

**STABILITY AND DECAY PROPERTIES OF SOLITARY-WAVE
SOLUTIONS TO THE GENERALIZED BO–ZK EQUATION**

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ABSTRACT. Studied here is the generalized Benjamin-Ono–Zakharov-Kuznetsov equation

$$u_t + u^p u_x + \alpha \mathcal{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+, \quad (1)$$

in two space dimensions. Here, \mathcal{H} is the Hilbert transform and subscripts denote partial differentiation. We classify when equation (1) possesses solitary-wave solutions in terms of the signs of the constants α and ε appearing in the dispersive terms and the strength of the nonlinearity. Regularity and decay properties of these solitary wave are determined and their stability is studied.

1. INTRODUCTION

This paper is concerned with existence and non-existence, stability and some decay properties of solitary-wave solutions of the two-dimensional generalized Benjamin-Ono–Zakharov-Kuznetsov equation (BO–ZK equation henceforth),

$$u_t + u^p u_x + \alpha \mathcal{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+. \quad (1.1)$$

Here $p > 0$, α and ε are non-zero real constants with ε normalized to ± 1 by appropriately rescaling the y -variable while \mathcal{H} is the Hilbert transform

$$\mathcal{H}u(x, y, t) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(z, y, t)}{x - z} dz,$$

in the x -variable, where p.v. denotes the Cauchy principal value.

When $p = 1$, this equation arises as a model for electromigration in thin nanoconductors on a dielectric substrate (see [27, 33]). Equation (1.1) may also be viewed as

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one of the natural, two-dimensional generalizations of the one-dimensional Benjamin-Ono equation in much the same way that the Kadomtsev-Petviashvili equation and the Zakharov-Kuznetsov equation generalize the Korteweg-de Vries equation.

The generalized Benjamin-Ono equation

$$u_t + u^p u_x + \alpha \mathcal{H} u_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+,$$

and its counterpart

$$u_t + u^p u_x + \alpha \mathcal{H} u_{xx} + \beta u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+,$$

taking into account surface tension effects between the two layers of fluid, have been considered by many authors. Well-posedness issues for the pure initial-value problem have attracted a lot of interest recently (see, *e.g.* [11, 30, 31, 42, 43, 46]). Questions about the existence and stability of solitary traveling-waves have been investigated in [1]–[7] and [28].

Theory for the generalized Zakharov-Kuznetsov equation

$$u_t + u^p u_x + \alpha u_{xxx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+,$$

is less abundant. Well-posedness was studied in [23, 24, 25, 41, 36, 45]. As far as we know, the only results concerning existence and nonlinear stability of solitary-wave solutions of this equation was provided in [16].

The solitary-wave solutions of interest here have the form $u(x, y, t) = \varphi(x - ct, y)$, where $c \neq 0$ is the speed of propagation and u belongs to a natural function space denoted \mathcal{S} and introduced presently. Substituting this form into (1.1), integrating once with respect to the variable $z = x - ct$ and assuming $\varphi(z, y)$ decays suitably for large values of $|z|$, it transpires that φ must satisfy

$$-c\varphi + \frac{1}{p+1}\varphi^{p+1} + \alpha \mathcal{H} \varphi_x + \varepsilon \varphi_{yy} = 0, \quad (1.2)$$

where we have replaced the variable z by x .

Remark 1.1. *When it is convenient, it may be assumed that (1.2) has the normalized form*

$$-\varphi + \frac{1}{p+1}\varphi^{p+1} + \mathcal{H} \varphi_x \pm \varphi_{yy} = 0 \quad (1.3)$$

by scaling the independent and dependent variables, viz.

$$u(x, y, t) = av(bx, dy, et)$$

where $a^p = c$, $e = b = c/\alpha$ and $d = \varepsilon/c^2$. If instead, we insist that $d > 0$, so $\varepsilon = +1$, then equation (1.2) may be taken in the form

$$-\varphi + \frac{1}{p+1}\varphi^{p+1} \pm \mathcal{H} \varphi_x + \varphi_{yy} = 0. \quad (1.4)$$

Of course, throughout, it will be presumed that the power p appearing in the nonlinearity is rational and has the form k/m where k and m are relatively prime and m is odd. This restriction allows us to define a branch of the mapping $w \mapsto w^{\frac{1}{m}}$ that is real on the real axis.

Attention is now turned to the structure of the paper. The theory begins by examining when solitary-wave solutions of (1.1) exist. As pointed out in [33], no exact formulas are known for solitary-wave solutions to (1.1), so an existence theory is needed before questions of stability can be addressed. Pohojaev-type identities are used to show that solitary-wave solutions do not exist for certain values of p and

signs of ε and α . In some of the cases where such solutions are not prohibited by elementary inequalities, a suitable minimization problem can be solved using Lions' concentration-compactness principle [37, 38] (see Theorem 2.1). For example, our results imply there are solitary-wave solutions when $c > 0$, $\alpha < 0$, $\varepsilon > 0$ and $0 < p < 4$. Moreover, these solutions are shown to be ground states.

With solitary waves in hand, their orbital stability is at issue. The variational approach of Cazenave and Lions [13] comes to the fore in Section 3 in establishing stability for the case $\alpha\varepsilon < 0$, $c\alpha < 0$, and $0 < p < 4/3$. Complementary instability results appeared in [20] for the same conditions on c , α and ε , but with $4/3 < p < 4$.

The regularity and decay properties of the solitary-wave solutions shown to exist in Section 2 are developed in Sections 4 and 5. Solitary-wave solutions are shown to be positive and real analytic. They are symmetric about their peak with respect to both the direction of propagation and the transverse direction. Moreover, solitary waves decay to zero algebraically in the direction of propagation and exponentially in the transverse direction. Some of the results in Section 4 inform the analysis of instability in [20].

In the theory developed here, the issue of well-posedness is not addressed. The presumption throughout is that suitable well-posedness obtains for these models. Detailed analysis of the initial-value problem appeared in [15] and [22].

Remark 1.2. *The scale-invariant Sobolev spaces for the BO-ZK equation (1.1) are $\dot{H}^{s_1, s_2}(\mathbb{R}^2)$, where $2s_1 + s_2 = \frac{3}{2} - \frac{2}{p}$ (see the definitions below). Hence a reasonable framework for studying local well-posedness of the BO-ZK equation (1.1) is the family of spaces $H^{s_1, s_2}(\mathbb{R}^2)$, $2s_1 + s_2 \geq \frac{3}{2} - \frac{2}{p}$.*

Remark 1.3. *The n -dimensional version of (1.1) is*

$$u_t + u^p u_{x_1} + \alpha \mathcal{H} u_{x_1 x_1} + \sum_{i=2}^n \varepsilon_i u_{x_1 x_i x_i} = 0, \quad (1.5)$$

where $t \in \mathbb{R}^+$, $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $\alpha, \varepsilon_i \in \mathbb{R}$, $i = 2, \dots, n$. The theory developed here has natural analogs for (1.5) which will be developed later.

Notation and Preliminaries. As already mentioned, the exponent p in (1.1) is taken to be a rational number of the form $p = k/m$, where m and k are relatively prime and m is odd. This allows the nonlinearity to be given a definition that is real-valued. The notation $\hat{f} = \hat{f}(\xi, \eta)$ means the Fourier transform,

$$\hat{f}(\xi, \eta) = \int_{\mathbb{R}^2} e^{-i(x\xi + y\eta)} f(x, y) dx dy$$

of $f = f(x, y)$. For any $s \in \mathbb{R}$, the space $H^s := H^s(\mathbb{R}^2)$ denotes the usual isotropic, $L^2(\mathbb{R}^2)$ -based, Sobolev space. For $s_1, s_2 \in \mathbb{R}$, the anisotropic Sobolev space $H^{s_1, s_2} := H^{s_1, s_2}(\mathbb{R}^2)$ is the set of all distributions f such that

$$\|f\|_{H^{s_1, s_2}}^2 = \int_{\mathbb{R}^2} (1 + \xi^2)^{s_1} (1 + \eta^2)^{s_2} |\hat{f}(\xi, \eta)|^2 d\xi d\eta < \infty.$$

The fractional Sobolev-Liouville spaces $H_p^{(s_1, s_2)} := H_p^{(s_1, s_2)}(\mathbb{R}^2)$, $1 \leq p < \infty$, are the set of all functions $f \in L^p(\mathbb{R}^2)$ such that

$$\|f\|_{H_p^{(s_1, s_2)}} = \|f\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^2 \|D_{x_i}^{s_i} f\|_{L^p(\mathbb{R}^2)} < \infty,$$

where $D_{x_i}^{s_i} f$ denotes the Bessel derivative of order s_i with respect to x_i (see *e.g.* [32], [39]). For short, $H_p^{(k)}(\mathbb{R}^2)$ denotes the space $H_p^{(k,k)}(\mathbb{R}^2)$.

The particular space $\mathcal{Z} := H^{\frac{1}{2},0}(\mathbb{R}^2) \cap H^{0,1}(\mathbb{R}^2) = H^{(\frac{1}{2},1)}(\mathbb{R}^2)$ arises naturally in the analysis to follow. It can be characterized alternatively as the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|\varphi\|_{\mathcal{Z}}^2 = \|\varphi\|_{L^2(\mathbb{R}^2)}^2 + \|\varphi_y\|_{L^2(\mathbb{R}^2)}^2 + \left\| D_x^{1/2} \varphi \right\|_{L^2(\mathbb{R}^2)}^2, \quad (1.6)$$

where $D_x^{1/2} \varphi$ denotes the fractional derivative of order 1/2 with respect to x , defined via its Fourier transform by $\widehat{D_x^{1/2} \varphi}(\xi, \eta) = |\xi|^{1/2} \widehat{\varphi}(\xi, \eta)$.

Remark 1.4. *By combining fractional Gagliardo-Nirenberg and Hölder's inequality one can deduce the existence of a positive constant C such that*

$$\|u\|_{L^{p+2}}^{p+2} \leq C \|u\|_{L^2}^{(4-p)/2} \|D_x^{1/2} u\|_{L^2}^p \|u_y\|_{L^2}^{p/2}, \quad 0 \leq p < 4. \quad (1.7)$$

This in turn implies the continuous embedding

$$\mathcal{Z} \hookrightarrow L^p(\mathbb{R}^2), \quad 0 \leq p < 4. \quad (1.8)$$

2. SOLITARY WAVES

This section is devoted to establishing existence and non-existence results for solitary-wave solutions of the BO-ZK equations. We begin with a non-existence result.

