+$, there is a positive number $c>0$ such that*

$$\phi=\phi(x-ct)=\sum\phi_n e^{i\frac{n\pi}{l}(x-ct)}$$

is a periodic traveling-wave solution of (1). Moreover, there are infinitely many Fourier coefficients ϕ_n which are strictly positive and Φ is infinitely smooth.

Proof. Since $\mathbf{\Phi}=\{\phi_n\}$ is a minimizer of (25), the Euler-Lagrange principle implies there is a constant $c\in\mathbb{R}$ such that

$$\phi(x)=\sum\phi_n e^{i\frac{n\pi}{l}x}$$

satisfies

$$\delta M(\phi)+c\delta E(\phi)=0. \tag{44}$$

Here, δ connotes the Euler derivative so that $\delta M(\phi) = L\phi - f(\phi)$ and $\delta E(\phi) = \phi$. Hence, (44) is as same as

$$L\phi - f(\phi) + c\phi = 0. \tag{45}$$

Thus, if $c > 0$, $\phi(x - ct)$ is a periodic traveling-wave solution of (1) propagating in the positive x-direction. Following Albert [2], one has

$$\begin{aligned}
\frac{d}{d\theta} M(\theta\phi)\Big|_{\theta=1} &= \Big\{ \frac{d}{d\theta} \int_{-l}^{l} \Big(\frac{\theta^2}{2} \phi L\phi - F(\theta\phi) \Big)\, dx \Big\}_{\theta=1} \\
&= \int_{-l}^{l} \Big(\phi L\phi - \sum_{j=3}^{p} j\gamma_j \phi^j \Big)\, dx \\
&= 2M(\phi) - \int_{-l}^{l} \sum_{j=3}^{p} (j-2)\gamma_j \phi^j \, dx.
\end{aligned} \tag{46}$$

Since $M(\phi) = \Gamma(\lambda) < 0$ from Lemma 3.1 and $\mathbf{\Phi} = \{\phi_n\} \in S_\lambda^+$, one deduces that

$$\int_{-l}^{l} \sum_{j=3}^{p} (j-2)\gamma_j \phi^j \, dx > 0.$$

Thus,

$$\frac{d}{d\theta} M(\theta\phi)\Big|_{\theta=1} < 0. \tag{47}$$

On the other hand, the chain rule implies that

$$\begin{aligned}
\frac{d}{d\theta} M(\theta\phi)\Big|_{\theta=1} &= \int_{-l}^{l} \delta M(\phi) \cdot \frac{d}{d\theta}(\theta\phi)\Big|_{\theta=1} dx \\
&= -c \int_{-l}^{l} \delta E(\phi) \cdot \phi \, dx = -c \int_{-l}^{l} \phi^2 \, dx = -2c\lambda.
\end{aligned} \tag{48}$$

It follows immediately from (47) and (48) that $c > 0$. Thus, the function ϕ defined via the minimizer Φ at the beginning of the proof is indeed a traveling-wave solution of (1). Furthermore, the Fourier coefficients ϕ_n of ϕ are non-negative and satisfy the relations

$$\Big(c + \alpha\Big(\frac{n\pi}{l}\Big)\Big)\phi_n = \sum_{j=2}^{p-1} (j+1)\gamma_{j+1} (*_j \mathbf{\Phi})_n \tag{49}$$

for all $n \in \mathbb{Z}$.

Attention is now turned to showing that the set $\{\phi_n : \phi_n > 0\}$ is infinite. By contradiction, suppose it is finite and let $N = \max\{|n| : \phi_n > 0\}$. Then $N \geq 1$, since the constant solution is known not to be a minimizer, and $\phi_n = \phi_{-n} = 0$ for $|n| > N$ and $\phi_N = \phi_{-N} > 0$. If $N = 1$, then, $\phi_1 = \phi_{-1} > 0$, but $\phi_{p-1} = 0$. In consequence, we have

$$0 = \Big(c + \alpha\Big(\frac{(p-1)\pi}{l}\Big)\Big)\phi_{p-1} \geq p\gamma_p \phi_1^{p-1} > 0.$$

Hence, the case $N = 1$ cannot happen. If $N > 1$, then $N^{p-1} > N$, so $\phi_{N^{p-1}} = 0$ and $\phi_N = \phi_{-N} > 0$ leads to

$$0 = \Big(c + \alpha\Big(\frac{N^{p-1}\pi}{l}\Big)\Big)\phi_{N^{p-1}} \geq p\gamma_p \phi_N^{p-1} > 0,$$

another contradiction. It follows that there must be an infinite number of positive components in $\mathbf{\Phi} = \{\phi_n\}$.

It remains to discuss the regularity of the minimizer $\mathbf{\Phi}$. By hypothesis (H3), there are two constants $\kappa_1, \kappa_2 > 0$ such that

$$c + \alpha(\xi) \geq -\kappa_1 + \kappa_2(1 + |\xi|)^{2s}.$$

This together with (49) yields

$$\kappa_2\Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{2s}\phi_n \leq \kappa_1\phi_n + \sum_{j=2}^{p-1}(j+1)\gamma_{j+1}(*_j\mathbf{\Phi})_n. \tag{50}$$

In case $s > \frac{1}{2}$, ℓ_2^s is an algebra and so $*_j\mathbf{\Phi} \in \ell_2^s$ for $j \geq 2$. Multiplying the last inequality by $(1 + |\frac{n\pi}{l}|)^{2s}\phi_n$, and summing over n, one obtains

$$\kappa_2\|\mathbf{\Phi}\|_{2s,l}^2 \leq \kappa_1\|\mathbf{\Phi}\|_{s,l}^2 + \sum_{j=2}^{p-1}(j+1)\gamma_{j+1}\| *_j \mathbf{\Phi}\|_{s,l}\|\mathbf{\Phi}\|_{s,l}.$$

Thus, $\mathbf{\Phi} \in \ell_2^{2s}$. Replacing s with $2s$ in the preceding argument allows the conclusion that $\mathbf{\Phi} \in \ell_2^{4s}$. Inductively, it is inferred that $\mathbf{\Phi} \in \ell_2^\infty$. In case $s \leq \frac{1}{2}$, the assumption (H3) on α indicates $p = 3$ and $s > \frac{1}{4}$, so (50) reduces to

$$\kappa_2\Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{2s}\phi_n \leq \kappa_1\phi_n + 3\gamma_3(\mathbf{\Phi} * \mathbf{\Phi})_n = \kappa_1\phi_n + 3\gamma_3\sum_k \phi_{n-k}\phi_k.$$

Let $\epsilon = \frac{1}{2}(s - \frac{1}{4}) > 0$, multiply the last inequality by $(1 + |\frac{n\pi}{l}|)^{2\epsilon}\phi_n$ and sum over n to obtain

$$\kappa_2\sum\Big(1+\Big|\frac{n\pi}{l}\Big|\Big)^{2s+2\epsilon}\phi_n^2 \leq \kappa_1\Big(1+\Big|\frac{n\pi}{l}\Big|\Big)^{2\epsilon}\phi_n^2 + 3\gamma_3\sum_n\Big(1+\Big|\frac{n\pi}{l}\Big|\Big)^{2\epsilon}\phi_n\sum_k\phi_{n-k}\phi_k.$$

Applying Shwarz' inequality to the last term on the right-hand side yields

$$\kappa_1\|\mathbf{\Phi}\|_{s+\epsilon,l}^2 \leq \kappa_1\|\mathbf{\Phi}\|_{\epsilon,l}^2 + 3\gamma_3\|\mathbf{\Phi}\|_{s,l}\Big\{\sum_n\Big(1+\Big|\frac{n\pi}{l}\Big|\Big)^{-2s+4\epsilon}\Big(\sum_k\phi_{n-k}\phi_k\Big)^2\Big\}^{\frac{1}{2}}. \tag{51}$$

Note that for each $n \in \mathbb{Z}$,

$$\begin{aligned}\sum_k\phi_{n-k}\phi_k &\leq \Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{-s}\sum_k\Big(1 + \tfrac{|n-k|\pi}{l}\Big)^s\Big(1 + \tfrac{|k|\pi}{l}\Big)^s\phi_{n-k}\phi_k \\ &\leq \Big(1 + \Big|\tfrac{n\pi}{l}\Big|\Big)^{-s}\|\mathbf{\Phi}\|_{s,l}^2.\end{aligned} \tag{52}$$

It is deduced from (51) that

$$\begin{aligned}\kappa_2\|\mathbf{\Phi}\|_{s+\epsilon,l}^2 &\leq \kappa_1\|\mathbf{\Phi}\|_{\epsilon,l}^2 + 3\gamma_3\|\mathbf{\Phi}\|_{s,l}^3\Big(\sum\Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{-4s+4\epsilon}\Big)^{\frac{1}{2}} \\ &= \kappa_1\|\mathbf{\Phi}\|_{\epsilon,l}^2 + 3\gamma_3\|\mathbf{\Phi}\|_{s,l}^3\Big(\sum\Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{-2s-\frac{1}{2}}\Big)^{\frac{1}{2}}.\end{aligned}$$

The series in the last parenthesis is convergent because $s > \frac{1}{4}$ and therefore $\mathbf{\Phi} \in \ell_2^{s+\epsilon}$. If $s+\epsilon > \frac{1}{2}$, we are back in the situation arising in the discussion of regularity for the case $s > \frac{1}{2}$ and it then follows that $\mathbf{\Phi} \in \ell_2^\infty$. Otherwise, iterate the last argument with s replaced by with $s+\epsilon = \frac{3}{2}s - \frac{1}{8}$. A finite number of such bootstrapping steps will lead to $\mathbf{\Phi} \in \ell_2^1$, and hence to the conclusion that $\mathbf{\Phi} \in \ell_2^\infty$. In consequence, ϕ is infinitely smooth and the proof is complete. □

Lemma 3.5. *Let $\{\mathbf{u}^{(k)}\}_k$ be a minimizing sequence for the variational problem (25). Then*

$$\lim_{k\to\infty} \inf_{\mathbf{\Phi}\in S_\lambda^+} \|\mathbf{u}^{(k)} - \mathbf{\Phi}\|_{s,l} = 0. \tag{53}$$

