Math 180: Calculus I **Fall 2014**

October 9

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We may use the following derivative rules now:

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 \le x \le 1$$
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \quad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \quad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

Also to find the derivative of an inverse function $f^{-1}(y)$ at $y = y_0$, if $y_0 = f(x_0)$ and $f'(x_0) \neq 0$ then

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

1. Evaluate the following derivatives:

   (a) $f(x) = \sin^{-1}(2x)$
   SOLUTION:
   $$f'(x) = \frac{1}{\sqrt{1-(2x)^2}}(2)$$

   (b) $f(x) = \cos(\sin^{-1}(2x))$
   SOLUTION:
   $$f'(x) = -\sin\left(\sin^{-1}(2x)\right)\frac{1}{\sqrt{1-(2x)^2}}(2) = \frac{-4x}{\sqrt{1-(2x)^2}}$$

   (c) $f(x) = \tan^{-1}(1/x)$
   SOLUTION:
   $$f'(x) = \frac{1}{1+(1/x)^2}(-x^{-2}) = -\frac{1}{x^2+1}$$

   (d) $f(x) = \csc^{-1}(\tan(e^x))$
   SOLUTION:
   $$f'(x) = -\frac{1}{|(\tan(e^x))|\sqrt{(\tan(e^x))^2-1}}(\sec^2(e^x))(e^x)$$

   (e) $f(x) = 1/\tan^{-1}(x^2+4)$
   SOLUTION:
   $$f'(x) = -1(\tan^{-1}(x^2+4))^{-2}\frac{1}{1+(x^2+4)^2}(2x)$$

2. Find the equation of the tangent line at the given point

   (a) $f(x) = \tan^{-1}(2x); (\frac{1}{2}, \frac{\pi}{4})$
   SOLUTION:
   $$f'(x) = \frac{1}{1+(2x)^2}(2)$$

So $f'(\frac{1}{2}) = \frac{1}{1+1}(2) = 1$. So the equation of the tangent line is simply

$$y - \frac{\pi}{4} = x - \frac{1}{2}.$$

(b) $f(x) = \sec^{-1}(e^x); (\ln 2, \frac{\pi}{3})$
SOLUTION:

$$f'(x) = \frac{1}{|e^x|\sqrt{(e^x)^2 - 1}}(e^x)$$

So $f'(\ln 2) = \frac{1}{2\sqrt{4-1}}(2) = \frac{1}{\sqrt{3}}$. The equation of the tangent line is

$$y - \frac{\pi}{3} = \frac{1}{\sqrt{3}}(x - \ln 2).$$

3. Find the derivative of $f^{-1}(x)$ at the given point.

(a) $f(x) = 3x + 4; (16, 4)$
SOLUTION: The corresponding point in $f(x)$ is $(4, 16)$, where the slope is $f'(4) = 3$. So at $(16, 4)$, the inverse function has a slope of $1/3$. The equation for the tangent line is

$$y - 4 = \frac{1}{3}(x - 16).$$

(b) $f(x) = x^2 - 2x - 3$ for $x \leq 1; (12, -3)$
SOLUTION: The corresponding point in $f(x)$ is $(-3, 12)$. $f'(x) = 2x - 2$, so $f'(-3) = -8$. Therefore the slope of the tangent line at $(12, -3)$ on f^{-1} is $1/8$. The equation for this line is

$$y + 3 = \frac{1}{8}(x - 12).$$

4. Use trig properties to prove the following identity. For what values of x is it true?

$$\cos(2\sin^{-1} x) = 1 - 2x^2$$

(Hint: $\cos 2\theta = 1 - 2\sin^2\theta$)
SOLUTION:

$$\begin{aligned}
\cos(2\sin^{-1} x) &= 1 - 2\sin^2(\sin^{-1} x) && \text{because } \cos 2\theta = 1 - 2\sin^2\theta \\
&= 1 - 2(\sin(\sin^{-1} x))^2 && \text{rewriting slightly} \\
&= 1 - 2x^2 && \text{because } \sin(\sin^{-1} x) = x
\end{aligned}$$

But this only holds for $-1 \leq x \leq 1$.

5. Consider $f(x) = \sin(2\sin^{-1} x)$.

(a) What is the domain of f? Find the derivative $f'(x)$.
SOLUTION: The domain is $-1 \leq x \leq 1$. By the chain rule,

$$f'(x) = \cos(2\sin^{-1} x)\frac{2}{\sqrt{1 - x^2}}$$

(b) Find the equation of the tangent line to the graph when $x = \frac{1}{2}$.
SOLUTION: Plugging in $x = 1/2$, we have

$$f'(1/2) = \cos\left(2\sin^{-1}\left(\frac{1}{2}\right)\right)\frac{2}{\sqrt{1 - \frac{1}{4}}} = \cos\left(2\frac{\pi}{3}\right)\frac{4}{\sqrt{3}} = \frac{-1}{2}\frac{4}{\sqrt{3}} = \frac{-2}{\sqrt{3}}$$

(c) Use $\sin 2\theta = 2\sin\theta\cos\theta$ and $1 = \cos^2\theta + \sin^2\theta$ to show $f(x) = 2x\sqrt{1-x^2}$.
SOLUTION:

$$
\begin{aligned}
f(x) &= \sin(2\sin^{-1}x) \\
&= 2\sin(\sin^{-1}x)\cos(\sin^{-1}x) && \text{by trig identity} \\
&= 2x\sqrt{1-\sin^2(\sin^{-1}x)} && \text{because } \cos\theta = \sqrt{1-\sin^2\theta} \text{ for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\
&= 2x\sqrt{1-(\sin(\sin^{-1}x)^2} && \text{rewriting slightly} \\
&= 2x\sqrt{1-x^2}
\end{aligned}
$$