Theorem 2.1. *Equation (1.2) cannot have a non-trivial solitary-wave solution unless either*

- (i) $\varepsilon = 1, c > 0, \alpha < 0, p < 4,$
- (ii) $\varepsilon = -1, c < 0, \alpha > 0, p < 4,$
- (iii) $\varepsilon = 1, c < 0, \alpha < 0, p > 4,$ or
- (iv) $\varepsilon = -1, c > 0, \alpha > 0, p > 4.$

Proof. This follows from some Pohojaev-type identities. If (1.2) is multiplied by φ , $x\varphi_x$ and $y\varphi_y$ and the results integrated over \mathbb{R}^2 , then the identities

$$\int_{\mathbb{R}^2} \left(-c\varphi^2 + \alpha\varphi\mathcal{H}\varphi_x - \varepsilon\varphi_y^2 + \frac{1}{p+1}\varphi^{p+2} \right) dx dy = 0, \quad (2.1)$$

$$\int_{\mathbb{R}^2} \left(c\varphi^2 + \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2} \right) dx dy = 0, \quad (2.2)$$

$$\int_{\mathbb{R}^2} \left(c\varphi^2 - \alpha\varphi\mathcal{H}\varphi_x - \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2} \right) dx dy = 0, \quad (2.3)$$

emerge. These formulas follow from the elementary properties of the Hilbert transform together with suitably chosen formal integrations by parts. The identities can be justified for functions of the minimal regularity required for them to make sense by first establishing them for smooth solutions and then using a standard truncation argument as in [17].

Summing (2.1) and (2.2) leads to

$$\int_{\mathbb{R}^2} \left(\alpha\varphi\mathcal{H}\varphi_x + \frac{p}{(p+1)(p+2)}\varphi^{p+2} \right) dx dy = 0, \quad (2.4)$$

whilst adding (2.2) and (2.3) yields

$$\int_{\mathbb{R}^2} \left(c\varphi^2 - \frac{\alpha}{2}\varphi\mathcal{H}\varphi_x - \frac{2}{p+1}\varphi^{p+2} \right) dx dy = 0. \quad (2.5)$$

If the integral of φ^{p+2} is eliminated between (2.4) and (2.5), there appears

$$\int_{\mathbb{R}^2} (2pc\varphi^2 + \alpha(4-p)\varphi\mathcal{H}\varphi_x) dx dy = 0. \quad (2.6)$$

On the other hand, adding (2.1) and (2.3) gives

$$\int_{\mathbb{R}^2} \left(2\varepsilon\varphi_y^2 - \frac{p}{(p+1)(p+2)}\varphi^{p+2} \right) dx dy = 0. \quad (2.7)$$

Finally, substituting (2.2) into (2.7), there obtains

$$\int_{\mathbb{R}^2} (pc\varphi^2 + \varepsilon(p-4)\varphi_y^2) dx dy = 0. \quad (2.8)$$

The advertised results follow immediately from (2.6) and (2.8). \square

For cases (i) and (ii) from Theorem 2.1, the existence of solitary-wave solutions of (1.1) is established in the next result.

Theorem 2.2. *Let $\alpha\varepsilon, c\alpha < 0$ and $p = \frac{k}{m} < 4$, where $m \in \mathbb{N}$ is odd and m and k are relatively prime. Then equation (1.2) admits a non-trivial solution $\varphi \in \mathcal{Z}$.*

Proof. The proof is based on the concentration-compactness principle [37, 38]. Suppose that $\alpha < 0$ (the proof for $\alpha > 0$ is similar). Without loss of generality, assume that $\alpha = -1$ and $c = 1$ so that $\varepsilon = +1$ (see Remark 1.1) and consider the minimization problem

$$I_\lambda = \inf \left\{ I(\varphi) ; \varphi \in \mathcal{Z}, J(\varphi) = \int_{\mathbb{R}^2} \varphi^{p+2} dx dy = \lambda \right\} \quad (2.9)$$

where $\lambda \neq 0$ and

$$I(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} (\varphi^2 + \varphi\mathcal{H}\varphi_x + \varphi_y^2) dx dy = \frac{1}{2} \|\varphi\|_{\mathcal{Z}}^2.$$

Clearly, $I_\lambda < \infty$ if there are elements $\varphi \in \mathcal{Z}$ such that $\int_{\mathbb{R}^2} \varphi^{p+2} dx dy = \lambda$.¹ The embedding (1.8) allows us to adduce a positive constant C such that

$$0 < |\lambda| = \left| \int_{\mathbb{R}^2} \varphi^{p+2} dx dy \right| \leq C \|\varphi\|_{\mathcal{Z}}^{p+2} = CI(\varphi)^{\frac{p+2}{2}},$$

from which one concludes that $I_\lambda \geq \left(\frac{|\lambda|}{C} \right)^{\frac{2}{p+2}} > 0$.

For suitable λ let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for I_λ . For $n = 1, 2, \dots$ and $r > 0$, define the concentration function $Q_n(r)$ associated to φ_n by

$$Q_n(r) = \sup_{(\bar{x}, \bar{y}) \in \mathbb{R}^2} \int_{B_r(\bar{x}, \bar{y})} \rho_n dx dy$$

¹Depending on p , this might require that $\lambda > 0$. Of course, I_λ is a number, but we will sometimes refer to it as the minimization problem. For example, the phrase “ $\{\phi_n\}$ is a minimizing sequence for the problem I_λ ” means that $J(\phi_n) = \lambda$ for all n and $I(\phi_n) \rightarrow I_\lambda$ as $n \rightarrow \infty$.

where $\rho_n = |\varphi_n|^2 + \left|D_x^{1/2}\varphi_n\right|^2 + |\partial_y\varphi_n|^2$ and $B_r(x, y)$ denotes the ball of radius $r > 0$ centered at $(x, y) \in \mathbb{R}^2$. If evanescence of the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ occurs, which is to say, for any $r > 0$,

$$\lim_{n \rightarrow +\infty} \sup_{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2} \int_{B_r(\tilde{x}, \tilde{y})} \rho_n \, dx dy = 0,$$

then embedding (1.8) implies that $\lim_{n \rightarrow \infty} \|\varphi_n\|_{L^{p+2}} = 0$, which contradicts the constraint imposed for the minimization problem. Thus, according to the concentration-compactness theorem, either dichotomy or compactness must occur for the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$.

The occurrence of dichotomy is ruled out next. Suppose that $\gamma \in (0, I_\lambda)$, where it is assumed that

$$\gamma = \lim_{r \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sup_{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2} \int_{B_r(\tilde{x}, \tilde{y})} \rho_n \, dx dy.$$

By the definition of γ , for a given $\epsilon > 0$, there exist $r_1 \in \mathbb{R}$ and $N \in \mathbb{N}$ such that

$$\gamma - \epsilon < Q_n(r) \leq Q_n(2r) < \gamma + \epsilon,$$

for any $r \geq r_1$ and $n \geq N$. Hence, there is a sequence $\{(\tilde{x}_n, \tilde{y}_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^2$ for which

$$\int_{B_r(\tilde{x}_n, \tilde{y}_n)} \rho_n \, dx dy > \gamma - \epsilon \quad \text{and} \quad \int_{B_{2r}(\tilde{x}_n, \tilde{y}_n)} \rho_n \, dx dy < \gamma + \epsilon.$$

Let ϕ, ψ lie in $C^\infty(\mathbb{R}^2)$ and suppose

- $\text{supp } \phi \subset B_2(0, 0)$, $\phi \equiv 1$ on $B_1(0, 0)$ and $0 \leq \phi \leq 1$,
- $\text{supp } \psi \subset \mathbb{R}^2 \setminus B_1(0, 0)$, $\psi \equiv 1$ on $\mathbb{R}^2 \setminus B_2(0, 0)$ and $0 \leq \psi \leq 1$.

Define the sequences $\{g_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$ by

$$g_n(x, y) = \phi_r((x, y) - (\tilde{x}_n, \tilde{y}_n))\varphi_n \quad \text{and} \quad h_n(x, y) = \psi_r((x, y) - (\tilde{x}_n, \tilde{y}_n))\varphi_n,$$

where

$$\phi_r(x, y) = \phi\left(\frac{(x, y)}{r}\right) \quad \text{and} \quad \psi_r(x, y) = \psi\left(\frac{(x, y)}{r}\right).$$

It is clear that $g_n, h_n \in \mathcal{L}$.

The following commutator estimate is helpful in obtaining the splitting lemma to follow.

Lemma 2.3 ([12, 14]). *Let $g \in C^\infty(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$. Then $[\mathcal{H}, g]\partial_x$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ with*

$$\|[\mathcal{H}, g]\partial_x f\|_{L^2(\mathbb{R})} \leq C\|g'\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

The splitting lemma proved next enables us to rule out the possibility of dichotomy occurring in the present context.

Lemma 2.4. *Let $\{g_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$ be as just defined. Then, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ with $\lim_{\epsilon \rightarrow 0} \delta(\epsilon) = 0$, $\mu \in (0, I_\lambda)$, $n_0 \in \mathbb{N}$ and $\rho \in (0, \lambda)$*

such that for all $n \geq n_0$,

$$|I(\varphi_n) - I(g_n) - I(h_n)| \leq \delta, \quad (2.10)$$

$$|I(g_n) - \mu| \leq \delta, \quad |I(h_n) - I_\lambda + \mu| \leq \delta, \quad (2.11)$$

$$|J(\varphi_n) - J(g_n) - J(h_n)| \leq \delta, \quad (2.12)$$

$$|J(g_n) - \rho| \leq \delta, \quad |J(h_n) - \lambda + \rho| \leq \delta. \quad (2.13)$$

Proof. Obviously, $\text{supp } g_n \cap \text{supp } h_n = \emptyset$. Write $g_n = \phi_r \varphi_n$ and $h_n = \psi_r \varphi_n$ so that

$$\begin{aligned} 2I(g_n) &= \int_{\mathbb{R}^2} \phi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathcal{H} \varphi_n + (\partial_y^2 \varphi_n)^2 \right] dx dy + 2 \int_{\mathbb{R}^2} \phi_r \varphi_n (\partial_y \phi_r) (\partial_y \varphi_n) dx dy \\ &\quad + \int_{\mathbb{R}^2} [(\partial_y \phi_r)^2 \varphi_n^2 + \varphi_n \phi_r \mathcal{H} (\varphi_n \partial_x \phi_r)] dx dy + \int_{\mathbb{R}^2} \varphi_n \phi_r [\mathcal{H}, \phi_r] \partial_x \varphi_n dx dy \end{aligned}$$

and

$$\begin{aligned} 2I(h_n) &= \int_{\mathbb{R}^2} \psi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathcal{H} \varphi_n + (\partial_y^2 \varphi_n)^2 \right] dx dy + 2 \int_{\mathbb{R}^2} \psi_r \varphi_n (\partial_y \psi_r) (\partial_y \varphi_n) dx dy \\ &\quad + \int_{\mathbb{R}^2} [(\partial_y \psi_r)^2 \varphi_n^2 + \varphi_n \psi_r \mathcal{H} (\varphi_n \partial_x \psi_r)] dx dy + \int_{\mathbb{R}^2} \varphi_n \psi_r [\mathcal{H}, \psi_r] \partial_x \varphi_n dx dy. \end{aligned}$$