Proof. Suppose (53) to be false. Then, there is a subsequence $\{\mathbf{u}^{(k_q)}\}_q$ of $\{\mathbf{u}^{(k)}\}_k$ and a number $\epsilon > 0$ such that for every $q \geq 1$,

$$\inf_{\mathbf{\Phi}\in S_\lambda^+} \|\mathbf{u}^{(k_q)} - \mathbf{\Phi}\|_{s,l} > \epsilon.$$

On the other hand, from Lemma 3.3, there is a subsequence $\{\mathbf{u}^{(k_{q_j})}\}_j$ of $\{\mathbf{u}^{(k_q)}\}_q$ and there is $\mathbf{\Phi}_1 \in S_\lambda^+$ such that $\mathbf{u}^{(k_{q_j})} \to \mathbf{\Phi}_1$ in ℓ_2^s as $j \to \infty$. This yields the absurdity

$$\epsilon \leq \lim_{j\to\infty} \inf_{\mathbf{\Phi}\in S_\lambda^+} \|\mathbf{u}^{(k_{q_j})} - \mathbf{\Phi}\|_{s,l} = \inf_{\mathbf{\Phi}\in S_\lambda^+} \|\mathbf{\Phi}_1 - \mathbf{\Phi}\|_{s,l} = 0,$$

a contradiction that establishes the lemma. □

Corollary 1. *The set S_λ^+ defined in (43) is compact.*

Theorem 3.6. *(Stability) The traveling-wave set S_λ^+ of (1) defined in (43) is stable in ℓ_2^s. Precisely, for any $\epsilon > 0$, there is a $\delta > 0$ such that if an initial value*

$$u(x,0) = \tilde{u}(x) = \sum \tilde{u}_n e^{i\frac{n\pi}{l}x}$$

of (1) satisfies

$$\inf_{\mathbf{\Phi}\in S_\lambda^+} \inf_{-l\leq\tau\leq l} \|\tilde{\mathbf{u}} - \mathbf{\Phi}(\cdot+\tau)\|_{s,l} \leq \delta,$$

then the solution $u(x,t) = \sum u_n(t)\, e^{i\frac{n\pi}{l}x}$ of (1) has the property

$$\inf_{\mathbf{\Phi}\in S_\lambda^+} \inf_{-l\leq\tau\leq l} \|\mathbf{u}(\cdot,t) - \mathbf{\Phi}(\cdot+\tau)\|_{s,l} \leq \epsilon$$

for all $t > 0$, where $\mathbf{\Phi}(\cdot+\tau) = \left\{\phi_n e^{i\frac{n\pi}{l}\tau}\right\}$ comprise the Fourier coefficients of $\phi(x+\tau)$.

Proof. Suppose the result is false. Then, there is a sequence $\{\tilde{\mathbf{u}}^{(k)}\}$ in ℓ_2^s, a sequence $\{t_k\}$ in $(0,\infty)$ and a number $\epsilon > 0$ such that

$$\inf_{\mathbf{\Phi}\in S_\lambda^+} \|\tilde{\mathbf{u}}^{(k)} - \mathbf{\Phi}\|_{s,l} \leq \frac{1}{k} \tag{54}$$

and

$$\inf_{\mathbf{\Phi}\in S_\lambda^+} \inf_{-l\leq\tau\leq l} \|\mathbf{u}^{(k)}(\cdot,t_k) - \mathbf{\Phi}(\cdot+\tau)\|_{s,l} \geq \epsilon. \tag{55}$$

Remember that $\mathbf{\Phi} \in S_\lambda^+$ implies $\mathbf{\Phi} \in \ell_2^s$, $E(\mathbf{\Phi}) = l\|\mathbf{\Phi}\|_2^2 = \lambda$ and the functional M defined in (11) or (23) attains its minimum $\Gamma(\lambda)$ at $\mathbf{\Phi}$. Hence, (54) dictates that

$$\lim_{k\to\infty} \|\tilde{\mathbf{u}}^{(k)}\|^2 = \frac{\lambda}{l},$$

or, equivalently,

$$\mu_k = \frac{\sqrt{\lambda}}{\sqrt{l}\|\tilde{\mathbf{u}}^{(k)}\|} \to 1 \qquad \text{as} \qquad k \to \infty \tag{56}$$

and

$$\|\mu_k \tilde{\mathbf{u}}^{(k)}\|^2 \equiv \frac{\lambda}{l} \quad \text{for every} \quad k.$$

Formulas (54) and (56) together with continuity of the functional M in ℓ_2^s imply

$$\lim_{k\to\infty} M(\mu_k \tilde{\mathbf{u}}^{(k)}) = \Gamma(\lambda).$$

Both functionals $E(u(\cdot,t))$ and $M(u(\cdot,t))$ are independent of t since u is a solution of (1). Hence,

$$E\big(\mu_k \mathbf{u}^{(k)}(\cdot,t_k)\big) = E\big(\mu_k \tilde{\mathbf{u}}^{(k)}\big) = l\|\mu_k \tilde{\mathbf{u}}^{(k)}\| = \lambda$$

and

$$\lim_{k\to\infty} M\big(\mu_k \mathbf{u}^{(k)}(\cdot,t_k)\big) = \lim_{k\to\infty} M\big(\mu_k \tilde{\mathbf{u}}^{(k)}\big) = \Gamma(\lambda).$$

This is to say, $\{\mu_k \mathbf{u}^{(k)}(\cdot,t_k)\}_k$ is a minimizing sequence for (24). By Lemma 3.6,

$$\lim_{k\to\infty} \inf_{\mathbf{\Phi}\in S_\lambda^+} \inf_{-l\le\tau\le l} \|\mu_k \mathbf{u}^{(k)}(\cdot,t_k) - \mathbf{\Phi}(\cdot+\tau)\|_{s,l} = 0.$$

In consequence, it must be the case that

$$\begin{aligned}&\lim_{k\to\infty} \inf_{\mathbf{\Phi}\in S_\lambda^+} \inf_{-l\le\tau\le l} \|\mathbf{u}^{(k)}(\cdot,t_k) - \mathbf{\Phi}(\cdot+\tau)\|_{s,l} \\ &\quad = \lim_{k\to\infty} \inf_{\mathbf{\Phi}\in S_\lambda^+} \inf_{-l\le\tau\le l} \|(1-\mu_k)\mathbf{u}^{(k)}(\cdot,t_k) + \mu_k \mathbf{u}^{(k)}(\cdot,t_k) - \mathbf{\Phi}(\cdot+\tau)\|_{s,l} = 0.\end{aligned} \tag{57}$$

This contradicts (55). The result is established. □

Corollary 2. *The traveling-wave solution set S_λ^+ of (1) is stable in H_l^s. Precisely, for any $\epsilon > 0$, there is $\delta > 0$ such that if the initial data $\tilde{u}$ lies in H_l^s with*

$$\inf_{\phi\in S_\lambda^+} \|\tilde{u} - \phi\|_{s,l} < \delta,$$

then the solution $u(\cdot,t)$ of (1) satisfies

$$\inf_{\phi\in S_\lambda^+} \inf_{-l\le\tau\le l} \|u(\cdot,t) - \phi(\cdot+\tau)\|_{s,l} < \delta$$

for all $t > 0$.

Remark 7. If the set S_λ^+ contains a single traveling wave and its collection of spatial translates, then whenever initial wave resembles it, that resemblance will remain for all time.

Theorem 3.7. *If $\alpha(\xi)$ is non-negative and monotone increasing on $\mathbb{R}^+$, then for every $\mathbf{\Phi} = \{\phi_n\} \in S_\lambda^+$, $\phi_n = \phi_{-n}$ is strictly positive and ϕ_n is monotone decreasing for $n \ge 0$.*

Proof. Arguing by contradiction, suppose there is a $\mathbf{\Phi} = \{\phi_n\}$ in S_λ^+ such that $\{\phi_n\}_{n\ge0}$ is not decreasing. Then, there are two integers $n_2 > n_1 \ge 0$ such that $\phi_{n_2} > \phi_{n_1}$. Define $\mathbf{\Psi} = \{\psi_n\}$ as follows;

$$\psi_n = \psi_{-n} = \phi_n \quad \text{if} \quad n \ne n_1, n_2$$

and

$$\psi_{n_1} = \psi_{-n_1} = \phi_{n_2},\ \psi_{n_2} = \psi_{-n_2} = \phi_{n_1}.$$

Then $\mathbf{\Psi} \in \ell_2^s$, it has the same ℓ_2-norm $\sqrt{\frac{\lambda}{l}}$, and by Riesz's rearrangement inequality,

$$(*_j \mathbf{\Psi})_0 \ge (*_j \mathbf{\Phi})_0.$$

Since α is increasing,

$$\sum \alpha\Big(\frac{n\pi}{l}\Big)\psi_n^2 < \sum \alpha\Big(\frac{n\pi}{l}\Big)\phi_n^2.$$

The last two inequalities allow the conclusion that $\Gamma(\lambda) = M(\mathbf{\Phi}) > M(\mathbf{\Psi})$, which contradicts the definition of $\Gamma(\lambda)$. Therefore, $\{\phi_n\}$ must be decreasing as $n \geq 0$ increases. The strict positivity of ϕ_n is a consequence of the fact that the Fourier coefficients are decreasing together with Theorem 3.4. The proof is concluded. □

Corollary 3. *Consider the generalized KdV-equation*

$$u_t + u_x + u^{p-2}u_x + u_{xxx} = 0.$$

If $p \leq 5$, then for l sufficiently large, there is a non-empty set S_l of cnoidal-wave solutions of period $2l$. For each $\phi = \sum_n \phi_n e^{i\frac{n\pi}{l}x} \in S_l$, its Fourier coefficients $\{\phi_n\}$ are strictly positive, even in the sense that $\phi_n = \phi_{-n} > 0$ and strictly monotone decreasing as $n \geq 0$ increases. Moreover, S_l is stable in H^1_l in the type-II sense.

