Math 180: Calculus I **Fall 2014**

October 30

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1. Two nonnegative numbers x and y have a sume of 23. What is the maximum possible product?
SOLUTION: We wish to maximize $p = xy$, but first we would like to express this as a function of a single variable. Since $x + y = 23$, we may write $y = 23 - x$. So we wish to maximize

$$p = x(23 - x) = -x^2 + 23x$$

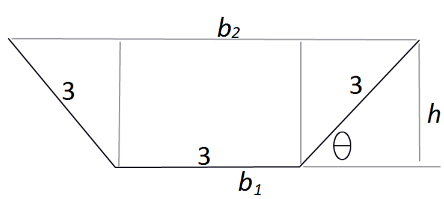
for $x \in [0, 23]$, since both x and y must be nonnegative. $p' = -2x + 23$ so we have $x = \frac{23}{2}$ is the only critical point. The product at the endpoints is 0, and $(\frac{23}{2})^2 > 0$ so this is the absolute maximum product possible.

2. A box's total dimensions (length + width + height) cannot exceed 108 in. If the box has a square base, what is the largest possible volume?
SOLUTION: We wish to maximize $v = lwh$. But first we want to express this as a single variable. Since the base is square, $l = w$. Also since the dimensions add up to 108, $l + w + h = 108$, which can be written $2l + h = 108$ since $l = w$. Then $h = 108 - 2l$. Finally we may state our optimization as follows: maximize:

$$v = l^2(108 - 2l) = -2l^3 + 108l^2$$

Where $l \in [0, 54]$, if the height is 108 inches, that is at least feasible, and a flat square 54×54 would also satisfy the requirements. The volume at the two endpoints is zero, so any critical point is a local maximum as long as its volume is > 0 (which of course it has to be!). Taking a first derivative, we get $v' = -6l^2 + 216l = 6l(-l + 36)$, so $l = 36$ is our critical point giving a $36 \times 36 \times 36$ cube, which maximizes volume.

3. A rain gutter is formed by taking a sheet of metal 9 in wide and bending it in thirds (making an isoceles trapezoid with an open top). What is the angle that maximizes cross-sectional area?
SOLUTION: The area of a trapezoid is $\frac{1}{2}(b_1 + b_2)h$. $b_1 = 3$ which is given.



The upper base depends on the angle θ. Consider the small triangles that are formed, say they have a base of x and height h. They have $\sin\theta = h/3$ and $\cos\theta = x/3$. Thus $h = 3\sin\theta, x = 3\cos\theta$. Then $b_2 = 3 + 2(3\cos\theta)$. Finally we have a formula for our area, we wish to maximize

$$a = \frac{1}{2}(3 + 3 + 2(3\cos\theta))(3\sin\theta) = 9\sin\theta + 9\cos\theta\sin\theta$$

and $\theta \in [0, \frac{\pi}{2}]$, since when $\theta > \frac{\pi}{2}$, $b_2 < 3$ and $h < 3$ so this is certainly not optimal. First derivative $a' = 9\cos\theta - 9\sin^2\theta + 9\cos^2\theta$, and set equal to zero we have

$$0 = \cos\theta + \cos^2\theta - \sin^2\theta$$

Since $\sin^2\theta = 1-\cos^2\theta$ from the Pythagorean Identity, we may let $u = \cos\theta$ and write

$$0 = u + u^2 - (1-u^2) = 2u^2 + u - 1 = (2u-1)(u+1)$$

So $u = \cos\theta = \frac{1}{2}, -1$, but since $\theta \in [0, \frac{\pi}{2}]$, it is impossible for $\cos\theta = -1$, so the only critical point is when $\cos\theta = \frac{1}{2}$ Which means $\theta = \frac{\pi}{3}$. And because at $\theta = 0$ and $\theta = \pi$ (the other critical point) the volume is zero, we must have $\theta = \frac{\pi}{3}$ is a local maximum. Or we can check:

$$a(\pi/3) = 9\frac{\sqrt{3}}{2} + 9\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = 9\frac{\sqrt{3}}{2}\left(1+\frac{1}{2}\right) = 9\frac{\sqrt{3}}{2}\frac{3}{2} = 9\left(\sqrt{\frac{27}{16}}\right) > 9$$

4. A rectangle with its base on the x-axis has two of its vertices on the parabola $y = 16 - x^2$. What are the dimensions that maximize area, and what is the maximum area?
SOLUTION: The coordinates of the lower corners are $(x, 0)$ and $(-x, 0)$ and the coordinates of the upper corners are $(x, 16-x^2)$ and $(-x, 16-x^2)$. So the base is $2x$ and height is $16-x^2$. So we wish to maximize
$$a = 2x(16-x^2) = -2x^3 + 32x$$
With $x \in [0, 4]$. Why 4 as an upper bound? Because this is where the parabola intersects the x axis.
The first derivative is $a' = -6x^2+32$. Setting this equal to zero we get $0 = 2(-3x^2+16)$ so $x = \pm\sqrt{16/3}$ Of course, only the positive solution is a feasible critical point. And because the area at the endpoints $x = 0$ and $x = 4$ is zero, this critical point is a local maximum and gives the absolute maximum area of $a(\sqrt{16/3}) = 2\sqrt{16/3}^3 + 32\sqrt{16/3}$.