Since $\|\phi_r\|_{L^\infty} = \|\psi_r\|_{L^\infty} = 1$, $\|\nabla \phi_r\|_{L^\infty} \leq \frac{1}{r} \|\nabla \phi\|_{L^\infty}$ and $\|\nabla \psi_r\|_{L^\infty} \leq \frac{1}{r} \|\nabla \psi\|_{L^\infty}$, it follows from Lemma 2.3 that

$$\left| I(g_n) - \frac{1}{2} \int_{\mathbb{R}^2} \phi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathcal{H} \varphi_n + (\partial_y^2 \varphi_n)^2 \right] dx dy \right| \leq \frac{1}{2} \delta(\epsilon)$$

and

$$\left| I(h_n) - \frac{1}{2} \int_{\mathbb{R}^2} \psi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathcal{H} \varphi_n + (\partial_y^2 \varphi_n)^2 \right] dx dy \right| \leq \frac{1}{2} \delta(\epsilon).$$

These inequalities imply (2.10), from which, one infers (taking subsequences if necessary) that there exists $\mu = \mu(\epsilon) \in [0, I_\lambda]$ such that $\lim_{n \rightarrow \infty} I(g_n) = \mu$. In consequence, we see that

$$|I(g_n) - I_\lambda + \mu| \leq \delta(\epsilon).$$

From (2.10) again, the fact that $\text{supp } g_n \cap \text{supp } h_n = \emptyset$ and the embedding (1.8), one obtains

$$|J(\varphi_n) - J(g_n) - J(h_n)| \leq C\delta(\epsilon)$$

for some constant C . It may therefore be presumed that there is a $\rho = \rho(\epsilon)$ and $\tilde{\rho} = \tilde{\rho}(\epsilon)$ such that

$$\lim_{n \rightarrow +\infty} J(g_n) = \rho(\epsilon), \quad \lim_{n \rightarrow +\infty} J(h_n) = \tilde{\rho}(\epsilon)$$

with $|\lambda - \rho(\epsilon) - \tilde{\rho}(\epsilon)| \leq \delta(\epsilon)$. If $\lim_{\epsilon \rightarrow 0} \rho(\epsilon) = 0$, then for ϵ sufficiently small, it must be that $J(h_n) > 0$ for n large enough. Hence, by considering $(\tilde{\rho}(\epsilon) J(h_n))^{\frac{1}{p+2}} h_n$, and noting that $J\left(\left(\tilde{\rho}(\epsilon) J(h_n)\right)^{\frac{1}{p+2}} h_n\right) = \tilde{\rho}(\epsilon)$, it transpires that

$$I_{\tilde{\rho}(\epsilon)} \leq \liminf_{n \rightarrow +\infty} I(h_n) \leq I_\lambda - \gamma + \delta(\epsilon),$$

which leads to a contradiction since $\lim_{\epsilon \rightarrow 0} \tilde{\rho}(\epsilon) = \lambda$. Thus $\rho = \lim_{\epsilon \rightarrow 0} \rho(\epsilon) > 0$. Necessarily $\rho < \lambda$, because the case $\rho = \lambda$ is ruled out in the same manner as just used to rule out $\rho = 0$, but with h_n replacing g_n in the argument. Since $\rho \in (0, \lambda)$, one infers that necessarily $\mu = \lim_{\epsilon \rightarrow +\infty} \mu(\epsilon) \in (0, I_\lambda)$. This completes the proof of the lemma. \square

Now, attention is returned to the proof that dichotomy cannot happen. The previous lemma implies that

$$I_\lambda \geq I_\rho + I_{\lambda-\rho}, \quad (2.14)$$

which contradicts the subadditivity of I_λ coming from the fact that $I_\lambda = \lambda^{2/(p+2)}I_1$. Hence dichotomy is ruled out.

The remaining case in the concentration-compactness principle is local compactness. Thus, there exists a sequence $\{(x_n, y_n)\}_{n \in \mathbb{N}} \subset \mathbb{R}^2$ such that for all $\epsilon > 0$, there are finite values $R > 0$ and $n_0 > 0$ with

$$\int_{B_R(x_n, y_n)} \rho_n \, dx dy \geq \iota_\lambda - \epsilon,$$

for all $n \geq n_0$, where

$$\iota_\lambda = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^2} \rho_n \, dx dy.$$

This implies that for n large enough,

$$\int_{B_R(x_n, y_n)} |\varphi_n|^2 \, dx dy \geq \int_{\mathbb{R}^2} |\varphi_n|^2 \, dx dy - 2\epsilon.$$

Since φ_n is bounded in the Hilbert space \mathcal{X} , there exists $\varphi \in \mathcal{X}$ such that a subsequence of $\{\varphi_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$ (denoted again by $\{\varphi_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$) converges weakly in \mathcal{X} to φ . It follows that

$$\begin{aligned} \int_{\mathbb{R}^2} |\varphi|^2 \, dx dy &\leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^2} |\varphi_n|^2 \, dx dy \\ &\leq \liminf_{n \rightarrow +\infty} \int_{B_R(x_n, y_n)} |\varphi_n|^2 \, dx dy + 2\epsilon \\ &= \liminf_{n \rightarrow +\infty} \int_{B_R(0,0)} |\varphi_n((x, y) - (x_n, y_n))|^2 \, dx dy + 2\epsilon. \end{aligned}$$

But, when restricted to bounded sets in \mathbb{R}^2 , \mathcal{X} is compactly embedded into L^2 . Consequently, $\{\varphi_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$ may be presumed to converges strongly in the Fréchet space $L^2_{loc}(\mathbb{R}^2)$. The last inequality above implies that this strong convergence also takes place in $L^2(\mathbb{R}^2)$ by what are, by now, standard arguments. Thus, because of the embedding (1.8), $\{\varphi_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$ also converges to φ strongly in $L^{p+2}(\mathbb{R}^2)$, whence $J(\varphi) = \lambda$ and

$$I_\lambda = \lim_{n \rightarrow +\infty} I(\varphi_n) = I(\varphi),$$

which is to say, φ is a solution of I_λ .

The Lagrange multiplier theorem now implies there exists $\theta \in \mathbb{R}$ such that

$$\varphi + \mathcal{H}\varphi_x - \varphi_{yy} = \theta(p+2)\varphi^{p+1} \quad (2.15)$$

as an equation in \mathcal{X}' (the dual space of \mathcal{X} in L^2 -duality). A change of scale yields a $\tilde{\varphi}$ which satisfies (1.2). \square

Remark 2.5. *Theorem 2.2 shows the existence of solitary-wave solutions of (1.1) in the cases (i) and (ii) in Theorem 2.1. The question of existence or nonexistence of solitary waves in cases (iii) and (iv) is currently open.*

Definition 2.6. A solution φ of equation (1.2) is called a ground state, if φ minimizes the action

$$\mathcal{S}(u) = \mathcal{E}(u) + c\mathcal{F}(u)$$

among all solutions of (1.2), where

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 \, dx dy$$

and

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\varepsilon u_y^2 - \alpha u \mathcal{H} u_x - \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dx dy.$$

Next, it is established that the minima obtained in Theorem 2.2 are precisely the ground-state solutions of (1.2). The proof is inspired by that of Lemma 2.1 in [18].

Theorem 2.7. In the context of equation (1.2) for solitary-wave solutions of the BO-ZK equation, let

$$\mathcal{K}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (cu^2 + u_y^2) \, dx dy - \frac{1}{(p+1)(p+2)} J(u)$$

with $J(u) = \int u^{p+2} \, dx dy$ as in (2.9). Up to a change of scale, the following assertions about a function $u^* \in \mathcal{L}$ are equivalent:

- (i) If $J(u^*) = \lambda^*$ then u^* is a minimizer of I_{λ^*} ,
- (ii) $\mathcal{K}(u^*) = 0$ and

$$\inf \left\{ \int_{\mathbb{R}^2} u \mathcal{H} u_x \, dx dy, u \in \mathcal{L}, u \neq 0, \mathcal{K}(u) = 0 \right\} = \int_{\mathbb{R}^2} u^* \mathcal{H} u_x^* \, dx dy,$$

- (iii) u^* is a ground state,
- (iv) $\mathcal{K}(u^*) = 0$ and

$$\inf \left\{ \mathcal{K}(u), u \in \mathcal{L}, u \neq 0, \int_{\mathbb{R}^2} u \mathcal{H} u_x \, dx dy = \int_{\mathbb{R}^2} u^* \mathcal{H} u_x^* \, dx dy \right\} = 0.$$

Proof. We set $\lambda^* = (2(p+1)I_1)^{\frac{p+2}{p}}$ and proceed with the proof.

(i) \Rightarrow (ii): Assume that u^* satisfies (i). Let $u \in \mathcal{L}$ with $u \neq 0$ and $\mathcal{K}(u) = 0$, from which it follows that $J(u) > 0$. Define

$$u_\mu(x, y) = u \left(\frac{x}{\mu}, y \right), \quad \text{with } \mu = \frac{J(u^*)}{J(u)},$$

so that $J(u_\mu) = J(u^*)$ and $\mathcal{K}(u_\mu) = 0$. Since u^* is a minimum of I_{λ^*} , it must be the case that $\mathcal{K}(u^*) = 0$ and

$$\mathcal{K}(u^*) + C_p J(u^*) + \frac{1}{2} \int_{\mathbb{R}^2} u^* \mathcal{H} u_x^* \, dx dy \leq \mathcal{K}(u_\mu) + C_p J(u_\mu) + \frac{1}{2} \int_{\mathbb{R}^2} u_\mu \mathcal{H} (u_\mu)_x \, dx dy,$$

where $C_p = \frac{1}{(p+1)(p+2)}$. This in turn implies that

$$\int_{\mathbb{R}^2} u^* \mathcal{H} u_x^* \, dx dy \leq \int_{\mathbb{R}^2} u \mathcal{H} u_x \, dx dy,$$

and (ii) holds.