Corollary 4. *The Benjamin-Ono equation*

$$u_t + u_x + uu_x + Hu_{xx} = 0,$$

where H is the usual Hilbert transform defined via its symbol by $\widehat{Hv}(\xi) = i\,\mathrm{sgn}(\xi)\,\widehat{v}(\xi)$, has a non-empty set S_l of cnoidal-wave solutions with period $2l$ for l sufficiently large. For every $\phi = \sum_n \phi_n e^{i\frac{n\pi}{l}x} \in S_l$, $\phi_n = \phi_{-n} > 0$ and the $\{\phi_n\}$ are strictly monotone decreasing as $n \geq 0$ increases. The set S_l is stable in $H^{\frac{1}{2}}_l$.

Corollary 5. *Consider the Benjamin equation*

$$u_t + u_x + uu_x + \alpha Hu_{xx} \pm \beta u_{xxx} = 0,$$

where $\alpha > 0, \beta > 0$ are constants and H is again the Hilbert transform. For either choice of the sign in front of β, there is, for l sufficiently large, a non-empty set S_l of cnoidal-wave solutions with period $2l$. The set S_l of such cnoidal waves is stable in H^1_l.

For the KdV-equation itself, $p = 3$ in Corollary 3, cnoidal-wave solutions are explicit and can be written in terms of elliptic functions. Properly normalized, they are unique. Benjamin [9] studied their stability (see also the recent work of Angulo *et al.* [6] for further development in this direction). The Benjamin-Ono equation also has well understood periodic and solitary traveling-wave solutions (see the original paper of Benjamin [7]). These, too, are known to be unique when properly scaled (see Amick and Toland [4]). The solitary-wave solutions were shown to be stable by Bennett *et al.* [11] while the analog of the KdV cnoidal-wave solutions are examined in Angulo Pava [5]. Explicit cnoidal-type traveling-wave soutions for more general nonlinear dispersive equations like (1) are not known, so the analysis using perturbation theory and spectral analysis that is featured in the work of Benjamin [8], [9] and Angulo *et al.* [6] appears unlikely to be successful.

4. Review of the variational problem associated with solitary waves.

Solitary-wave solutions of (1) and their stability have been investigated using the variational problem (12). Since the operator L is defined by its Fourier symbol, the functionals M_∞ and E_∞ in (13) may be written in the alternative form

$$M_\infty(\eta) = \int_{-\infty}^{\infty} \frac{1}{2}\alpha(2\pi\xi)|\widehat{\eta}(\xi)|^2\,d\xi - \sum_{j=3}^{p} \gamma_j(*_j\widehat{\eta})(0)$$

and (58)

$$E_\infty(\eta) = \frac{1}{2}\|\eta\|^2 = \frac{1}{2}\int_{-\infty}^{\infty} |\widehat{\eta}(\xi)|^2\,d\xi = \frac{1}{2}(\widehat{\eta} * \widehat{\eta})(0).$$

Since $\gamma_j \geq 0$, the constraints may be further restricted as follows:

$$\Gamma_\infty(\lambda) = \inf \Big\{ M_\infty(\eta) : \ \eta \in H^s, \, \widehat{\eta}(-\xi) = \widehat{\eta}(\xi) \geq 0, \, E_\infty(\eta) = \lambda \Big\}. \tag{59}$$

Denote the set of minimizers by $S^+_{\lambda,\infty}$, that is

$$S^+_{\lambda,\infty} = \Big\{ \eta \in H^s : \, \widehat{\eta}(-\xi) = \widehat{\eta}(\xi) \geq 0, \, E(\eta) = \lambda, \, M_\infty(\eta) = \Gamma_\infty(\lambda) \Big\}. \tag{60}$$

Under various hypotheses on the nonlinearity f and the dispersion operator L, it is known that each minimizer in $S^+_{\infty,\lambda}$ is associated with a solitary-wave solution of (1) and the set of all such solitary waves is stable. For detailed studies, see Weinstein [29] and Albert [2].

Proposition 4. *The functional M_∞ is a continuous, real-valued mapping of H^s.*

Proof. Hypotheses (H1)-(H3) guarantee that the imbeddings $H^s \to L_j$ are continuous for $3 \leq j \leq p$ and $s > \frac{p-2}{4}$. The result follows readily. Indeed, this mapping is locally Lipschitz continuous, but we will not need this fact. □

Proposition 5. $S^+_{\lambda,\infty} \subset H^\infty$.

Proof. If $S^+_{\infty,\lambda} = \emptyset$, the result is vacuously true. If $S^+_{\infty,\lambda} \neq \emptyset$, then for any $\eta \in S^+_{\infty,\lambda}$, there is a number c such that

$$\delta M_\infty(\eta) + c\delta E_\infty(\eta) = 0$$

by the Lagrange multiplier principle. As in the proof of Theorem 3.4, one first shows that $c > 0$ and that η satisfies

$$(c + L)\eta = f(\eta),$$

so that $u(x,t) = \eta(x - ct)$ is a solitary-wave solution of (1). A bootstrapping argument similar to that provided in Theorem 3.4 then establishes that $\eta \in H^\infty$. □

Lemma 4.1. *The value $\Gamma_\infty(\lambda)$ defined in (59) satisfies*

$$-\infty < \Gamma_\infty(\lambda) < 0.$$

Furthermore, for every minimizing sequence $\{\eta_k\}_k$, there is a number $R > 0$ such that

$$\sup_k \|\eta_k\|_s < R.$$

Proof. First, we check that $\Gamma_\infty(\lambda) > -\infty$. Since $s > \frac{p-2}{4} \geq \frac{j-2}{2j}$ for all $j \in [3,p]$, Proposition 2 asures the imbeddings $H^s \hookrightarrow L_j$ are continuous and that for every $\eta \in H^s$ and $r_j \in (\frac{j-2}{2s}, j]$,

$$\Big| \int_{-\infty}^{\infty} \eta^j(x)\, dx \Big| \leq |\eta|_j^j \leq |\widehat{\eta}|^j_{\frac{j}{j-1}} \leq \tilde{\beta}_j \|\eta\|^{j-r_j} \|\eta\|_s^{r_j}$$

where

$$\tilde{\beta}_j = \Big(\int_{-\infty}^{\infty} (1 + |\xi|)^{-\frac{2sr_j}{j-2}}\, d\xi \Big)^{\frac{j-2}{2}}. \tag{61}$$

Thus,

$$M_\infty(\eta) \geq \frac{1}{2} \int_{-\infty}^{\infty} \alpha(2\pi\xi) |\widehat{\eta}(\xi)|^2 \, d\xi - \sum_{j=3}^{p} \gamma_j \tilde{\beta}_j \|\eta\|^{j-r_j} \|\eta\|_s^{r_j}.$$

From (H3), there are two non-negative numbers C_1 and C_2 such that the symbol α of the disperson operator L satisfies $\alpha(\xi) \geq -C_1 + 2C_2(1+|\xi|)^{2s}$ for all $\xi \in \mathbb{R}$. One deduces from this that

$$M_\infty(\eta) \geq -\frac{1}{2}C_1\|\eta\|^2 + C_2\|\eta\|_s^2 - \sum_{j=3}^{p} \gamma_j \tilde{\beta}_j \|\eta\|^{j-r_j}\|\eta\|_s^{r_j}. \tag{62}$$

It follows that

$$\begin{aligned}\Gamma_\infty(\lambda) &= \inf\{M_\infty(\eta) : \eta \in H^s(\mathbb{R}), \|\eta\|^2 = 2\lambda\}\\ &\geq \inf\Big\{ -C_1\lambda + C_2 z^2 - \sum_{j=3}^{p} \gamma_j \tilde{\beta}_j (2\lambda)^{\frac{j-r_j}{2}} z^{r_j} \,:\, z \geq 0\Big\}.\end{aligned}$$

Because $\frac{j-2}{2s} \leq \frac{p-2}{2s} < 2$ for $3 \leq j \leq p$, all the r_j can be chosen strictly less than 2. Consequently,

$$\Gamma_\infty(\lambda) \geq \inf\Big\{ -C_1\lambda + C_2 z^2 - \sum_{j=3}^{p} \gamma_j \tilde{\beta}_j (2\lambda)^{\frac{j-r_j}{2}} z^{r_j} \,:\, z \geq 0\Big\} > -\infty.$$

It is also the case that $\Gamma_\infty(\lambda) < 0$. Given a function η, define $\eta_\theta(x) = \theta^{\frac{1}{2}}\eta(\theta x)$ for $\theta > 0$, so that $\widehat{\eta_\theta}(\xi) = \theta^{-\frac{1}{2}}\widehat{\eta}(\theta^{-1}\xi)$, or, what is the same, $\widehat{\eta}(\xi) = \theta^{\frac{1}{2}}\widehat{\eta}_\theta(\theta\xi)$. Then η and η_θ have the same L_2-norm, so they are either both within or outside the constraint set of the variational problem (59). It follows that

$$\begin{aligned}\Gamma_\infty(\lambda) &= \inf\Big\{ \int_{-\infty}^{\infty} \frac{1}{2}\alpha(2\pi\theta^{-1}\xi)\,|\widehat{\eta_\theta}(\xi)|^2\,d\xi - \sum_{j=3}^{p}\gamma_j\theta^{-\frac{j-2}{2}}(*_j\widehat{\eta_\theta})(0) \,:\\ &\qquad\qquad \eta \in H^s,\, \widehat{\eta}(-\xi) = \widehat{\eta}(\xi) \geq 0,\, \|\eta\|^2 = 2\lambda\Big\}\\ &= \inf\Big\{ \int_{-\infty}^{\infty} \frac{1}{2}\alpha(2\pi\theta^{-1}\xi)\,|\widehat{v}(\xi)|^2\,d\xi - \sum_{j=3}^{p}\gamma_j\theta^{-\frac{j-2}{2}}(*_j\widehat{v})(0) \,:\\ &\qquad\qquad v \in H^s,\, \widehat{v}(-\xi) = \widehat{v}(\xi) \geq 0,\, \|v\|^2 = 2\lambda\Big\}.\end{aligned} \tag{63}$$

Choose a member $v \in H^s$ that lies within the constraint set of (63) such that $\widehat{v}$ is continuous and has support contained in $(-1, 1)$. Then, for every $\theta > 0$,

$$\begin{aligned}\Gamma_\infty(\lambda) &\leq \int_{-1}^{1} \frac{1}{2}\alpha(2\pi\theta^{-1}\xi)\widehat{v}^2(\xi)\,d\xi - \sum_{j=3}^{p}\gamma_j\theta^{-\frac{j-2}{2}}\,(*_j\widehat{v})(0)\\ &= \theta^{-\frac{p_0-2}{2}}\Big\{ \int_{-1}^{1} \frac{1}{2}\theta^{\frac{p_0-2}{2}}\alpha(2\pi\theta^{-1}\xi)\widehat{v}^2(\xi)\,d\xi - \sum_{j=3}^{p}\gamma_j\theta^{-\frac{j-p_0}{2}}\,(*_j\widehat{v})(0)\Big\}.\end{aligned}$$

On the other hand for $\theta > 0$ sufficiently large, hypothesis (H2) implies that the right-hand side of the last inequality is strictly negative, whence

$$\Gamma_\infty(\lambda) < 0. \tag{64}$$

The first part of the Lemma is verified.