5. A piece of wire 60 in. long is cut. One part is bent to make a square, the other is used to make a circle. Where should it be cut to maximize the combined area? Minimize combined area?
SOLUTION: Say we cut the wire at x inches, using x inches for the squareand $60 - x$ for the circle. Each side of the square will be $x/4$ so its area is
$$A_S = \frac{x^2}{16}$$
The circle will have a circumfrence of $60 - x$ and because $C = 2\pi r$, $r = (60-x)/(2\pi)$. The area of the circle is
$$A_C = \pi r^2 = \pi\left(\frac{60-x}{2\pi}\right)^2 = \frac{(60-x)^2}{4\pi}$$
The combined area $A(x) = A_S + A_C$. To find critical points we check the first derivative:
$$A'(x) = \frac{2x}{16} + \frac{2(60-x)(-1)}{4\pi} = \frac{\pi x}{8\pi} + \frac{4x-240}{8\pi} = \frac{(4+\pi)x - 240}{8\pi}$$
So the critical point is $x = 240/(4+\pi) \approx 33.606$. The first derivative is negative for $x = 0$ and positive for $x = 60$, so this critical point must be a local minimum. So the maximum combined area must be at one of the endpoints. $A(60) = 15^2 = 225$, while $A(0) \approx 286.479$.
To maximize combined area, use the entire wire for a circle. To minimize combined area, use 33.606 inches for the square.

6. Two right circular cones of height h and radius r are placed on either end of a cylinder of height h and radius r, making a doubly-pointed object. If the surface area is fixed at A, what dimensions h and r will maximize the total volume? (*Hint*: the volume of a cone is $v = \frac{1}{3}\pi r^2 h$, lateral surface area $s = \pi r l$ where l is the length from the base to the vertex. For a cylinder $v = \pi r^2 h$ and lateral surface

area $s = 2\pi hr$)
SOLUTION: The surface area of this object is

$$A = 2\pi rl + 2\pi hr$$

And because the cone is a right circular cone, $l = \sqrt{r^2 + h^2}$ giving

$$A = 2\pi r(\sqrt{r^2 + h^2} + h)$$

We may write: $\frac{A}{2\pi r} = \sqrt{r^2 + h^2} + h$. By squaring both sides we get

$$\left(\frac{A}{2\pi r}\right)^2 = (r^2 + h^2) + 2h\sqrt{r^2 + h^2} + h^2$$

and we may substitute $\sqrt{r^2 + h^2} = \frac{A}{2\pi r} - h$ into the equation to get

$$\left(\frac{A}{2\pi r}\right)^2 = (r^2 + h^2) + 2h\left(\frac{A}{2\pi r} - h\right) + h^2 = r^2 + h\frac{A}{\pi r}$$

Now we can solve for h, giving us

$$h = \left(\frac{A^2}{4\pi^2 r^2} - r^2\right)\left(\frac{\pi r}{A}\right) = \frac{A}{4\pi r} - \frac{\pi r^3}{A}$$

Since $h \geq 0$,

$$\begin{aligned} 0 &\leq \frac{A}{4\pi r} - \frac{\pi r^3}{A} \\ \frac{\pi r^3}{A} &\leq \frac{A}{4\pi r} \\ r^4 &\leq \frac{A^2}{4\pi^2} \\ r &\leq \sqrt{\frac{A}{2\pi}} \end{aligned}$$

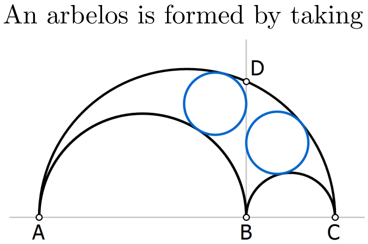
Meanwhile, the volume is

$$V = \pi r^2 h + 2\frac{\pi}{3}r^2 h = \frac{5\pi}{3}r^2 h = \frac{5\pi}{3}r^2\left(\frac{A}{4\pi r} - \frac{\pi r^3}{A}\right) = \frac{5A}{12}r - \frac{5\pi^2}{3A}r^5$$

$$V'(r) = \frac{5A}{12} - \frac{25\pi^2}{3A}r^4$$

Set equal to zero, we have only one critical point, $r = \sqrt{A}/(\sqrt[4]{20}\sqrt{\pi})$. The corresponding value of h is found by pluggint this in. After simplification, we have $h = \sqrt{A}\sqrt[4]{20}/(5\sqrt{\pi})$. Because the volume is zero at both endpoints, these are the maximum dimensions.

7. An arbelos is formed by taking a semicircle and removing two tangent semicircles as so:



Say $AC = 1$. At what position of B will the area of the arbelos be maximized? Show that the area of the arbelos is th same as a circle with diameter BD.

SOLUTION: Let the length from A to B be x. The area of a semicircle with diameter d is given by $A(d) = \frac{1}{2}\pi\left(\frac{d}{2}\right)^2 = \pi d/8$. So The area of the arbelos is

$$A(x) = \frac{\pi}{8} - \frac{\pi x^2}{8} - \frac{\pi(1-x)^2}{8} = \frac{\pi}{8}(1 - x^2 - (1-x)^2) = \frac{\pi}{8}(-2x^2 + 2x) = \frac{\pi}{4}x(1-x)$$

It can be easily shown that $x(1-x)$ is maximized when $x = \frac{1}{2}$.

The distance from B to the center of the larger circle is $|x - \frac{1}{2}|$. Since the radius of this circle is $1/2$, BD is found by the pythagorean theorem: $BD = \sqrt{1/2 - (x - 1/2)^2} = \sqrt{x(1-x)}$. A circle with this diameter has area $\pi\frac{x(1-x)}{4}$, the exact same as the area of the arbelos.