(ii) \Rightarrow (iii) : If u^* satisfies (ii), then there is a Lagrange multiplier θ such that

$$cu^* - u_{yy}^* + \theta \mathcal{H}u_x^* - \frac{1}{p+1}(u^*)^{p+1} = 0.$$

By multiplying the above equation by u^* , integrating by parts and using that $\mathcal{K}(u^*) = 0$, we can see that θ is positive. Hence the scale change $u_*(x, y) = u^*(x/\theta, y)$ satisfies equation (1.2).

On the other hand, the identity $S(u) = \mathcal{K}(u) + \frac{1}{2} \int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy$ shows that if u is a solution of (1.2), then

$$S(u) = \frac{1}{2} \int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy \geq \frac{1}{2} \int_{\mathbb{R}^2} u^* \mathcal{H}u_x^* dx dy = \frac{1}{2} \int_{\mathbb{R}^2} u_* \mathcal{H}(u_*)_x dx dy = S(u^*),$$

whence u^* is a ground state.

(iii) \Rightarrow (i) : From the proof of Theorem 2.1, one sees that if u is a solution of (1.2), then $\mathcal{K}(u) = 0$ and

$$I(u) = \frac{1}{2} \left(1 + \frac{2}{p} \right) \int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy. \quad (2.16)$$

Hence if u^* is a ground state, then u^* minimizes both $I(u)$ and $\int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy$ among all solutions of (1.2). Let $\lambda = J(u)$ and \tilde{u} be a minimum of I_λ . Then

$$I_\lambda = I(\tilde{u}) \leq I(u^*) \quad (2.17)$$

and there is a positive number θ such that

$$c\tilde{u} - \tilde{u}_{yy} + \mathcal{H}\tilde{u}_x = \frac{\theta}{p+1} \tilde{u}^{p+1}.$$

Using the equations satisfied by \tilde{u} and u^* , inequality (2.17) is written as

$$I_\lambda = \frac{\lambda\theta}{p+1} \leq \frac{\lambda}{p+1},$$

from which it is deduced immediately that $\theta \leq 1$. On the other hand, $u_* = \theta^p \tilde{u}$ satisfies equation (1.2), and since u^* is a ground state, it must be the case that

$$I(u^*) \leq I(u_*) \leq \theta^{2p} I(\tilde{u}),$$

so that $\theta \geq 1$. In consequence, $u^* = \tilde{u}$ is a minimum of I_λ with $\lambda = \lambda^*$.

(ii) \Rightarrow (iv) : Let $u \in \mathcal{L}$ with $\int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy = \int_{\mathbb{R}^2} u^* \mathcal{H}u_x^* dx dy$. Suppose that $\mathcal{K}(u) < 0$. Since $\mathcal{K}(\tau u) > 0$ for $\tau > 0$ sufficiently small, then there is a $\tau_0 \in (0, 1)$ such that $\mathcal{K}(\tau_0 u) = 0$. Thus by setting $\tilde{u} = \tau_0 u$, one has $\tilde{u} \in \mathcal{L}$, $\mathcal{K}(\tilde{u}) = 0$ and

$$\int_{\mathbb{R}^2} \tilde{u} \mathcal{H}\tilde{u}_x dx dy < \int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy = \int_{\mathbb{R}^2} u^* \mathcal{H}u_x^* dx dy,$$

which contradicts (ii) and shows that u^* satisfies (iv) because $\mathcal{K}(u^*) = 0$.

(iv) \Rightarrow (ii) : Let $u \in \mathcal{L}$ with $\mathcal{K}(u) = 0$ and $u \neq 0$. Suppose that

$$\int_{\mathbb{R}^2} u \mathcal{H}u_x dx dy < \int_{\mathbb{R}^2} u^* \mathcal{H}u_x^* dx dy.$$

Since $\mathcal{K}(\tau u) < 0$ for $\tau > 1$, there is a $\tau_0 > 1$ with

$$\int_{\mathbb{R}^2} (\tau_0 u) \mathcal{H}(\tau_0 u)_x dx dy = \int_{\mathbb{R}^2} u^* \mathcal{H}u_x^* dx dy$$

and $\mathcal{K}(\tau_0 u) < 0$. This contradicts (iv). Hence $\int_{\mathbb{R}^2} u \mathcal{H} u_x \, dx dy \geq \int_{\mathbb{R}^2} u^* \mathcal{H} u_x^* \, dx dy$ and (ii) holds. \square

Remark 2.8. *Note that the proof of the above theorem shows that, indeed, (i) and (iii) are equivalent and imply (ii) and (iv), which are also equivalent. The converse holds modulo a scale change.*

3. STABILITY

The notion of orbital stability employed here is the standard one.

Definition 3.1. *Let φ_c be a solitary-wave solution of (1.1). We say that φ_c is orbitally stable if for all $\eta > 0$, there is a $\delta > 0$ such that for any $u_0 \in H^s(\mathbb{R}^2)$, $s > 2$, with $\|u_0 - \varphi_c\|_{\mathcal{X}} < \delta$, the corresponding solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies*

$$\sup_{t \geq 0} \inf_{r \in \mathbb{R}^2} \|u(t) - \varphi_c(\cdot - r)\|_{\mathcal{X}} < \eta.$$

Some of the arguments below can be found in [3] where the stability of solitary waves for the generalized BO equation has been established. Hereafter, without loss of generality, we take $\alpha = -1$ so that $\varepsilon = +1$, and $c > 0$.

The following theorem is a consequence of Theorem 2.2 and it will be used to obtain the stability results.

Theorem 3.2. *Let $\lambda \neq 0$.*

- (i) *Every minimizing sequence for the problem I_λ converges, up to translations, in \mathcal{X} to an element in the set*

$$M_\lambda = \{\varphi \in \mathcal{X}; I(\varphi) = I_\lambda, J(\varphi) = \lambda\}$$

of minimizers for I_λ .

- (ii) *Let $\{\varphi_n\}$ be a minimizing sequence for I_λ . Then, it must be the case that*

$$\lim_{n \rightarrow +\infty} \inf_{\psi \in M_\lambda, z \in \mathbb{R}^2} \|\varphi_n(\cdot + z) - \psi\|_{\mathcal{X}} = 0, \quad (3.1)$$

$$\lim_{n \rightarrow +\infty} \inf_{\psi \in M_\lambda} \|\varphi_n - \psi\|_{\mathcal{X}} = 0. \quad (3.2)$$

Proof. Part (i) follows immediately from the proof of Theorem 2.2. The equality (3.1) is proved by contradiction. Indeed, if (3.1) does not hold, then there exists a subsequence of the sequence $\{\varphi_n\}$, denoted again by $\{\varphi_n\}$, and an $\varepsilon > 0$ such that

$$\varpi = \inf_{\psi \in M_\lambda, r \in \mathbb{R}^2} \|\varphi_n(\cdot + r) - \psi\|_{\mathcal{X}} \geq \varepsilon,$$

for all n sufficiently large. On the other hand, since $\{\varphi_n\}$ is a minimizing sequence for I_λ , part (i) implies that there exists a sequence $\{r_n\} \subset \mathbb{R}^2$ such that, up to a subsequence, $\varphi_n(\cdot + r_n) \rightarrow \varphi$ in \mathcal{X} , as $n \rightarrow \infty$. Hence, for n large enough, it is inferred that

$$\frac{\varepsilon}{2} \geq \|\varphi_n(\cdot + r_n) - \varphi\|_{\mathcal{X}} \geq \varpi \geq \varepsilon,$$

which is a contradiction.

The proof of (3.2) follows from (3.1), the fact that if $\psi \in M_\lambda$ then $\psi(\cdot + r) \in M_\lambda$ for all $r \in \mathbb{R}^2$, and the equality

$$\inf_{\psi \in M_\lambda} \|\varphi_n - \psi\|_{\mathcal{X}} = \inf_{\psi \in M_\lambda, r \in \mathbb{R}^2} \|\varphi_n - \psi(\cdot - r)\|_{\mathcal{X}} = \inf_{\psi \in M_\lambda, r \in \mathbb{R}^2} \|\varphi_n(\cdot + r) - \psi\|_{\mathcal{X}}.$$

This completes the proof of the theorem. \square

The next lemma shows that there exists a $\lambda > 0$ such that every element in the set of minimizers satisfies (1.2).

Lemma 3.3. *If $\lambda = (2(p+1)I_1)^{\frac{p+2}{p}}$ in the minimization problem (2.9), then any $\varphi \in M_\lambda$ is a solitary-wave solution of (1.2).*

For λ as in the preceding lemma, define the set

$$\mathcal{N}_c = \{\varphi \in \mathcal{Z}; J(\varphi) = 2(p+1)I(\varphi) = \lambda\}.$$

It is clear that $M_\lambda = \mathcal{N}_c$; the latter notation simply emphasizes the dependence upon the wave speed c . Next, for any $c > 0$ and any $\varphi \in \mathcal{N}_c$, define the function $d: \mathbb{R} \rightarrow \mathbb{R}$ by

$$d(c) = E(\varphi) + c\mathcal{F}(\varphi). \quad (3.3)$$

Lemma 3.4. *The function d in (3.3) is constant on \mathcal{N}_c and differentiable and strictly increasing for $c > 0$. Moreover, $d''(c) > 0$ if and only if $0 < p < \frac{4}{3}$.*

Proof. It is straightforward to check that

$$d(c) = I(\varphi) - \frac{1}{(p+1)(p+2)}J(\varphi) = \frac{p}{2(p+1)(p+2)}J(\varphi) = \frac{p(2(p+1))^{\frac{2}{p}}}{p+2}I_1^{\frac{p+2}{p}}. \quad (3.4)$$

It is plain that d is constant on \mathcal{N}_c . From the second equality in (3.4) and the definition of J , one obtains

$$d(c) = \frac{p}{2(p+1)(p+2)}c^{\frac{2}{p}-\frac{1}{2}}J(\psi), \quad (3.5)$$

where $\psi(x, y) = c^{-\frac{1}{p}}\varphi\left(\frac{x}{c}, \frac{y}{\sqrt{c}}\right)$. Note that ψ satisfies (1.2), with $c = 1$. But, from (2.4) and (2.6), one infers that

$$\frac{1}{(p+1)(p+2)}J(\varphi) = \frac{2c}{4-p}\mathcal{F}(\varphi).$$

Thus, from (3.5) follows the formula

$$d'(c) = c^{(\frac{2}{p}-\frac{3}{2})}\mathcal{F}(\psi),$$

whence

$$d''(c) = \left(\frac{2}{p} - \frac{3}{2}\right)c^{(\frac{2}{p}-\frac{5}{2})}\mathcal{F}(\psi).$$

This proves the lemma. \square

A study is initiated of the behavior of d in a neighborhood of the set \mathcal{N}_c .