Attention is now turned to the second part. Since $\{\eta_k\}_k$ is a minimizing sequence for (59), $\lim_{k\to\infty} M_\infty(\eta_k) = \Gamma_\infty(\lambda)$ and so $M_\infty(\eta_k) < 0$ for k sufficiently large. For such large indices k, (62) implies that

$$0 > M_\infty(\eta_k) \geq C_2\|\eta_k\|_s^2 - C_1\lambda - \sum_{j=3}^{p} \gamma_j \tilde{\beta}_j \big(2\lambda\big)^{\frac{j-r_j}{2}} \|\eta_k\|_s^{r_j}$$

where $2\lambda = \|\eta_k\|^2$, r_j could take any value in $\big(\frac{j-2}{2s}, j\big]$ and $\tilde{\beta}_j$ depends on r_j as in (61). Since $\frac{j-2}{2s} < 2$ as pointed out earlier, choose r_j close enough to $\frac{j-2}{2s}$ so that $r_j < 2$. Then, were the sequence $\{\eta_k\}_k$ to be unbounded, there would be a subsequence $\{\eta_{k_q}\}_q$ whose H^s–norm tends to $+\infty$ as $q \to \infty$. For large values of q, we would then have

$$0 > M_\infty(\eta_{k_q}) \geq C_2\|\eta_{k_q}\|_s^2 - C_1\lambda - \sum_{j=3}^{p} \gamma_j \tilde{\beta}_j \big(\frac{\lambda}{2}\big)^{\frac{j-r_j(j-1)}{2}} \|\eta_{k_q}\|_s^{r_j(j-1)} \to \infty$$

as $q \to \infty$. This contradiction leaves only the conclusion that the sequence $\{\eta_k\}_k$ is bounded and the lemma is proved. □

5. **The long-wavlength limit of periodic traveling waves.** It is well known that the classical cnoidal-wave solutions of Boussinesq and Korteweg and de Vries converge to solitary-wave solutions as the period length grows unboundedly. Bona [13] showed that the same is true not only of the cnoidal waves, but a wide class of solutions of the Korteweg-de Vries equation. This latter result is important in principle since most of the numerical simulations of solutions of these sorts of equations are actually performed with periodic boundary conditions, even though one is attempting to approximate solutions on the entire real axis.

The present section is concerned with the question of the long-wavelength limit in the context of the more general cnoidal-type travelingr -wave solutions whose existence has been established in Section 3. In Section 3, it was shown that for any fixed $\lambda > 0$, then for each $l > 0$ sufficiently large, the variational problem (25) has non-trivial minimizers. In the current section, we reconsider (25) and treat the half-period l as a variable while $\lambda > 0$ is still held fixed. To emphasize the dependence on the half-period, an l is hung on the notation for the variational problem, *viz.*

$$\Gamma_l(\lambda) = \inf\Big\{ M_l(\mathbf{u}) = l\sum \alpha\Big(\frac{n\pi}{l}\Big)|u_n|^2 - 2l\sum_{j=3}^{p}\gamma_j(*_j\mathbf{u})_0 \ : \\ u_n = u_{-n} \geq 0,\, \mathbf{u} = (u_n) \in \ell_2^s,\ E_l(\mathbf{u}) = l\|\mathbf{u}\|^2 = \lambda\Big\}. \tag{65}$$

The set of minimizers of this variational problem also depends on l and this, too, is recorded with an additional subscript, *viz.*

$$S_{\lambda,l}^{+} = \Big\{\mathbf{\Phi}_l = \{\phi_{l,n}\} \in \ell_2^s :\ \phi_{l,n} = \phi_{l,-n} \geq 0,\ l\|\mathbf{\Phi}\|^2 = \lambda,\ M_l(\phi) = \Gamma_l(\lambda)\Big\}. \tag{66}$$

A mapping $T = T_l$ is now introduced that allows us to compare periodic solutions with solitary-wave solutions. First, write a periodic function u of period $2l$ in terms of its Fourier coefficients thusly;

$$u(x) = \sum u_n e^{i\frac{n\pi}{l}x} = \sum_{n=-\infty}^{\infty} u_n e^{i\frac{n\pi}{l}x}.$$

Define a new real-valued function Tu on all of $\mathbb{R}$ by

$$Tu\,(x) = \int_{-\infty}^{\infty} \omega_u(\xi)\, e^{2\pi i x\xi}\, d\xi. \tag{67}$$

Thus, Tu is simply the inverse Fourier transform of the function ω_u. The function ω_u is a step function defined in terms of u by

$$\omega_u(\xi) = \begin{cases} 2lu_0 & \text{if} \quad -\frac{\frac{1}{2}}{2l} \le \xi \le \frac{\frac{1}{2}}{2l}, \\ 2lu_n & \text{if} \quad \frac{n-\frac{1}{2}}{2l} < \xi \le \frac{n+\frac{1}{2}}{2l} \text{ and } n \ge 1, \\ 2lu_n & \text{if} \quad \frac{n-\frac{1}{2}}{2l} \le \xi < \frac{n+\frac{1}{2}}{2l} \text{ and } n \le -1. \end{cases} \tag{68}$$

Remark 8. An elementary calculation reveals that Tu has the alternative representation

$$Tu(x) = u(x) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\frac{\pi x}{l}\tau}\, d\tau = u(x)\frac{\sin\left(\frac{\pi x}{2l}\right)}{\frac{\pi x}{2l}} \tag{69}$$

The operator T acts as a bridge from periodic function spaces like H_l^σ, say, to the corresponding Sobolev space H^σ. The details of an analysis of $T = T_l$ will be presented in a sequence of lemmas and propositions.

Proposition 6. *Let $\sigma \ge 0$ and H_l^σ be the periodic function space introduced in Section 2. Then T is a bounded linear operator from H_l^σ to H^σ. Moreover, for any $u \in H_l^\sigma$,*

$$\|Tu\| = \sqrt{2l}\|\mathbf{u}\| = \Big\{ \int_{-l}^{l} u^2(x)\, dx \Big\}^{\frac{1}{2}}. \tag{70}$$

If we further assume $2l \ge \pi$, then

$$\begin{aligned} \|u\|_{H_l^\sigma} - \frac{2^{2\sigma}\sigma\pi}{2l}\|u\|_{H_l^{\sigma-\frac{1}{2}}} \le &\|Tu\|_\sigma \le \|u\|_{H_l^\sigma} + \frac{2^{2\sigma}\sigma\pi}{2l}\|u\|_{H_l^{\sigma-\frac{1}{2}}} \quad \text{if} \quad \sigma > \frac{1}{2}, \\ \|u\|_{H_l^\sigma} - \frac{\sigma\pi}{l}\|u\| \le &\|Tu\|_\sigma \le \|u\|_{H_l^\sigma} + \frac{\sigma\pi}{l}\|u\| \quad \text{if} \quad 0 < \sigma \le \frac{1}{2}. \end{aligned} \tag{71}$$

Remark 9. If $l < \frac{\pi}{2}$ and $\sigma > \frac{1}{2}$, the first inequality in (71) still holds, but the same proof yields a coefficient of $\|u\|_{H_l^{\sigma-\frac{1}{2}}}$ having the value $(1+\frac{\pi}{2l})^{2\sigma-1}\sigma\pi/l$. As interest is focussed on large values of l, it is presumed henceforth that $l \ge \frac{\pi}{2}$.

