Math 180: Calculus I **Fall 2014**

November 4

TA: Brian Powers

1. Use linear approximation to first find the derivative at $x = a$, then estimate f at the given point.

   (a) $f(x) = 12 - x^2; \quad a = 2, f(2.1)$
   SOLUTION: $f'(x) = -2x$, and at $x = a$, $f(2) = 12 - 4 = 8$, $f'(2) = -4$. So a linear approximation is
   $$f(2.1) \approx f(2) + 0.1f'(2) = 8 + 0.1(-4) = 8 - .4 = 7.6$$

   (b) $f(x) = \ln(1 + x); \quad a = 0, f(0.1)$
   SOLUTION: $f'(x) = \frac{1}{1+x}$ and $f(0) = 0, f'(0) = 1$. So the linear approximation is
   $$f(0.1) \approx f(0) + 0.1f'(0) = 0 + 0.1(1) = 0.1$$

   (c) $f(x) = (8 + x)^{-1/3}; \quad a = 0, f(-0.1)$
   SOLUTION: $f'(x) = -\frac{1}{3}(8 + x)^{-4/3}$, $f(0) = \frac{1}{2}$ and $f'(0) = -\frac{1}{3 \cdot 16} = -\frac{1}{48}$. So the linear approximation is
   $$f(-0.1) \approx f(0) - 0.1f'(0) = \frac{1}{2} + 0.1\frac{1}{48} = \frac{241}{480}$$

2. Approximate the change in volume of a sphere when its radius changes from $r = 5$ to $r = 5.1$.
   The volume of the sphere is $v(r) = \frac{4}{3}\pi r^3$. $v'(r) = 4\pi r^2$. $v'(5) = 100\pi$. So the approximation is
   $$v(5.1) - v(5) \approx 0.1(100\pi) = 10\pi \text{ units}^3$$

3. Find the differential $dy = f'(x)dx$ for

   (a) $f(x) = 3x^2 - 4x$
   SOLUTION: $f'(x) = 6x - 4$, so $dy = (6x - 4)dx$.

   (b) $f(x) = \sin^2 x$
   SOLUTION: $f'(x) = 2\cos x \sin x$, so $dy = (2\cos x \sin x)dx$.

4. Write a linear approximation equation L of f at a. Do linear approximations for x near a over-estimate or under-estimate? (*Hint*: Look at concavity)

   (a) $f(x) = \frac{2}{x}; a = 1$
   SOLUTION: $f'(x) = -\frac{2}{x^2}$ and $f''(x) = \frac{4}{x^3}$. At $x = 1$, the linear approximation is
   $$f(x) \approx f(a) + f'(a)(x - a) = -2 + 4(x - 2) = 4x - 4$$
   Because at $x = 1$ the second derivative is positive, the curve is concave up, so the linear approximation is underestimating the true value.

   (b) $f(x) = \sqrt{2}\cos x; a = \frac{\pi}{4}$
   SOLUTION: $f'(x) = -\sqrt{2}\sin x$, $f''(x) = -\sqrt{2}\cos x$. So the linear approximation is
   $$f(x) \approx f(\pi/4) + f'(\pi/4)(x - \pi/4) = \sqrt{2}\frac{\sqrt{2}}{2} - \sqrt{2}\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) = -x + \frac{5\pi}{4}$$
   Since the second derivative is negative at $x = \pi/4$, the curve is concave down, and the linear approximation is over-estimating.