Lemma 3.5. *Let $c > 0$. Then, there exists a positive number ϵ and a C^1 -map $v: \mathcal{B}_\epsilon(\mathcal{N}_c) \rightarrow (0, +\infty)$ defined by*

$$v(u) = d^{-1}\left(\frac{p}{2(p+1)(p+2)}J(u)\right),$$

such that $v(\varphi) = c$ for every $\varphi \in \mathcal{N}_c$, where

$$\mathcal{B}_\epsilon(\mathcal{N}_c) = \left\{ \varphi \in \mathcal{Z}; \inf_{\psi \in \mathcal{N}_c} \|\varphi - \psi\|_{\mathcal{Z}} < \epsilon \right\}.$$

Proof. By definition, \mathcal{N}_c is a bounded set in \mathcal{Z} . Moreover,

$$\mathcal{N}_c \subset B(0, r) \subset \mathcal{Z},$$

where $r = (2(p+1))^{\frac{2}{p}} I_1^{\frac{p+2}{p}}$ and $B(0, r)$ is the ball of radius $r > 0$ centered at the origin in \mathcal{Z} . Let $\rho > 0$ be sufficiently large that $\mathcal{N}_c \subset B(0, \rho) \subset \mathcal{Z}$. Since the function $u \mapsto J(u)$ is uniformly continuous on bounded sets, there exists $\epsilon > 0$ such that if $u, v \in B(0, \rho)$ and $\|u - v\|_{\mathcal{Z}} < 2\epsilon$ then $|J(u) - J(v)| < \rho$. Considering the neighborhoods $\mathcal{I} = (d(c) - \rho, d(c) + \rho)$ and $\mathcal{B}_\epsilon(\mathcal{N}_c)$ of $d(c)$ and \mathcal{N}_c , respectively, we have that if $u \in \mathcal{B}_\epsilon(\mathcal{N}_c)$ then $J(u) \in \mathcal{I}$. Therefore v is well defined on $\mathcal{B}_\epsilon(\mathcal{N}_c)$ and satisfies $v(\varphi) = c$, for all $\varphi \in \mathcal{N}_c$. \square

Here is the crucial inequality in the study of stability.

Lemma 3.6. *Let $c > 0$ and suppose that $d''(c) > 0$. Then for all $u \in \mathcal{B}_\epsilon(\mathcal{N}_c)$ and any $\varphi \in \mathcal{N}_c$,*

$$\mathcal{E}(u) - \mathcal{E}(\varphi) + v(u) (\mathcal{F}(u) - \mathcal{F}(\varphi)) \geq \frac{1}{4} d''(c) |v(u) - c|^2.$$

Proof. For $\omega > 0$, let I_ω be the functional

$$I_\omega(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} (\omega \varphi^2 + \varphi \mathcal{H} \varphi_x + \varphi_y^2) \, dx dy.$$

It follows that

$$\mathcal{E}(u) + v(u) \mathcal{F}(u) = I_{v(u)}(u) - \frac{1}{(p+1)(p+2)} J(u).$$

Let φ_ω denote any element of \mathcal{N}_ω . It is easy to see that $J(u) = J(\varphi_{v(u)})$ because $d(v(u)) = \frac{p}{2(p+1)(p+2)} J(u)$ for $u \in \mathcal{B}_\epsilon(\mathcal{N}_c)$ and $d(v(u)) = \frac{p}{2(p+1)(p+2)} J(\varphi_{v(u)})$. It thus transpires that

$$I_{v(u)}(u) \geq I_{v(u)}(\varphi_{v(u)}).$$

A Taylor expansion of d around the value c yields

$$\begin{aligned} \mathcal{E}(u) + v(u) \mathcal{F}(u) &\geq I_{v(u)}(\varphi_{v(u)}) - \frac{1}{(p+1)(p+2)} J(\varphi_{v(u)}) \\ &= d(v(u)) \geq d(c) + \mathcal{F}(\varphi)(v(u) - c) + \frac{1}{4} d''(c) |v(u) - c|^2 \\ &= \mathcal{E}(\varphi) + v(u) \mathcal{F}(\varphi) + \frac{1}{4} d''(c) |v(u) - c|^2, \end{aligned}$$

and the lemma follows. \square

Before proving stability, we state a well-posedness result for (1.1). This can be proved in several standard ways, for example by using a parabolic regularization (see [26] and [15]).

Theorem 3.7. *Let $s > 2$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0, T]; H^s(\mathbb{R}^2))$ of equation (1.1) with $u(0) = u_0$. In addition, $u(t)$ depends continuously on u_0 in the H^s -norm and satisfies $\mathcal{E}(u(t)) = \mathcal{E}(u_0)$, $\mathcal{F}(u(t)) = \mathcal{F}(u_0)$, for all $t \in [0, T)$.*

When $0 < p < \frac{4}{3}$, the stability in \mathcal{Z} of the set of minimizers \mathcal{N}_c is established next.

Theorem 3.8. *Let $c > 0$, $s > 2$, $0 < p < \frac{4}{3}$ and $\lambda = (2(p+1)I_1)^{\frac{p+2}{p}}$. Then the set $\mathcal{N}_c = M_\lambda$ is \mathcal{L} -stable with regard to the flow of the BO-ZK equation. That is, for any positive ϵ , there is a positive $\delta = \delta(\epsilon)$ such that if $u_0 \in H^s$ and $\inf_{\varphi \in \mathcal{N}_c} \|u_0 - \varphi\|_{H^s} \leq \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies*

$$\sup_{t \geq 0} \inf_{\psi \in \mathcal{N}_c} \|u(t) - \psi\|_{\mathcal{X}} \leq \epsilon.$$

Proof. Assume that \mathcal{N}_c is \mathcal{L} -unstable with regard to the flow of the BO-ZK equation. Then, there is a sequence of initial data $u_k(0) \in H^s(\mathbb{R}^2)$ such that

$$\inf_{\varphi \in \mathcal{N}_c} \|u_k(0) - \varphi\|_{H^s} \leq \frac{1}{k} \quad \text{and} \quad \sup_{t \in [0, T]} \inf_{\psi \in \mathcal{N}_c} \|u_k(t) - \psi\|_{\mathcal{X}} \geq \epsilon, \quad (3.6)$$

where $u_k(t)$ is the solution of (1.1) with initial data $u_k(0)$. By continuity in t , for all k large enough, there are times t_k such that

$$\inf_{\varphi \in \mathcal{N}_c} \|u_k(t_k) - \varphi\|_{\mathcal{X}} = \frac{\epsilon}{2}. \quad (3.7)$$

Since \mathcal{E} and \mathcal{F} are conserved quantities, it follows from (3.6) that

$$|\mathcal{E}(u_k(t_k)) - \mathcal{E}(\varphi_k)| = |\mathcal{E}(u_k(0)) - \mathcal{E}(\varphi_k)| \rightarrow 0, \quad (3.8)$$

$$|\mathcal{F}(u_k(t_k)) - \mathcal{F}(\varphi_k)| = |\mathcal{F}(u_k(0)) - \mathcal{F}(\varphi_k)| \rightarrow 0, \quad (3.9)$$

as $k \rightarrow +\infty$. In this circumstance, Lemma 3.6 implies that

$$\mathcal{E}(u_k(t_k)) - \mathcal{E}(\varphi_k) + v(u_k(t_k))(\mathcal{F}(u_k(t_k)) - \mathcal{F}(\varphi_k)) \geq \frac{1}{4} d''(c) |v(u_k(t_k)) - c|^2,$$

for all k large enough. Since $\{u_k(t_k)\}$ is uniformly bounded in k , the right-hand side of the last inequality goes to zero as $k \rightarrow +\infty$ on account of (3.8) and (3.9). This in turn implies that $v(u_k(t_k)) \rightarrow c$ as $k \rightarrow +\infty$. Hence, by the definition of v and continuity of d , we must have

$$\lim_{k \rightarrow +\infty} J(u_k(t_k)) = \frac{2(p+1)(p+2)}{p} d(c). \quad (3.10)$$

On the other hand, Lemma 3.4 implies that

$$\begin{aligned} I(u_k(t_k)) &= \mathcal{E}(u_k(t_k)) + c\mathcal{F}(u_k(t_k)) + \frac{1}{(p+1)(p+2)} J(u_k(t_k)) \\ &= d(c) + \mathcal{E}(u_k(t_k)) - E(\varphi_k) + c(\mathcal{F}(u_k(t_k)) - \mathcal{F}(\varphi_k)) \\ &\quad + \frac{1}{(p+1)(p+2)} J(u_k(t_k)). \end{aligned}$$

The limit (3.10) then yields

$$\lim_{k \rightarrow +\infty} I(u_k(t_k)) = \frac{p+2}{p} d(c) = (2(p+1))^{\frac{2}{p}} I_1^{\frac{p+2}{p}}. \quad (3.11)$$

Defining

$$\vartheta_k(t_k) = \left(J(u_k(t_k)) \right)^{-\frac{1}{p+2}} u_k(t_k),$$

it is seen that $J(\vartheta_k(t_k)) = 1$. Combining (3.10), (3.11) and Lemma 3.4 leads to

$$\lim_{k \rightarrow +\infty} I(\vartheta_k(t_k)) = I_1. \quad (3.12)$$

Hence $\{\vartheta_k(t_k)\}$ is a minimizing sequence for I_1 . Thus, from Theorem 3.2, there exists a sequence $\{\psi_k\} \subset M_1$ such that

$$\lim_{k \rightarrow +\infty} \|\vartheta_k(t_k) - \psi_k\|_{\mathcal{X}} = 0. \quad (3.13)$$

The Lagrange multiplier theorem then implies there is a sequence $\{\theta_k\} \subset \mathbb{R}$ such that

$$\mathcal{H}(\psi_k)_x + c\psi_k - (\psi_k)_{yy} = \theta_k(p+2)\psi_k^{p+1}. \quad (3.14)$$