Proof. For $u \in H_l^\sigma$, the Fourier transform $\widehat{Tu}$ of Tu is almost everywhere equal to the step-function ω_u given in (68). Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} (1+|2\pi\xi|)^{2\sigma}|\widehat{Tu}(\xi)|^2\, d\xi &= \sum_{n=-\infty}^{\infty} \int_{\frac{n-\frac{1}{2}}{2l}}^{\frac{n+\frac{1}{2}}{2l}} (1+|2\pi\xi|)^{2\sigma}|2lu_n|^2\, d\xi \\ &= \sum_{n=-\infty}^{\infty} |2lu_n|^2 \int_{\frac{n-\frac{1}{2}}{2l}}^{\frac{n+\frac{1}{2}}{2l}} (1+|2\pi\xi|)^{2\sigma}\, d\xi. \end{aligned} \tag{72}$$

The mean-value theorem for integrals of positive functions implies that for each n, there is a $\lambda_n \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\int_{\frac{n-\frac{1}{2}}{2l}}^{\frac{n+\frac{1}{2}}{2l}} (1+|2\pi\xi|)^{2\sigma}\, d\xi = \Big(1+\frac{\pi}{l}|n+\lambda_n|\Big)^{2\sigma}\frac{1}{2l} \le \Big(1+\frac{\pi}{2l}\Big)^{2\sigma}\Big(1+\Big|\frac{n\pi}{l}\Big|\Big)^{2\sigma}\frac{1}{2l}. \tag{73}$$

It follows readily that

$$\int_{-\infty}^{\infty}(1+|2\pi\xi|)^{2\sigma}|\widehat{Tu}(\xi)|^2\,d\xi \le \Big(1+\frac{\pi}{2l}\Big)^{2\sigma}2l\|\mathbf{u}\|_{\sigma,l}^2 = \Big(1+\frac{\pi}{2l}\Big)^{2\sigma}\|u\|_{H_l^\sigma}^2 < \infty.$$

It thus transpires that $Tu \in H^\sigma$. When T is considered as a linear operator from H_l^σ to H^σ, its operator norm is bounded thursly; $\|T\| \le \Big(1+\frac{\pi}{2l}\Big)^{\sigma}$. By Parseval's relation,

$$\begin{aligned}\|Tu\|^2 &= \int_{-\infty}^{\infty}|\widehat{Tu}(\xi)|^2\,d\xi = \sum\int_{\frac{n-\frac12}{2l}}^{\frac{n+\frac12}{2l}}|2lu_n|^2\,d\xi \\ &= 2l\sum|u_n|^2 = 2l\|\mathbf{u}\|^2 = \int_{-l}^{l}u^2(x)\,dx.\end{aligned} \tag{74}$$

This is the case $\sigma = 0$ in the preceding estimate and thus (70) holds. To prove (71), substitute the left-hand equality in (73) into (72) to deduce that

$$\|Tu\|_\sigma^2 - \|u\|_{H_l^\sigma}^2 = 2l\sum|u_n|^2\Big(1+\frac{\pi}{l}|n+\lambda_n|\Big)^{2\sigma} - 2l\sum|u_n|^2\Big(1+\frac{\pi}{l}|n|\Big)^{2\sigma}.$$

The Mean-Value Theorem assures that for each n, there is $\delta_n \in [0,1]$ such that

$$\begin{aligned}\|Tu\|_\sigma^2 - \|u\|_{H_l^\sigma}^2 =& 2l\sum_{n\ge1}2\sigma|u_n|^2\Big(1+\frac{\pi}{l}n+\frac{\pi}{l}\delta_n\lambda_n\Big)^{2\sigma-1}\frac{\pi}{l}\lambda_n \\ &+ 2l\sum_{n\le-1}2\sigma|u_n|^2\Big(1-\frac{\pi}{l}n-\frac{\pi}{l}\delta_n\lambda_n\Big)^{2\sigma-1}\Big(-\frac{\pi}{l}\lambda_n\Big) \\ &+ 2l\,2\sigma|u_0|^2\Big(1+\frac{\pi}{l}\delta_0|\lambda_0|\Big)^{2\sigma-1}\frac{\pi}{l}|\lambda_0|.\end{aligned}$$

If $\sigma > \frac12$, then $|\frac{\pi}{l}\delta_n\lambda_n| < 1$ since $2l \ge \pi$. In consequence of this observation, we have

$$\begin{aligned}\Big|\|Tu\|_\sigma^2 - \|u\|_{H_l^\sigma}^2\Big| &\le \sigma 2^{2\sigma}2l\sum|u_n|^2\Big(1+\frac{\pi}{l}|n|\Big)^{2\sigma-1}\frac{\pi}{2l} \\ &= \frac{\pi\sigma 2^{2\sigma}}{2l}2l\|\mathbf{u}\|_{\sigma,l}^2 = \frac{\pi\sigma2^{2\sigma}}{2l}\|u\|_{H_l^{\sigma-\frac12}}^2.\end{aligned}$$

In case $0 < \sigma \le \frac12$, then it transpires that

$$\Big|\|Tu\|_\sigma^2 - \|u\|_{H_l^\sigma}^2\Big| \le \frac{\sigma\pi}{l}2l\sum|u_n|^2 = \frac{\sigma\pi}{l}2l\|\mathbf{u}\|^2 = \frac{\sigma\pi}{l}\|u\|^2.$$

The last two inequalities are equivalent to those in (71). □

Lemma 5.1. *Viewed as a function of l, the value $\Gamma_l(\lambda)$, associated to the variational problem (65) is uniformly bounded below as $l \to \infty$. That is to say,*

$$\liminf_{l\to\infty}\Gamma_l(\lambda) > -\infty. \tag{75}$$

Furthermore, there is an $R > 0$ such that

$$\sup_{l>0}\Big\{\|\phi\|_{H_l^s} = \sqrt{2l}\|\mathbf{\Phi}\|_{s,l} \,:\, \mathbf{\Phi} = \{\phi_n\} \in S_{\lambda,l}^+ \ne \emptyset\Big\} \le R, \tag{76}$$

where $S_{\lambda,l}^+$ is the set of minimizers of the variational problem as in (66) and as before, if $\mathbf{\Phi} = \{\phi_n\}$, then $\phi(x) = \sum\phi_n e^{in\pi x/l}$.

Proof. Theorem 3.4 guarantees that the set $S^+_{\lambda,l} \neq \emptyset$ for $l > 0$ sufficiently large. Moreover, every $\mathbf{\Phi} \in S^+_{\lambda,l}$ satisfies

$$\|\mathbf{\Phi}\| = \frac{\lambda}{l} \quad \text{and} \quad \Gamma_l(\lambda) = M_l(\mathbf{\Phi}).$$

In the inequality (33), replace $\mathbf{u}$ by $\mathbf{\Phi}$ and let $z = \sqrt{2l}\|\mathbf{\Phi}\|_{s,l}$. Then, the inequalityr (33) yields

$$\Gamma_l(\lambda) = M_l(\mathbf{\Phi}) \geq -C_1\lambda + C_2 z^2 - \sum_{j=3}^{p} \gamma_j (2\lambda)^{\frac{j-(j-1)r_j}{2}} (2l)^{-\frac{j-2}{2}} \beta_j z^{r_j(j-1)} \tag{77}$$

where $r_j \in (\frac{j-2}{2s(j-1)}, \frac{j}{j-1}]$ is chosen close to the left-hand end point so that $r_j(j-1) < 2$ for all the relevant j and the corresponding β_j are given in (31). Viewed as a function of l, β_j has the property that

$$\lim_{l\to\infty} \frac{1}{2l} \beta_j^{\frac{2}{j-2}} = \int_{-\infty}^{\infty} (1 + |2\pi\xi|)^{-\frac{2sr_j(j-1)}{j-2}} \, d\xi < \infty. \tag{78}$$

This in turn means that $(2l)^{-\frac{j-2}{2}}\beta_j$ is bounded by some constant $\tilde{\beta}_j$, say. We have thus shown that

$$\Gamma_l(\lambda) \geq \inf\Big\{ -C_1\lambda + C_2 z^2 - \sum_{j=3}^{p} \gamma_j (2\lambda)^{\frac{j-(j-1)r_j}{2}} \tilde{\beta}_j z^{r_j(j-1)} \, : \, z \geq 0 \Big\}.$$

The fact that $r_j(j-1) < 2$ for all j guarantees the right-hand side is a finite number, clearly independent of l. The inequality (75) is thereby established.

To prove (76), simply substitute the fact $\Gamma_l(\lambda) < 0$ into (77) to obtain

$$-C_1\lambda + C_2 z^2 - \sum_{j=3}^{p} \gamma_j (2\lambda)^{\frac{j-(j-1)r_j}{2}} (2l)^{-\frac{j-2}{2}} \beta_j z^{r_j(j-1)} < 0.$$

The conditions $r_j(j-1) < 2$ for all j together with (78) imply that there must be an $R > 0$ which is independent of l such that $z = \|\phi\|_{s,l} = \sqrt{2l}\|\mathbf{\Phi}\|_{s,l} \leq R$. (Otherwise the left-hand side tends to infinity as z tends to infinity.) Thus, (76) is valid and the lemma is proved. □

Lemma 5.2. *Assume (H4) in addition to hypotheses (H1)-(H3). Then, for any* $\epsilon > 0$, *there exists an* $l_\epsilon > 0$ *such that if* $l \geq l_\epsilon$, *then each* $\phi_l(x) = \sum_n \phi_{l,n} e^{i\frac{n\pi}{l}x}$ *with* $\mathbf{\Phi}_l = \{\phi_{l,n}\} \in S^+_{\lambda,l}$, *has the property that*

$$\Big|\Gamma_l(\lambda) - M_\infty\big(T\phi_l\big)\Big| \leq \epsilon \tag{79}$$

where the operator T *is defined in* (68) *and* (67) *and* M_∞ *is as in* (58). *Hence, for* l *sufficiently large,*

$$\Gamma_l(\lambda) \geq \Gamma_\infty(\lambda) - \epsilon. \tag{80}$$

Proof. Since ϕ_l has Fourier coefficients $\mathbf{\Phi}_l = \{\phi_{l,n}\} \in S^+_{\lambda,l}$, it follows that

$$\begin{aligned}
&\Gamma_l(\lambda)\\
=&M_l(\phi_l) = l\sum_{n=-\infty}^{\infty}\alpha\Big(\frac{n\pi}{l}\Big)\phi_{l,n}^2 - 2l\sum_{j=3}^{p}\gamma_j\sum_{n_1+\cdots+n_j=0}\phi_{l,n_1}\cdots\phi_{l,n_j}\\
=&\,l\sum_{n=-\infty}^{\infty}\alpha\Big(\frac{n\pi}{l}\Big)\phi_{l,n}^2 - 2l\sum_{j=3}^{p}\gamma_j\sum_{n_1,\cdots,n_{j-1}=-\infty}^{\infty}\phi_{l,n_1}\cdots\phi_{l,n_{j-1}}\phi_{l,-(n_1+\cdots+n_{j-1})}\\
=&\,l\sum_{n=-\infty}^{\infty}\int_{\frac{n-\frac12}{2l}}^{\frac{n+\frac12}{2l}}\Big(\alpha(2\pi\xi)+\alpha\Big(\frac{n\pi}{l}\Big)-\alpha(2\pi\xi)\Big)2l\,\phi_{l,n}^2\,d\xi - \sum_{j=3}^{p}\gamma_j\int_{-\infty}^{\infty}\big(T\phi_l\big)^j(x)\,dx\\
&+\sum_{j=3}^{p}\gamma_j\int_{-\infty}^{\infty}\big(T\phi_l\big)^j(x)\,dx\\
&-\ 2l\sum_{j=3}^{p}\gamma_j\sum_{n_1,\cdots,n_{j-1}=-\infty}^{\infty}\phi_{l,n_1}\cdots\phi_{l,n_{j-1}}\phi_{l,-(n_1+\cdots+n_{j-1})}.
\end{aligned}$$

