Math 180: Calculus I **Fall 2014**

October 23

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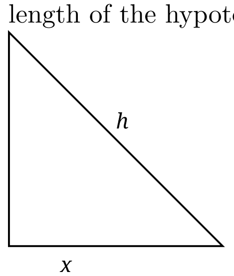
1. The hypotenuse of an isosceles right triangle is decreasing in length at a rate of 4 m/s.

   (a) At what rate is the area of the triangle changing when the legs are 5m long?
   SOLUTION: First we should define the variables. Let x be the length of the legs, and h the length of the hypotenuse.

   Since the legs of the isosceles right triangle are in ratio of $1 : \sqrt{2}$ to the hypotenuse, the rate of change of the legs will be proportional, i.e. $x = h/\sqrt{2}$. Since $\frac{dh}{dt} = -4$,

   $$\frac{dx}{dt} = \frac{dh}{dt}\frac{1}{\sqrt{2}} = -\frac{4}{\sqrt{2}} = -2\sqrt{2}$$

   Because the area $A = x^2/2$,

   $$\frac{dA}{dt} = \frac{dA}{dx}\frac{dx}{dt} = x(-2\sqrt{2} = -2\sqrt{2}x \text{ m/s}$$

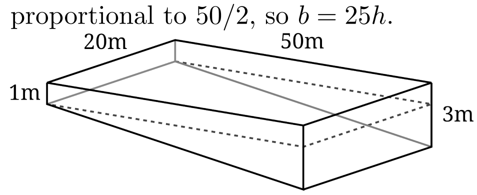
   When $x = 5$, $dA/dt = -2\sqrt{2}5 = -10\sqrt{2}$ m^2/s.

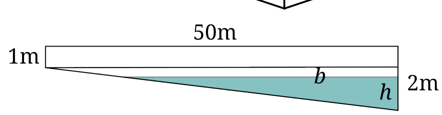
   (b) At what rate are the length of the legs of the triangle changing?
   SOLUTION: The legs are decreasing at a constant rate of $2\sqrt{2}$ m/s.

   (c) At what rate is the area of the triangle changing when the area is 4 m^2?
   SOLUTION: The area is 4 m^2 when $x^2/2 = 4 \Rightarrow x^2 = 8$ so $x = 2\sqrt{2}$. At that time $dA/dt = -(2\sqrt{2})^2 = -8$ m^2/s.

2. A swimming pool is 50m long and 20m wide. Its depth decreases linearly along the length from 3m to 1m. It is initially empty and filled with water at 1 m^3/min.

   (a) How fast is the water level rising 250 minutes after the filling begins?
   SOLUTION: First of all, we need to talk about the dimensions of this. The inclined portion of the pool has height 2m and length 50m. As a height h is filled in, it will have a length b that is proportional to 50/2, so $b = 25h$.

The volume when height $h \leq 2$ has been filled up is then $V(h) = 20\frac{1}{2}(25h)h = 250h^2$. For $2 < h \leq 3$, $V(h) = 250(2)^2 + 20(50)(h-2) = 1000h - 1000$.

$$V(h) = \begin{cases} 250h^2 & \text{if } h \leq 2 \\ 1000h - 1000 & \text{if } 2 < h \leq 3 \end{cases}$$

After 250 minutes, the volume will be 250 m^3, which means $h = 1$. We have

$$\frac{dV}{dt} = 1, \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = 2 \cdot 250h\frac{dh}{dt} = 500 \text{ when } h = 1$$

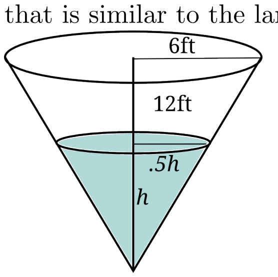
Which means that

$$1 = 500\frac{dh}{dt}$$

So $\frac{dh}{dt} = 1/500$ m/min, or 2mm/min

(b) How long will it take to fill the pool?
SOLUTION: This is easier. We just have to calculate the total volume. The cross section of the pool has area $1 \cdot 50 + \frac{1}{2} \cdot 50 \cdot 2 = 100\text{m}^2$, so the volume is $100 \cdot 20 = 2000$ m^3, so it takes 2000 minutes to fill the pool.

3. An inverted conical water tank with height of 12ft and radius of 6ft is drained through a hole in the vertex at a rate of 2 ft^3/sec. What is the rate of change of the water depth when the water depth is 3ft?
SOLUTION: Let h be the height of the water in the cone. Because the volume of water is a cone that is similar to the large cone, the radius is going to be $.5h$.



So the volume of water is

$$V(h) = \frac{1}{3}\pi(.5h)^2h = \frac{\pi}{12}h^3$$

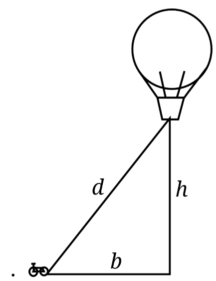
Furthermore, we have

$$-2 = \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

We can solve for $dh/dt = -8/(\pi h^2)$, so when $h = 3$, $dh/dt = -8/(9\pi)$ ft/sec.

4. A hot-air balloon is 150 ft above the ground when a motorcycle (traveling in a straight line on a horizontal road) passes directly underneath it going 58.67 ft/s. If the balloon rises vertically at a rate of 10 ft/s, what is teh rate of change of the distance beween the motorcycle and the balloon 10 seconds later?
SOLUTION: Let h be the height of the ballon at time t, and b be the distance the motorcycle has traveled by time t. So

$$h = 150 + 10t, b = 58.67t$$

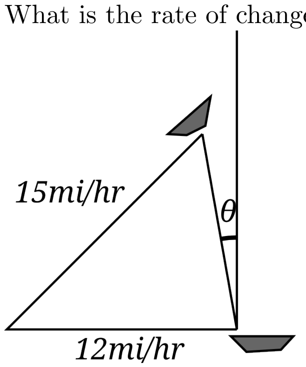


The distance between the balloon and the motorcycle is $d = \sqrt{h^2 + b^2}$, so we can take the derivative

$$\frac{dd}{dt} = \frac{1}{2}(h^2 + b^2)^{-1/2}\left(2h\frac{dh}{dt} + 2b\frac{db}{dt}\right) = \frac{20h + 117.34b}{2\sqrt{h^2 + b^2}}$$

10 seconds later, $h = 250, b = 586.7$, so we can evaluate $dd/dt \approx 57.89$ ft/sec.

5. A boat leaves a port traveling due east at 12 mi/hr and at the same time another boat leaves traveling northeast at 15 mi/hr. The angle θ of the line between the two boats is measureed from due north. What is the rate of change of this angle 30 minutes after they leave port? 2 hr after they leave port?



SOLUTION: The position of each boat can be calculated as an (x, y) coordinate with $(0, 0)$ as the port they leave from. The first boat, heading east will be at $(12t, 0)$ while the second boat will be at $(15\sin(\pi/4)t, 15\sin(\pi/4)t) = (7.5\sqrt{2}t, 7.5\sqrt{2}t)$.

The angle can be found by using an inverse trig function. For example, the inverse tangent. The opposite side of the angle will be $(12t - 7.5\sqrt{2}t)$ while the adjacent side will be $7.5\sqrt{2}t$. So

$$\theta = \tan^{-1}\left(\frac{(12 - 7.5\sqrt{2})t}{7.5\sqrt{2}t}\right) = \tan^{-1}\left(\frac{12 - 7.5\sqrt{2}}{7.5\sqrt{2}}\right) \approx \tan^{-1}(0.13137085) \approx 7.484^\circ$$

Because the angle is constant, the rate of change is 0 always.