In other words, $2I_1 = \theta_k(p+2)$, which implies $\theta_k = \theta$ for all k . Write $\varphi_k = \mu\psi_k$ with

$$\mu^p = \theta(p+1)(p+2) = 2(p+1)I_1.$$

Then the φ_k satisfy (1.2) and $2(p+1)I(\varphi_k) = J(\varphi_k) = \mu^{p+2}$ so that $\varphi_k \in \mathcal{N}_c$ for all k . Additionally, (3.10)-(3.13) and Lemma 3.4 together allow the conclusion

$$\begin{aligned} \|u_k(t_k) - \varphi_k\|_{H^s} &= J(u_k(t_k))^{\frac{1}{p+2}} \left\| J(u_k(t_k))^{-\frac{1}{p+2}} (u_k(t_k) - \varphi_k) \right\|_{H^s} \\ &\leq J(u_k(t_k))^{\frac{1}{p+2}} \left(\|\vartheta_k(t_k) - \mu^{-1}\varphi_k\|_{H^s} + \mu^{-1}\|\varphi_k\|_{H^s} - J(u_k(t_k))^{-\frac{1}{p+2}} \right). \end{aligned}$$

This in turn implies that

$$\lim_{k \rightarrow +\infty} \|u_k(t_k) - \varphi_k\|_{\mathcal{X}} = 0,$$

which contradicts (3.7) and completes the proof of the Theorem. \square

4. DECAY AND REGULARITY

To investigate the regularity and the spatial asymptotics of the solitary-wave solutions of (1.1), it is convenient to take the Fourier transform of equation (1.2) for the solitary-wave in both x and y . If (ξ, η) are the variables dual to (x, y) by way of the Fourier transform, then (1.2) implies that

$$\widehat{\varphi} = \frac{\widehat{g}}{c - \alpha|\xi| + \varepsilon\eta^2}, \quad \text{where } g = -\frac{1}{p+1}\varphi^{p+1}. \quad (4.1)$$

Taking the inverse Fourier transform then yields

$$\varphi = -\frac{1}{p+1} \int_{\mathbb{R}^2} K(x-s, y-t) \varphi^{p+1}(s, t) ds dt. \quad (4.2)$$

Properties of the integral kernel K in (4.2) will be central in the analysis to follow. Here are few standard properties of anisotropic Sobolev spaces that will be helpful in expressing useful aspects of K .

Lemma 4.1. *If $s_i > 1/2$, for $i = 1, 2$, then H^{s_1, s_2} is an algebra.*

Lemma 4.2. *Let $s_{ij}, 1 \leq i, j \leq 2$ and $\theta \in [0, 1]$ be given real numbers with $s_{1j} \leq s_{2j}$, $j = 1, 2$. Define $\varrho_j = \theta s_{1j} + (1-\theta)s_{2j}$ for $j = 1, 2$. Then, H^{ϱ_1, ϱ_2} is an interpolation space between $H^{s_{11}, s_{12}}$ and its subspace $H^{s_{21}, s_{22}}$. Moreover, if $f \in H^{s_{21}, s_{22}}$, then*

$$\|f\|_{H^{\varrho_1, \varrho_2}} \leq \|f\|_{H^{s_{11}, s_{12}}}^\theta \|f\|_{H^{s_{21}, s_{22}}}^{1-\theta}. \quad (4.3)$$

Remark 4.3. *Since $\widehat{K}(\xi, \eta) = \frac{1}{c - \alpha|\xi| + \eta^2}$, the Residue Theorem allows us to write the kernel K as an integral, namely*

$$K(x, y) = K_c(x, y) = C \int_0^{+\infty} \frac{|\alpha|\sqrt{t}}{\alpha^2 t^2 + x^2} e^{-\left(ct + \frac{y^2}{4t}\right)} dt, \quad (4.4)$$

where $C > 0$ is independent of α , x and y . Fubini's theorem can then be used to show that

$$\|K\|_{L^1} = C \int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\alpha|\sqrt{t}}{\alpha^2 t^2 + x^2} e^{-(ct + \frac{y^2}{4t})} dx dy dt = C(\alpha) \int_0^{+\infty} e^{-ct} dt.$$

In consequence of representation (4.4), the following facts about K become clear.

Lemma 4.4. *The kernel K is positive, an even function of both x and y , monotone decreasing in both $|x|$ and $|y|$, tends to zero as $|(x, y)| \rightarrow \infty$ and is C^∞ away from the origin. Furthermore, $\widehat{K} \in L^p(\mathbb{R}^2)$ for any $p \in (3/2, +\infty]$ and $K \in L^p(\mathbb{R}^2)$, for any $p \in [1, 3)$. (However, while $K(x, y)$ is symmetric in both x and y , it is not radially symmetric.)*

Lemma 4.5. *$K \in H^{s_1, 0}(\mathbb{R}^2) \cap H^{0, s_2}(\mathbb{R}^2)$ for any $s_1 < \frac{1}{4}$ and $s_2 < \frac{1}{2}$. Moreover, $K \in H^{r, s}(\mathbb{R}^2) \cap H^{s_1, s_2}(\mathbb{R}^2)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0, 1]$.*

Lemma 4.6. (i) $\widehat{K} \in H^{s_1, 0}(\mathbb{R}^2) \cap H^{0, s_2}(\mathbb{R}^2)$, for any $s_1 < \frac{3}{2}$ and $s_2 \in \mathbb{R}$.
 Moreover, $\widehat{K} \in H^{r, s}(\mathbb{R}^2) \cap H^{(s_1, s_2)}(\mathbb{R}^2)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0, 1]$.
 (ii) $\widehat{K} \in H_p^{(s_1, s_2)}(\mathbb{R}^2)$, for any $s_1 < 1 + \frac{1}{p}$, $p \geq 2$ and $s_2 \in \mathbb{R}$.
 (iii) $|x|^{s_1} |y|^{s_2} K \in L^p(\mathbb{R}^2)$, for any $s_1, s_2 \geq 0$ and $1 \leq p \leq \infty$ such that $s_1 < 2 - \frac{1}{p}$ and $2s_1 + s_2 > 1 - \frac{3}{p}$.

With these facts about K in hand, the solitary-wave solutions of the BO-ZK equation (1.1) now become the focus of attention.

Theorem 4.7. *Let p be a positive integer. Any solitary-wave solution φ of (1.1) belongs to $H_r^{(k)}$, for all $k \in \mathbb{N}$ and all $r \in [1, +\infty]$. In particular, the solitary-wave solutions of the BO-ZK equation are continuous, bounded and tend to zero at infinity.*

Proof. Formula (4.1) implies that $\varphi \in H^{\frac{1}{2}, 1}(\mathbb{R}^2) \cap H^{0, 2}(\mathbb{R}^2) \cap H^{1, 0}(\mathbb{R}^2)$. Lemma 4.2 and the embedding (1.8) then imply that $\varphi \in H^{s, 2(1-s)}(\mathbb{R}^2)$, for any $s \in [0, 1]$. A bootstrapping argument and the use of Lemmas 4.2 and 4.1 completes the proof. \square

More detailed aspects of the solitary-wave solutions of (1.1) are now addressed. Interest will focus first upon their symmetry properties. For $u : \mathbb{R}^2 \rightarrow \mathbb{R}^+$, $u^\#$ will denote the Steiner symmetrization of u with respect to $\{x = 0\}$ and u^* the Steiner symmetrization of u with respect to $\{y = 0\}$ (see, for example, [10, 29, 44]). Notice that $u^{\#\#} = u^{**}$ is a function symmetric with respect to both the x - and y -axis.

Lemma 4.8. *If $f \in \mathcal{X}$, then $|f|$ lies in \mathcal{X} and $I(|f|) \leq I(f)$.*

Proof. If $g = |f|$, then for any $c > 0$,

$$\langle f, K * f \rangle \leq \langle g, K * g \rangle,$$

where $K = K_c$. It thus transpires that

$$\int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left| \widehat{f}(\xi, \eta) \right|^2 d\xi d\eta = \langle f, K * f \rangle \leq \langle g, K * g \rangle = \int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta.$$

Since $\|\widehat{f}\|_{L^2} = \|\widehat{g}\|_{L^2}$, it follows that

$$\int_{\mathbb{R}^2} c(1 - c\widehat{K}) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \leq \int_{\mathbb{R}^2} c(1 - c\widehat{K}) \left| \widehat{f}(\xi, \eta) \right|^2 d\xi d\eta. \quad (4.5)$$

Taking the limit as $c \rightarrow +\infty$ on both sides of (4.5), the Monotone Convergence Theorem yields

$$\int_{\mathbb{R}^2} (|\xi| + \eta^2) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \leq \int_{\mathbb{R}^2} (|\xi| + \eta^2) \left| \widehat{f}(\xi, \eta) \right|^2 d\xi d\eta, \quad (4.6)$$

which shows that $|f| \in \mathcal{Z}$ and that $I(|f|) \leq I(f)$. \square

Corollary 4.9. *For $c > 0$, there is always a non-negative solitary-wave solution φ_c of the BO-ZK equation.*

Proof. Theorem 2.7 assures that there are solitary-wave solutions ψ , say. The last result shows that if $\psi \in M_\lambda$, then so is $\varphi = |\psi|$. \square

If $p = \frac{k}{m}$ where m is odd and k and m relatively prime it follows from the formula

$$\varphi = \frac{1}{p+1} K * \varphi^{p+1} \quad (4.7)$$

that if k is odd, then necessarily all solitary-wave solutions are non-negative. This is false if k is even, however. Indeed, in this case, if φ is a solitary wave, then so is $-\varphi$. Hence, when k is even, there are always at least two solitary-wave solutions, one positive and one negative. Of course, when k is even, it is also the case that $J(|f|) = J(f)$.