The construction of $T\phi_l$ and definition of the dispersive operator L provide

$$\int_{-\infty}^{\infty}(T\phi_l)L(T\phi_l)\,dx = \sum_{n=-\infty}^{\infty}\int_{\frac{n-\frac12}{2l}}^{\frac{n+\frac12}{2l}}\alpha(2\pi\xi)|2l\phi_{l,n}|^2\,d\xi$$

and, for $j \geq 2$,

$$\begin{aligned}
\int_{-\infty}^{\infty}\big(T\phi_l\big)^j(x)\,dx &= \Big(\widehat{T}\phi_l * \cdots * \widehat{T}\phi_l\Big)(0)\\
&= 2l\sum_{n_1,\cdots,n_{j-1}=-\infty}^{\infty}\phi_{l,n_1}\cdots\phi_{l,n_{j-1}}\phi_{l,-(n_1+\cdots+n_{j-1})}.
\end{aligned}$$

It thus appears that $\Gamma_l(\lambda)$ can be expressed as

$$\Gamma_l(\lambda) = M_\infty\big(T\phi_l\big) + \Delta_l \tag{81}$$

where

$$\Delta_l = \frac12\sum_{n=-\infty}^{\infty}\int_{\frac{n-\frac12}{2l}}^{\frac{n+\frac12}{2l}}\Big[\alpha\Big(\frac{n\pi}{l}\Big)-\alpha\Big(2\pi\xi\Big)\Big]|2l\phi_n|^2\,d\xi.$$

The symmetries $\alpha(-\xi) = \alpha(\xi)$ and $\phi_{l,-n} = \phi_{l,n}$ together with $\alpha(0) = 0$ and the foregoing calculations imply that

$$\Delta_l = \sum_{n=1}^{\infty}\int_{\frac{n-\frac12}{2l}}^{\frac{n+\frac12}{2l}}\Big[\alpha\Big(\frac{n\pi}{l}\Big)-\alpha\Big(2\pi\xi\Big)\Big]\,|2l\phi_n|^2\,d\xi - \int_0^{\frac{1}{4l}}\alpha(2\pi\xi)|2l\phi_{l,0}|^2\,d\xi.$$

The mean-value theorem and a Taylor expansion may be invoked to ascertain that there are $\lambda_n \in [-1,1]$ for $n = 0, 1, \cdots$ such that

$$|\Delta_l| \leq \sum_{n=1}^{\infty}\Big|\alpha'\Big(\frac{n\pi}{l}+\frac{\lambda_n\pi}{2l}\Big)\Big|\,|\phi_{l,n}|^2 + \Big|\alpha\Big(\frac{\lambda_0\pi}{4l}\Big)\Big|\,|\phi_{l,0}|^2\,l.$$

Hypothesis (H4) together with (H2) and (H3) imply that

$$|\Delta_l| \le \theta \sum_{n=1}^{\infty} \Big|1 + \frac{n\pi}{l} + \frac{\pi}{2l}\Big|^{2s-1} |\phi_{l,n}|^2 + \theta_0 \Big|\frac{\pi}{4l}\Big|^{2\tilde{s}} |\phi_{l,0}|^2\, l$$

for l sufficiently large, where θ_0 and θ are two positive numbers and s and $\tilde{s}$ are as in (H3) and (H2), respectively. Since $l > \frac{\pi}{2}$, this inequality may be further extended to

$$|\Delta_l| \le \theta \sum_{n=1}^{\infty} 2^{2s} \Big|1 + \frac{n\pi}{l}\Big|^{2s-1} |\phi_{l,n}|^2 + \theta_0 \Big|\frac{\pi}{4l}\Big|^{2\tilde{s}} |\phi_{l,0}|^2\, l.$$

Lemma 4.3 asserts that there is an $R > 0$ such that

$$\|\phi\|_{H_l^s} = \Big(2l \sum \Big(1 + \Big|\frac{n\pi}{l}\Big|\Big)^{2s} |\phi_n|^2\Big)^{\frac{1}{2}} < R$$

uniformly for $\phi \in \cup_{l>0} S_{\lambda,l}$. Hence, there is a constant Θ independent of l such that

$$|\Delta_l| \le \frac{\Theta R^2}{(2l)^{2s_0}} \tag{82}$$

where $s_0 = \min\{\frac{1}{2}, \tilde{s}\}$. This in turn implies

$$\Big|\Gamma_l(\lambda) - M_\infty(T\phi_l)\Big| \le \frac{\Theta R^2}{(2l)^{2s_0}}.$$

From Proposition 6, we know $T\phi_l \in H^s$ and $\|T\phi_l\|^2 = \frac{\lambda}{2}$, so $M_\infty(T\phi_l) \ge \Gamma_\infty(\lambda)$. If we define $l_\epsilon = \frac{1}{2}\Big(\frac{\Theta R^2}{\epsilon}\Big)^{\frac{1}{2s_0}}$, then when $l > l_\epsilon$, each ϕ_l whose Fourier coefficients $\mathbf{\Phi}_l \in S^+_{\lambda,l}$, satisy both (79) and (80). □

Lemma 5.3. *Consider the variational problems (59) and (65). For any $\epsilon \in (0,1)$, there is a sufficiently large number l_ϵ such that for each $l > l_\epsilon$, there exists a periodic function $\psi_l = \sum_n \psi_{l,n} e^{i\frac{n\pi}{l}x}$, say, whose Fourier coefficients $\{\psi_{l,n}\}$ lie in the constraint set of (65) and which is such that*

$$\Gamma_\infty(\lambda) \ge M_l(\psi_l) - \epsilon \ge \Gamma_l(\lambda) - \epsilon. \tag{83}$$

The following result will be used to prove this lemma.

Proposition 7. *For any $\epsilon > 0$, there exists an η^*_ϵ which lies in the constraint set of (59) whose Fourier transform $\widehat{\eta^*_\epsilon}$ lies in C^∞_c and for which*

$$\Gamma_\infty(\lambda) > M_\infty(\eta^*_\epsilon) - \epsilon. \tag{84}$$

Here, the space $C^\infty_c = C^\infty_c(\mathbb{R})$ is the space of infinitely smooth functions with compact support.

Proof. Since $\Gamma_\infty(\lambda)$ is finite according to Lemma 4.3, for any $\epsilon > 0$, there is $\eta \in H^s$ with $\widehat{\eta}(\xi) = \widehat{\eta}(-\xi) \ge 0$ and $E_\infty(\eta) = \frac{1}{2}\|\eta\|^2 = \lambda$ such that

$$\Gamma_\infty(\lambda) > M_\infty(\eta) - \frac{1}{2}\epsilon. \tag{85}$$

Standard arguments (see Lemma 6.2 in the Appendix for a sketch) imply that there is a function $\eta_\epsilon \in H^s$ such that $\widehat{\eta_\epsilon} \in C^\infty_c$ is an even, non-negative function for which

$$\|\eta_\epsilon - \eta\|^2_s = \int_{-\infty}^{\infty} (1 + |2\pi\xi|)^{2s} \big|\widehat{\eta}_\epsilon(\xi) - \widehat{\eta}(\xi)\big|^2\, d\xi < \frac{1}{4}\epsilon^2. \tag{86}$$

If $\eta_\epsilon^* = \frac{\sqrt{2\lambda}}{\|\eta_\epsilon\|}\eta_\epsilon$, then η_ϵ^* lies in the constraint set of (59) and its Fourier transform $\widehat{\eta_\epsilon^*} \in C_c^\infty$. The inequalities (86) and (62) conspire to show that, for ϵ sufficientlly small,

$$|M_\infty(\eta_\epsilon^*) - M_\infty(\eta)| < \frac{1}{2}\epsilon.$$

Then, the inequality (85) implies immediately that

$$\Gamma_\infty(\lambda) \geq M_\infty(\eta_\epsilon^*) - \epsilon$$

for sufficiently small ϵ and the proposition is proved. □

Attention is now given to a proof of Lemma 5.3.

Proof. Let $\eta_\epsilon = \eta_\epsilon^*$ be as in Proposition 7 and suppose its support lies in $(-R_\epsilon, R_\epsilon)$. Construct a periodic function $P_l\eta_\epsilon$ of period $2l$ by specifying its Fourier series, *viz.*

$$P_l\eta_\epsilon(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l}\widehat{\eta}_\epsilon\Big(\frac{n}{2l}\Big)e^{i\frac{n\pi}{l}x}.$$

Since the support of $\widehat{\eta_\epsilon}$ is compact, this series features only a finite number of non-zero terms. Let $N = \lceil 2lR_\epsilon - \frac{1}{2}\rceil$, so that

$$P_l\eta_\epsilon(x) = \sum_{n=-N}^{N} \frac{1}{2l}\widehat{\eta}_\epsilon\Big(\frac{n}{2l}\Big)e^{i\frac{n\pi}{l}x}. \tag{87}$$

Obviously, $P_l(\eta_\epsilon)$ is periodic and infinitely smooth. Viewing (87) as a Riemann sum, it is inferred that

$$\lim_{l\to\infty}\|P_l\eta_\epsilon\|^2_{L_2(-l,l)} = 2l\lim_{l\to\infty}\sum_{n=-N}^{N}\Big|\frac{1}{2l}\widehat{\eta}_\epsilon\Big(\frac{n}{2l}\Big)\Big|^2 = \int_{-R_\epsilon}^{R_\epsilon}\big|\widehat{\eta}_\epsilon(\xi)\big|^2\,d\xi = 2\lambda \tag{88}$$

and

$$\begin{aligned}\lim_{l\to\infty}\|P_l\eta_\epsilon\|^2_{H_l^s} &= 2l\lim_{l\to\infty}\sum_{n=-N}^{N}\Big(1+\Big|\frac{n\pi}{l}\Big|\Big)^{2s}\Big|\frac{1}{2l}\widehat{\eta}_\epsilon\Big(\frac{n}{2l}\Big)\Big|^2 \\ &= \int_{-R_\epsilon}^{R_\epsilon}(1+|2\pi\xi|)^{2s}\big|\widehat{\eta}_\epsilon(\xi)\big|^2\,d\xi \;= \|\eta_\epsilon\|_s^2.\end{aligned} \tag{89}$$

It is further deduced that

$$\begin{aligned}&M_\infty(\eta_\epsilon)\\ =&\frac{1}{2}\int_{-\infty}^{\infty}\alpha(2\pi\xi)|\widehat{\eta}_\epsilon(\xi)|^2\,d\xi - \sum_{j=3}^{p}\gamma_j\int_{-\infty}^{\infty}(\eta_\epsilon(x))^j\,dx\\ =&\frac{1}{2}\int_{-R_\epsilon}^{R_\epsilon}\alpha(2\pi\xi)|\widehat{\eta}_\epsilon(\xi)|^2\,d\xi\\ &-\sum_{j=3}^{p}\gamma_j\int_{-R_\epsilon}^{R_\epsilon}\cdots\int_{-R_\epsilon}^{R_\epsilon}\widehat{\eta}_\epsilon(\xi_1)\cdots\widehat{\eta}_\epsilon(\xi_{j-1})\,\widehat{\eta}_\epsilon(-(\xi_1+\cdots\xi_{j-1})\,d\xi_1\cdots d\xi_{j-1}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\lim_{l\to\infty}\sum_{n=-N}^{N}\alpha\Big(\frac{n\pi}{l}\Big)\Big|\widehat{\eta}_\epsilon\Big(\frac{n\pi}{l}\Big)\Big|^2\frac{1}{2l}\\
&\quad -\sum_{j=3}^{p}\gamma_j\sum_{n_1=-N}^{N}\cdots\sum_{n_{j-1}=-N}^{N}\widehat{\eta}_\epsilon\Big(\frac{n_1\pi}{2l}\Big)\cdots\widehat{\eta}_\epsilon(\frac{n_{j-1}\pi}{2l})\,\widehat{\eta}_\epsilon \qquad (90)\\
&\qquad \Big(-\frac{(n_1+\cdots+n_{j-1})\pi}{2l}\Big)\Big(\frac{1}{2l}\Big)^{j-1}\\
&= \lim_{l\to\infty}M_l(P_l\eta_\epsilon).
\end{aligned}
$$

In consequence, for given $\epsilon>0$, there is an $l_{\epsilon,1}>0$ sufficiently large such that when $l\ge l_{\epsilon,1}$,

$$M_\infty(\eta_\epsilon) > M_l(P_l\eta_\epsilon)-\epsilon,$$

whence

$$\Gamma_\infty(\lambda)\ge M_\infty(\eta_\epsilon)-\epsilon > M_l(P_l\eta_\epsilon)-2\,\epsilon.$$

Define the periodic function ψ_l by

$$\psi_l=\frac{\sqrt{2\lambda}}{\|P_l\eta_\epsilon\|}P_l\eta_\epsilon. \qquad (91)$$

Then, $\psi_l\in H_l^s$ and $\|\psi_l\|=\sqrt{2\lambda}$ so that $M_l(\psi_l)\ge\Gamma_l(\lambda)$ and

$$\Gamma_\infty(\lambda)\ge M_l(\psi_l)+M_l(P_l\eta_\epsilon)-M_l(\psi_l)-2\epsilon\ge\Gamma_l(\lambda)+M_l(P_l\eta_\epsilon)-M_l(\psi_l)-2\epsilon. \qquad (92)$$

It remains to show the quantity $M_l(P_l\eta_\epsilon)-M_l(\psi_l)$ tends to zero as l tends to infinity. Toward this end, denote the quantity $\frac{\sqrt{2\lambda}}{\|P_l\eta_\epsilon\|}$ by μ_l and note that

$$
\begin{aligned}
M_l(P_l\eta_\epsilon)-M_l(\psi_l) &= (1-\mu_l^2)\frac{1}{4l}\sum_{n=-N}^{N}\alpha\Big(\frac{n\pi}{l}\Big)\Big|\widehat{\eta}_\epsilon\Big(\frac{n\pi}{2l}\Big)\Big|^2 \qquad (93)\\
&\quad -\sum_{j=3}^{p}\gamma_j\int_{-l}^{l}(1-\mu_l^j)\big(P_l\eta_\epsilon(x)\big)^j\,dx.
\end{aligned}
$$

Thanks to (86), we have $\lim_{l\to\infty}\mu_l=1$. As it is known that both α and $\widehat{\eta}_\epsilon$ are continuous, so,

$$\lim_{l\to\infty}\frac{1}{2l}\sum_{n=-N}^{N}\alpha\Big(\frac{n\pi}{2l}\Big)\Big|\widehat{\eta}_\epsilon\Big(\frac{n\pi}{2l}\Big)\Big|^2=\int_{-R_\epsilon}^{R_\epsilon}\alpha(2\pi\xi)|\widehat{\eta}_\epsilon(\xi)|^2\,d\xi$$

and

$$
\begin{aligned}
\lim_{l\to\infty}\int_{-l}^{l}\big(P_l\eta_\epsilon(x)\big)^j\,dx &= \lim_{l\to\infty}\Big(\frac{1}{2l}\Big)^{j-1}\sum_{n_1+\cdots+n_{j-1}=0}\widehat{\eta}_\epsilon\Big(\frac{n_1\pi}{2l}\Big)\cdots\widehat{\eta}_\epsilon\Big(\frac{n_j\pi}{2l}\Big)\\
&= (*_j\widehat{\eta}_\epsilon)(0)
\end{aligned}
$$

for all $j\in[3,p]$. It is then immediate that

$$\lim_{l\to\infty}\big(M_l(P_l\eta_\epsilon)-M_l(\psi_l)\big)=0. \qquad (94)$$

The advertised result now follows. □

A consequence of the last lemma is the next result.

Theorem 5.4. *Assume Hypotheses (H1)-(H4) hold. For any $\epsilon > 0$, there is an $l_\epsilon >$ such that for all $l > l_\epsilon$,*

$$\Gamma_\infty(\lambda) - \epsilon \le \Gamma_l(\lambda) \le \Gamma_\infty(\lambda) + \epsilon,$$

or, what is same,

$$\lim_{l\to\infty} \Gamma_l(\lambda) = \Gamma_\infty(\lambda).$$

Theorem 5.5. *Continue to assume Hypotheses (H1)-(H4) hold. Let $\{l_k\}_k$ be a positive increasing sequence, unbounded, and for each k, let $\phi_{l_k} \in S^+_{l_k,\lambda}$. Then, the sequence $\{T\phi_{l_k}\}_k$ is a minimizing sequence for the variational problem (59)*

Proof. This is a direct consequence of the inequalities

$$\Gamma_\infty(\lambda) - \epsilon \le M_\infty(T\phi_{l_k}) - \epsilon \le \Gamma_{l_k}(\lambda) = M_{l_k}(\phi_{l_k}) \le \Gamma_\infty(\lambda) + \epsilon$$

which hold for k large enough that $l_k > l_\epsilon$. □

Remark 10. Since the sequence $\{T\phi_{l_k}\}_k$ in Theorem 5.5 is a minimizing sequence for the variational problem (59), any limit point of the sequence is a solitary-wave solution of (1) for an appropriate propagation speed $c > 0$.

Theorem 5.6. *Hypotheses (H1) through (H4) are still in force. Let $\{l_k\}_k$ be a positive, increasing and unbounded sequence of periods. Suppose there is a corresponding sequence of functions $\{\phi_{l_k}\}_k = \{\phi_k\}_k \in S^+_{\lambda,l_k}$ whose Fourier coefficients $\{\phi_{k,n}\}_n$ satisfy*

$$\sum_{n=-\infty}^{\infty} 2l_k|\phi_{k,n+1} - \phi_{k,n}| < B \tag{95}$$

where B is a constant independent of l_k. Then there is a function $\eta \in H^s$ and a subsequence of $\{\phi_k\}_k$, still denoted by $\{\phi_k\}_k$, which converges to η as follows:

$$\begin{aligned}
&\lim_{k\to\infty} \phi_k(x) = \eta(x) \quad \textit{for } x \in \mathbb{R} \textit{ if } s > \frac{1}{2},\\
&\lim_{k\to\infty} \int_{-l_k}^{l_k} |\phi_k(x) - \eta(x)|^r\, dx = 0 \quad \textit{for any} \quad r > 1 \textit{ if } s = \frac{1}{2},\\
&\textit{and}\\
&\lim_{k\to\infty} \int_{-l_k}^{l_k} |\phi_l(x) - \eta(x)|^r\, dx = 0 \quad \textit{for } 2 \le r < \frac{2}{1-2s} \quad \textit{if } 0 < s < \frac{1}{2}.
\end{aligned} \tag{96}$$

For $s > \frac{1}{2}$, the convergence is uniform on any compact domain. Furthermore, η is a solitary-wave solution of (1).