Lemma 4.10. *If $f \in \mathcal{Z}$ is non-negative, its Steiner symmetrizations f^\sharp and f^* also lie in \mathcal{Z} . Moreover, $I(f^\sharp) \leq I(f)$ and $I(f^*) \leq I(f)$.*

Proof. Remark first that $K^\sharp = K = K^*$. The Reisz-Sobolev rearrangement inequality (see [10, 29, 44]) implies that

$$\begin{aligned} & \int_{\mathbb{R}^4} f(x, y) f(s, t) K(x - s, y - t) ds dt dx dy \\ & \leq \int_{\mathbb{R}^4} f^\sharp(x, y) f^\sharp(s, t) K(x - s, y - t) ds dt dx dy. \end{aligned}$$

In the Fourier transformed variables, this amounts to

$$\int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left| \widehat{f}(\xi, \eta) \right|^2 d\xi d\eta \leq \int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left| \widehat{f}^\sharp(\xi, \eta) \right|^2 d\xi d\eta.$$

On the other hand, the fact that symmetrization does not change the measure theoretic properties of f implies that

$$\|\widehat{f}\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} = \|f^\sharp\|_{L^2(\mathbb{R}^2)} = \|\widehat{f}^\sharp\|_{L^2(\mathbb{R}^2)}.$$

This together with the analysis in Lemma 4.8 shows that $f^\sharp \in \mathcal{Z}$ and that $I(f^\sharp) \leq I(f)$. The same argument applies to f^* . \square

Since Steiner symmetrization preserves the L^{p+2} -norm, it follows that $J(\varphi) = J(\varphi^\sharp)$. In consequence of Lemma 4.10,

$$I_\lambda \leq I(\varphi^\sharp) \leq I(\varphi) = I_\lambda.$$

Therefore $\varphi^\sharp \in M_\lambda$. The same argument shows that $\varphi^* \in M_\lambda$. \square

Corollary 4.11. *There are non-negative, solitary-wave solutions of the BO-ZK equation (1.1) that are symmetric with respect to both the propagation direction and the transverse direction and are monotone decreasing in both $|x|$ and $|y|$.*

Proof. By Theorems 2.2 and 4.7, there is a non-negative function φ satisfying (1.2). Since Steiner symmetrization preserves the L^{p+2} -norm, it follows that $J(\varphi) = J(\varphi^\sharp) = J(\varphi^{\sharp*})$. On the other hand, because of Lemma 4.10, the double rearrangement $\varphi^{\sharp*}$ has the property that

$$I_\lambda \leq I(\varphi^{\sharp*}) \leq I(\varphi^\sharp) \leq I(\varphi) = I_\lambda.$$

Therefore, $\varphi^{\sharp*}$ is a non-negative solitary-wave solution of equation (1.1) which is symmetric with respect to both $\{x = 0\}$ and $\{y = 0\}$ and which is monotone decreasing with respect to both $|x|$ and $|y|$. \square

Remark 4.12. *One may also obtain symmetry properties of the solitary-wave solutions of (1.1) by using the reflection method and a unique continuation argument (see [40] and [21]).*

5. SPATIAL ASYMPTOTICS

Attention is now turned to the spatial decay properties of the solitary-wave solutions of (1.1). In this analysis, we follow the lead of [9].

Lemma 5.1. *Let $j \in \mathbb{N}$. Suppose also that ℓ and m are two constants satisfying $0 < \ell < m - j$. Then there exists $C > 0$, depending only on ℓ and m , such that for all $\epsilon \in (0, 1]$, we have*

$$\int_{\mathbb{R}^j} \frac{|a|^\ell}{(1 + \epsilon|a|)^m(1 + |b - a|)^m} da \leq \frac{C|b|^\ell}{(1 + \epsilon|b|)^m}, \quad \forall b \in \mathbb{R}^j, |b| \geq 1, \quad (5.1)$$

and

$$\int_{\mathbb{R}^j} \frac{da}{(1 + \epsilon|a|)^m(1 + |b - a|)^m} \leq \frac{C}{(1 + \epsilon|b|)^m}, \quad \forall b \in \mathbb{R}^j. \quad (5.2)$$

The proof of this elementary lemma is essentially the same as the proof of Lemma 3.1.1 in [9] (see [19]).

Theorem 5.2. *Let φ be a solitary-wave solution of (1.2).*

- (i) *For all $q \in (3/2, +\infty)$, $\ell \in [0, 1)$ $\varrho \geq 0$, $|x|^\ell |y|^\varrho \varphi(x, y) \in L^q(\mathbb{R}^2)$.*
- (ii) *For all $q \in (3/2, +\infty)$ and any $\theta \in [0, 1)$, $|(x, y)|^\theta \varphi(x, y) \in L^q(\mathbb{R}^2)$.*
- (iii) *And finally, $\varphi \in L^1(\mathbb{R}^2)$.*

Proof. (i) For $q \in (1, 3)$ and $1 - \frac{1}{q} < s_1 < 2 - \frac{1}{q}$, let $\ell \in [0, s_1 - 1 + \frac{1}{q})$. Also, for $s_2 > 1 - \frac{1}{q}$, choose $\varrho \in [0, s_2 - 1 + \frac{1}{q})$. For $0 < \epsilon < 1$, define h_ϵ by

$$h_\epsilon(x, y) = \mathcal{A}(x, y) \varphi(x, y),$$

where

$$\mathcal{A}(x, y) = \frac{|x|^\ell |y|^\varrho}{(1 + \epsilon|x|)^{s_1} (1 + \epsilon|y|)^{s_2}},$$

Then, by using the explicit representation of h_ϵ , it is easy to check that $h_\epsilon \in L^{q'}(\mathbb{R}^2)$, where $q' = \frac{q}{q-1}$. Hölder's inequality and (4.2) then implies that

$$|\varphi(x, y)| \leq C(s_1, s_2, q) \left(\int_{\mathbb{R}^2} |\mathcal{G}_{x,y}(z, w)|^{q'} dz dw \right)^{\frac{1}{q'}},$$

where

$$\mathcal{G}_{x,y}(z, w) = \frac{g(\varphi)(z, w)}{(1 + |x - z|)^{s_1} (1 + |y - w|)^{s_2}},$$

$g(t) = \frac{t^{p+1}}{p+1}$ and

$$C := C(s_1, s_2, p) = \|(1 + |x|)^{s_1} (1 + |y|)^{s_2} K\|_{L^q(\mathbb{R}^2)} < \infty.$$

This last constant is finite thanks to Lemma 4.6. Since the solitary wave φ converges to the rest state as $|(x, y)| \rightarrow +\infty$, it follows that for every $\delta > 0$, there exists $R_\delta > 1$ such that if $|(x, y)| \geq R_\delta$, then

$$|g(\varphi)(x, y)| \leq \delta |\varphi(x, y)|.$$

An application of Hölder's inequality yields

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'} dx dy &= \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'-r} \mathcal{A}^r(x, y) |\varphi(x, y)|^r dx dy \\ &\leq C^r \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'-r} \mathcal{A}^r(x, y) \|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)}^r(x, y) dx dy \\ &\leq C^r \|h_\epsilon\|_{L^{q'}(\mathbb{R}^2 \setminus B(0, R_\delta))}^{q'-r} \left\| \mathcal{A} \|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)} \right\|_{L^{q'}(\mathbb{R}^2 \setminus B(0, R_\delta))}^r. \end{aligned}$$

Because $h_\epsilon \in L^{q'}(\mathbb{R}^2)$, the latter inequality implies

$$\|h_\epsilon\|_{L^{q'}(\mathbb{R}^2 \setminus B(0, R_\delta))}^r \leq C^r \left\| \mathcal{A} \|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)} \right\|_{L^{q'}(\mathbb{R}^2 \setminus B(0, R_\delta))}^r,$$

which is to say,

$$\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'} dx dy \leq C^{q'} \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \mathcal{A}^{q'}(x, y) \|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)}^{q'} dx dy.$$

Fubini's theorem and Lemma 5.1 combine to show that

$$\begin{aligned} &\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \mathcal{A}^{q'}(x, y) \|\mathcal{G}_{x,y}\|_{L^{q'}(\mathbb{R}^2)}^{q'}(x, y) dx dy \\ &= \int_{\mathbb{R}^2} |g(\varphi)(z, w)|^{q'} \left(\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \frac{\mathcal{A}^{q'}(x, y)}{(1 + |x - z|)^{q' s_1} (1 + |y - w|)^{q' s_2}} dx dy \right) dz dw \\ &\leq C \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |g(\varphi)(z, w)|^{q'} \mathcal{A}^{q'}(z, w) dz dw \\ &+ \int_{B(0, R_\delta)} |g(\varphi)(z, w)|^{q'} \left(\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \frac{\mathcal{A}^{q'}(x, y)}{(1 + |x - z|)^{q' s_1} (1 + |y - w|)^{q' s_2}} dx dy \right) dz dw, \end{aligned} \tag{5.3}$$

where we used (5.1) (with $j = 1$) to show that for $|(z, w)|$ large,

$$\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \frac{\mathcal{A}^{q'}(x, y)}{(1 + |x - z|)^{q' s_1} (1 + |y - w|)^{q' s_2}} dx dy \leq C \mathcal{A}^{q'}(z, w).$$

The second integral on the right-hand side of (5.3) is bounded by a constant, say C' , depending on φ and R_δ , but independent of ϵ . Therefore, by using the fact that $|g(\varphi)(x, y)| \leq \delta |\varphi(x, y)|$ on $\mathbb{R}^2 \setminus B(0, R_\delta)$, there obtains

$$\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'} dx dy \leq C^{q'} \left(C \delta^{q'} \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'} dx dy + C' \right).$$

Choosing δ such that $C\delta C^{\frac{1}{q'}} < 1$, the last inequality entails that

$$\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |h_\epsilon(x, y)|^{q'} dx dy \leq C'', \quad (5.4)$$

where C'' is a constant independent of ϵ . Letting $\epsilon \rightarrow 0$ in (5.4) and applying Lebesgue's dominated convergence theorem, one deduces

$$\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |x|^{\ell q'} |y|^{\rho q'} |\varphi(x, y)|^{q'} dx dy \leq C.$$

Hence $|x|^\ell |y|^\rho \varphi(x, y) \in L^{q'}(\mathbb{R}^2)$, for $q' = \frac{q}{q-1}$.

In the limits $q \rightarrow 1$ and $q \rightarrow 3$, we have $\ell \rightarrow 1$ and $q' \in (3/2, +\infty)$. This proves part (i) of the theorem.

(ii) This follows directly from (i).