Proof. Based on the condition (95), the following three points present themselves;

$$|x\phi_k(x)| \ \le\ B \quad \text{unformly for } x \in (-l_k, l_k), \tag{97}$$

$$|xT\phi_k(x)| \ \le\ B \quad \text{uniformly for}\, x \in \mathbb{R}, \tag{98}$$

where the mapping T is as introduced in (67) and

$$|T\phi_k(x) - \phi_k(x)| \le \frac{B}{l_k} \qquad \text{for} \qquad x \in (-l_k, l_k). \tag{99}$$

Suppose for the moment these are valid statements. Since $\{T\phi_k\}_k$ is a minimizing sequence for (59), by Lemma 4.1, it is bounded in H^s and there is a subsequence, still denoted by $\{T\phi_k(x)\}_k$, weakly convergent to some function η, say, in H^s. This together with (98) indicates that $\{T\phi_k(x)\}$ converges strongly to η in L_r for all r as described in (96), and depending on s of course. Hence, $E_\infty(\eta) = \frac{1}{2}\|\eta\|^2 = \lambda$ and consequently, $M_\infty(\eta) \le \Gamma_\infty(\lambda)$. Thus, η is a minimizer for (59). It follows that the H^s–norm of $\{T\phi_k\}$, converges to the H^s–norm of η. Coupled with the weak convergence, this implies strong convergence, which is to say,

$$\lim_{l\to\infty} \|T\phi_k - \eta\|_s = 0.$$

Next, it is shown that (96) holds as stated. If $s > \frac{1}{2}$, then we have

$$\begin{aligned} |\phi_k(x) - \eta(x)| &\le |\phi_k(x) - T\phi_k(x)| + |T\phi_k(x) - \eta(x)| \\ &\le \frac{B}{l_k} + C\|T\phi_k - \eta\|_s \quad \text{for} \quad x \in (-l_k, l_k) \end{aligned} \tag{100}$$

where C is the embedding constant from H^s to L_∞. In consequence,

$$\lim_{k\to\infty} \phi_k(x) = \eta(x)$$

for $x \in \mathbb{R}$. The uniform convergence on compact domain is straightforward. If $s = \frac{1}{2}$, then for $r > 1$

$$\begin{aligned} \Big(\int_{-l}^{l} |\phi_k(x) - \eta(x)|^r\,dx\Big)^{\frac{1}{r}} &\le \Big(\int_{-l}^{l} |\phi_k(x) - T\phi_k(x)|^r\,dx\Big)^{\frac{1}{r}} \\ &\quad + \Big(\int_{-l}^{l} |T\phi_k(x) - \eta(x)|^r\,dx\Big)^{\frac{1}{r}} \\ &\le 2B(2l_k)^{-1+\frac{1}{r}} + \Big(\int_{-\infty}^{\infty} |T\phi_k(x) - \eta(x)|^r\,dx\Big)^{\frac{1}{r}} \\ &\le 2B(2l_k)^{-1+\frac{1}{r}} + C\|T\phi_k - \eta\|_s, \end{aligned}$$

where C is the embedding constant from H^s to L_r. When $k \to \infty$ so that $l_k \to \infty$, the second result featured in (96) emerges. If $0 < s < \frac{1}{2}$, a similar argument yields the third statement in (96).

Because η is a minimizer for (59), there is a positive constant $c > 0$ such that $\eta(x,t) = \eta(x - ct)$ is a solitary-wave solution of (1). (The detailed argument can be found in Albert 1999.)

It remains to show (97)–(99). Write ϕ_k in terms of its Fourier series,

$$\phi_k(x) = \sum_n \phi_{k,n} e^{i\frac{n\pi}{l_k}x} = \sum_n \phi_{k,n+1} e^{i\frac{(n+1)\pi}{l_k}x} = e^{i\frac{\pi}{l_k}x} \sum_n \phi_{k,n+1} e^{i\frac{n\pi}{l_k}x}. \tag{101}$$

It follows that

$$\Big(e^{-i\frac{\pi}{l_k}x} - 1\Big)\phi_k(x) = \sum_n (\phi_{k,n+1} - \phi_{k,n}) e^{i\frac{n\pi}{l_k}x} \tag{102}$$

and therefore,

$$x\phi_k(x) = \frac{x e^{i\frac{\pi}{l_k}x}}{2l_k(1 - e^{i\frac{\pi}{l_k}x})} \sum_n 2l_k(\phi_{k,n+1} - \phi_{k,n}) e^{i\frac{n\pi}{l_k}x}. \tag{103}$$

It is elementary that for $x \in [-l_k, l_k]$,

$$\Big|\frac{x e^{i\frac{\pi}{l_k}x}}{2l_k(1 - e^{i\frac{\pi}{l_k}x})}\Big| = \Big|\frac{x}{4l_k \sin\big(\frac{\pi}{2l_k}x\big)}\Big| \le \frac{1}{4}.$$

Combining this with the hypothesis (95) yields (97). Recall that $T\phi_k$ has the alternative expression $T\phi_k(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\frac{\pi x}{l_k}\tau}\, d\tau \phi_k(x)$. Hence, $|xT\phi_{l_k}(x)| \le |x\phi_k(x)| \le B$ and (5.34) is proved. The latter representation for T also implies that

$$|T\phi_k(x) - \phi_k(x)| = \Big|\Big(1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{i\frac{\pi x}{l_k}\tau}\, d\tau\Big)\phi_k(x)\Big| \le \frac{\pi}{4l_k}|x\phi_k(x)| \le \frac{B}{l_k},$$

thereby establishing (99). ☐

6. Appendix.

Lemma 6.1. *If $j \ge 3$ is an integer, then*

$$\sigma_j = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{j}{j-2k}\binom{2k}{k}\Big(\frac{1}{2}\Big)^{j/2+k} \ge \sqrt{2}$$

where $\lfloor j/2 \rfloor$ is the largest integer which is less than or equal to $j/2$.

Proof. We argue by induction on j. First, $\sigma_3 = \big(\frac{1}{2}\big)^{\frac{3}{2}} + 6\big(\frac{1}{2}\big)^{\frac{3}{2}+1} = \sqrt{2}$ and

$$\sigma_4 = \Big(\frac{1}{2}\Big)^{\frac{4}{2}} + 12\Big(\frac{1}{2}\Big)^{\frac{4}{2}+1} + 6\Big(\frac{1}{2}\Big)^{\frac{4}{2}+2} > \sqrt{2}.$$

Even and odd values of j are considered separately. Suppose $\sigma_{2j+1}, \sigma_{2j+2} \ge \sqrt{2}$ for some $j \ge 1$. Compute as follows:

$$\begin{aligned}
\sigma_{2j+3} &= \sum_{k=0}^{j+1} \binom{2j+3}{2j+3-2k}\binom{2k}{k}\Big(\frac{1}{2}\Big)^{\frac{2j+3}{2}+k} \\
&= \Big(\frac{1}{2}\Big)^{\frac{2j+3}{2}} + \sum_{k=1}^{j+1} \binom{2j+3}{2j+3-2k}\binom{2k}{k}\Big(\frac{1}{2}\Big)^{\frac{2j+3}{2}+k} \\
&= \Big(\frac{1}{2}\Big)^{\frac{2j+3}{2}} + \sum_{k=0}^{j} \binom{2j+1}{2j+1-2k}\binom{2k}{k}\frac{(2j+3)(2j+2)}{(k+1)^2}\Big(\frac{1}{2}\Big)^{\frac{2j+3}{2}+k} \\
&> \sigma_{2j+1} \ge \sqrt{2}.
\end{aligned}$$

Similarly, it is seen that

$$\begin{aligned}
\sigma_{2j+4} &= \Big(\frac{1}{2}\Big)^{\frac{2j+4}{2}} + \sum_{k=1}^{j+2} \binom{2j+4}{2j+4-2k}\binom{2k}{k}\Big(\frac{1}{2}\Big)^{\frac{2j+4}{2}+k} \\
&= \Big(\frac{1}{2}\Big)^{\frac{2j+4}{2}} + \sum_{k=0}^{j+1} \binom{2j+2}{2j+2-2k}\binom{2k}{k}\frac{(2j+4)(2j+3)}{(k+1)^2}\Big(\frac{1}{2}\Big)^{\frac{2j+4}{2}+k} \\
&> \sigma_{2j+2} \ge \sqrt{2}.
\end{aligned}$$

The induction is complete and the proof concluded. ☐

Lemma 6.2. *Suppose $\eta \in H^s(\mathbb{R})$ has its Fourier transform $\widehat{\eta}$ even and non-negative. Then given $\epsilon > 0$, there is a function $\varphi \in H^s(\mathbb{R})$ such that $\|\eta - \varphi\|_{H^s} \le \epsilon$ and $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$ with $\varphi(\xi) = \varphi(-\xi) \ge 0$ for all $\xi \in \mathbb{R}$.*

Proof. Let $\rho : \mathbb{R} \to \mathbb{R}$ be the usual mollifer defined to be

$$\rho(x) = C\, exp\Big(\frac{1}{x^2-1}\Big) \text{ for } |x| < 1$$

and $\rho(x) \equiv 0$ for $|x| \geq 1$. The function ρ lies in $C_c^\infty(\mathbb{R})$ and the constant C is chosen so that

$$\int_{-\infty}^{\infty} \rho(x)\, dx \;=\; 1.$$

For $\epsilon > 0$, let

$$\rho_\epsilon(x) = \frac{1}{\epsilon}\rho\Big(\frac{x}{\epsilon}\Big) \text{ and let } \chi_\epsilon(x) = \chi_{[-\frac{1}{\epsilon},\frac{1}{\epsilon}]}(x)$$

be the characteristic function of the interval $[-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$. Define $\varphi = \varphi_\delta$ via its Fourier transform, *viz.*

$$\widehat{\varphi}_\delta(\xi) \;=\; \rho_\delta * (\chi_\delta \cdot \widehat{\eta}).$$

Clearly $\varphi \in H^s(\mathbb{R})$ and $\widehat{\varphi} \in C_c^\infty(\mathbb{R})$ is even and non-negative. The standard theory (see, for example, Appendix C of Evan's text [22]) is straightforwardly adapted to show that $\widehat{\varphi}_\delta \to \widehat{\eta}$ in the weighted space $L_2((1+|2\pi\xi|)^{2s}\, d\xi)$ as $\delta \to 0$. The result in view follows. □

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