(iii) Let $s > 1$ and g, δ and R_δ be as defined in the proof of (i). For $\epsilon > 0$ let \mathcal{A} be

$$\mathcal{A}_\epsilon(x, y) = \frac{1}{(1 + \epsilon|(x, y)|)^s}.$$

Fubini's Theorem, Lemma 5.1 and the fact that $\varphi, \mathcal{A}_\epsilon \in L^2(\mathbb{R}^2)$ so that the product $\varphi \mathcal{A}_\epsilon \in L^1(\mathbb{R}^2)$ allow us to adduce the inequalities

$$\begin{aligned} & \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |\varphi(x, y)| \mathcal{A}_\epsilon(x, y) dx dy \\ & \leq \int_{\mathbb{R}^2} |g(\varphi)(z, w)| \left(\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \mathcal{A}_\epsilon(x, y) K(x-z, y-w) dx dy \right) dz dw \\ & \leq \int_{\mathbb{R}^2} |g(\varphi)(z, w)| \left(\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \mathcal{A}_1^{-2}(x-z, y-w) K^2(x-z, y-w) dx dy \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\mathbb{R}^2 \setminus B(0, R_\delta)} \mathcal{A}_1^2(x-z, y-w) \mathcal{A}_\epsilon^2(x, y) dx dy \right)^{\frac{1}{2}} dz dw \\ & \leq C(s) C^{\frac{1}{2}} \int_{\mathbb{R}^2} |g(\varphi)(z, w)| \mathcal{A}_\epsilon(z, w) dz dw \\ & \leq C(s) C^{\frac{1}{2}} \delta \int_{\mathbb{R}^2 \setminus B(0, R_\delta)} |\varphi(z, w)| \mathcal{A}_\epsilon(z, w) dz dw \\ & \quad + C(s) C^{\frac{1}{2}} \int_{B(0, R_\delta)} |g(\varphi)(z, w)| dz dw. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, Fatou's lemma together with the restriction on δ leads to the conclusion that $\varphi \in L^1(\mathbb{R}^2)$. \square

Theorem 5.2, identity (4.7) and the elementary inequality

$$|t|^\theta \leq C(|t-s|^\theta + |s|^\theta), \quad \text{for } \theta \geq 0. \quad (5.5)$$

imply the following.

Corollary 5.3. *Suppose that $\varphi \in L^\infty(\mathbb{R}^2)$ satisfies (1.2) and $\varphi \rightarrow 0$ at infinity. Then*

- (i) $|x|^\ell |y|^\rho \varphi(x, y) \in L^\infty(\mathbb{R}^2)$, for all $\ell \in [0, 1)$ and any $\rho \geq 0$,
- (ii) $|(x, y)|^\theta \varphi(x, y) \in L^\infty(\mathbb{R}^2)$, for all $\theta \in [0, 1)$.

The aim now is to display even stronger decay properties in the x -variable for solitary-wave solutions of the *BO-ZK* equation. These results are developed in a sequence of lemmas.

Lemma 5.4. $|x|^2|y|^\varrho K \in L^\infty(\mathbb{R}^2)$, for any $\varrho \geq 0$.

Proof. In view of the explicit form of K , the proof is straightforward. \square

Corollary 5.5. $|x|^\ell|y|^\varrho\varphi(x, y) \in L^\infty(\mathbb{R}^2)$, for any $0 \leq \ell \leq 2$ and any $\varrho \geq 0$.

Proof. The proof is based on a standard bootstrapping argument. Decay in the y -variable is not in question, so without loss of generality, take it that that $\varrho = 0$. Setting $\gamma_1 = \min\{2, p+1\}$ and making use of the inequality

$$|x|^{\gamma_1}|\varphi| \lesssim |x|^{\gamma_1}|K| * |g(\varphi)| + |K| * ||x|^{\gamma_1}|g(\varphi)||, \quad (5.6)$$

where $g(t) = \frac{t^{p+1}}{p+1}$, we obtain from Corollary 5.3, Lemma 5.4 and Theorem 5.2 that $|x|^{\gamma_1}\varphi \in L^\infty(\mathbb{R}^2)$. The proof is complete if $\gamma_1 = 2$. If $\gamma_1 < 2$, then define $\gamma_2 = \min\{2, (p+1)^2\}$ and repeat the above argument to show $|x|^{\gamma_2}\varphi \in L^\infty(\mathbb{R}^2)$. Continuing in this manner, one concludes that $|x|^2\varphi \in L^\infty(\mathbb{R}^2)$ after a finite number of steps. \square

The following corollary follows from (5.5), Corollary 5.3 and Theorem 5.2.

Corollary 5.6. (i) $|x|^\ell|y|^\varrho\varphi(x, y) \in L^1(\mathbb{R}^2)$, for all $\ell \in [0, 1)$ and any $\varrho \geq 0$,
(ii) $|(x, y)|^\theta\varphi(x, y) \in L^1(\mathbb{R}^2)$, for all $\theta \in [0, 1)$.

Lemma 5.7. For any $1 \leq r, q < \infty$, there is $\sigma_0 > 0$ such that for all $\sigma \in [0, \sigma_0)$ and $s \in (\frac{1}{2} - \frac{1}{r} - \frac{1}{2q}, 2 - \frac{1}{r})$, we have

$$|x|^s e^{\sigma|y|} K \in L_x^r L_y^q(\mathbb{R}^2) \cap L_y^q L_x^r(\mathbb{R}^2).$$

Proof. It suffices to choose $\sigma_0 = \sqrt{\frac{c}{q}}$, where c is the wave velocity and use (4.4). \square

The next result is a consequence of another of Young's inequalities, namely

$$\|f * g\|_{L_y^q L_x^r(\mathbb{R}^2)} \leq \|f\|_{L_y^{q_1} L_x^{r_1}(\mathbb{R}^2)} \|g\|_{L_y^{q_2} L_x^{r_2}(\mathbb{R}^2)},$$

where $1 \leq r, q, r_1, q_1, r_2, q_2 \leq \infty$, $1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and $1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Corollary 5.8. $\varphi \in L_x^r L_y^q(\mathbb{R}^2) \cap L_y^q L_x^r(\mathbb{R}^2)$, for any $1 \leq r \leq \infty$ satisfying

$$\frac{1}{r} + \frac{1}{2q} > \frac{1}{2}.$$

Here is the main result about the spatial decay of the solitary-wave solutions.

Theorem 5.9. Let $\sigma_0 > 0$ be in Lemma 5.7. Then, for any $\sigma \in [0, \sigma_0)$ and any $0 \leq s < 3/2$, it transpires that $|x|^s e^{\sigma|y|}\varphi(x, y) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

Proof. Without loss of generality, assume that $s = 0$. By using Lemma 5.7 and the proof of Corollary 3.14 in [9], with natural modifications, it may be seen that there is a $\tilde{\sigma} \geq \sigma_0$ such that $e^{\sigma|y|}\varphi(x, y) \in L^1(\mathbb{R}^2)$, for any $\sigma < \tilde{\sigma}$. The inequality

$$|\varphi(x, y)| e^{\sigma|y|} \leq \int_{\mathbb{R}^2} |K(x-z, y-w)| e^{\sigma|y-w|} |\varphi(z, w)| e^{\sigma|w|} |\varphi(z, w)|^p dz dw \quad (5.7)$$

and the facts $\varphi(x, y) e^{\sigma|y|} \in L^1(\mathbb{R}^2)$, $\varphi \in L^\infty(\mathbb{R}^2)$ and $K(x, y) e^{\sigma|y|} \in L^2(\mathbb{R}^2)$, for any $\sigma < \sigma_0$, entails that $\varphi(x, y) e^{\sigma|y|} \in L^\infty(\mathbb{R}^2)$, for the same range of σ . \square

Finally, the following theorem deals with the analyticity of the solitary-wave solutions. Of course, for this, one needs to restrict p so that $z \mapsto z^p$ is analytic in a full neighborhood of the origin in \mathbb{C} .

Theorem 5.10. *Let $1 \leq p < 4$ be an integer. Then, there is a $\sigma > 0$ and a holomorphic function f of two variables z_1 and z_2 , defined in the domain*

$$\mathcal{H}_\sigma = \{(z_1, z_2) \in \mathbb{C}^2; |\operatorname{Im}(z_1)| < \sigma, |\operatorname{Im}(z_2)| < \sigma\}$$

such that $f(x, y) = \varphi(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

Similar results are obtained by the same method for related evolution equations in [34] and [9]. Results of this nature for dispersive equations made via Gevrey-space analysis appear in [8] (and see also the reference therein).

Proof. By the Cauchy-Schwarz inequality, Theorem 4.7 implies that $\widehat{\varphi} \in L^1(\mathbb{R}^2)$. Equation (1.2) implies in turn that

$$|\xi| |\widehat{\varphi}|(\xi, \eta) \leq \overbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}^{p+1}(\xi, \eta), \quad (5.8)$$

$$|\eta| |\widehat{\varphi}|(\xi, \eta) \leq \underbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}_{p+1}(\xi, \eta). \quad (5.9)$$

Denote by \mathcal{T}_1 the correspondence $\mathcal{T}_1(|\widehat{\varphi}|) = |\widehat{\varphi}|$ and, for $m \geq 1$, $\mathcal{T}_{m+1}(|\widehat{\varphi}|) = \mathcal{T}_m(|\widehat{\varphi}|) * |\widehat{\varphi}|$. A straightforward induction yields

$$r^m |\widehat{\varphi}|(\xi, \eta) \leq (m-1)! (p+1)^{m-1} \mathcal{T}_{mp+1}(|\widehat{\varphi}|)(\xi, \eta), \quad (5.10)$$

where $r = |(\xi, \eta)|$. It follows that

$$\begin{aligned} r^m |\widehat{\varphi}|(\xi, \eta) &\leq (m-1)! (p+1)^{m-1} \|\mathcal{T}_{mp+1}(|\widehat{\varphi}|)\|_{L^\infty(\mathbb{R}^2)} \\ &\leq (m-1)! (p+1)^{m-1} \|\mathcal{T}_{mp}(|\widehat{\varphi}|)\|_{L^2(\mathbb{R}^2)} \|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)} \\ &\leq (m-1)! (p+1)^{m-1} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^{mp} \|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)}^2. \end{aligned}$$

Let

$$a_m = \frac{(p+1)^{m-1} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^{mp} \|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)}^2}{m},$$

so that

$$\frac{a_{m+1}}{a_m} \longrightarrow (p+1) \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^p,$$

as $m \rightarrow +\infty$. In consequence, the series $\sum_{m=0}^{\infty} t^m r^m |\widehat{\varphi}|(\xi, \eta)/m!$ converges uniformly in $L^\infty(\mathbb{R}^2)$ provided $0 < t < \sigma = \frac{1}{p+1} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^{-p}$. Hence $e^{tr} \widehat{\varphi}(\xi, \eta) \in L^\infty(\mathbb{R}^2)$, for $t < \sigma$. Now define the function

$$f(z_1, z_2) = \int_{\mathbb{R}^2} e^{i(\xi z_1 + \eta z_2)} \widehat{\varphi}(\xi, \eta) d\xi d\eta.$$

By the Paley-Wiener Theorem, f is well defined and analytic in \mathcal{H}_σ while Plancherel's Theorem assures that $f(x, y) = \varphi(x, y)$ for all $(x, y) \in \mathbb{R}^2$. This proves the theorem. \square

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