Math 180: Calculus I **Fall 2014**

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1. Evaluate the following limits using the facts that:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

   (a) $\lim_{x\to 0}\frac{\tan 5x}{x}$
   SOLUTION:

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan 5x}{x} &= \lim_{x\to 0}\frac{\sin 5x}{x\cos x} && \text{definition of tan}\\
&= \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin 5x}{x}\right) && \text{Limit product rule}\\
&= (1)\left(\lim_{x\to 0}\frac{\sin 5x}{x}\right) && \text{Evaluate first limit}\\
&= 5\left(\lim_{x\to 0}\frac{\sin 5x}{5x}\right) && \text{multiply by } \frac{5}{5}\\
&= 5\left(\lim_{u\to 0}\frac{\sin u}{u}\right) && \text{use substitution } u = 5x\\
&= 5(1) = 5
\end{aligned}$$

   (b) $\lim_{x\to -3}\frac{\sin(x+3)}{x^2+8x+15}$
   SOLUTION:

$$\begin{aligned}
\lim_{x\to -3}\frac{\sin(x+3)}{x^2+8x+15} &= \lim_{x\to -3}\frac{\sin(x+3)}{(x+3)(x+5)} && \text{factor denominator}\\
&= \lim_{u\to 0}\frac{\sin(u)}{(u)(u+2)} && \text{use substitution } u = x+3\\
&= \left(\lim_{u\to 0}\frac{\sin u}{u}\right)\left(\lim_{u\to 0}\frac{1}{u+2}\right) && \text{limit product rule}\\
&= (1)\left(\frac{1}{2}\right) = \frac{1}{2}
\end{aligned}$$

2. Evaluate the derivatives dy/dx

   (a) $y = \frac{\cos x}{\sin x + 1}$
   SOLUTION: Use the quotient rule with $u = \cos x$, v=$\sin x + 1$, so $u' = -\sin x$, $v' = \cos x$.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{u'v - v'u}{v^2}\\
&= \frac{(-\sin x)(\sin x + 1) - (\cos x)(\cos x)}{(\sin x + 1)^2}
\end{aligned}$$

(b) $y = \csc x$ using the quotient rule
SOLUTION: First rewrite $\csc x = \frac{1}{\sin x}$, so $u = 1$, $v = \sin x$, $u' = 0$, $v' = \cos x'$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{u'v - v'u}{v^2} \\ &= \frac{(0)(\sin x) - (\cos x)(1)}{\sin^2 x} \\ &= -\frac{\cos x}{\sin^2 x} \\ &= -\frac{\cos x}{\sin x}\frac{1}{\sin x} \\ &= -\cot x \csc x \end{aligned}$$

(c) $y = \frac{\cot x}{1+\csc x}$
SOLUTION: Quotient rule: $u = \cot x, v = 1 + \csc x, u' = -\csc^2 x, v' = -\cot x \csc x$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{u'v - v'u}{v^2} \\ &= \frac{(-\csc^2 x)(1 + \csc x) - (-\cot x \csc x)(\cot x)}{(1 + \csc x)^2} \end{aligned}$$

(d) $y = \frac{x \cos x}{1+x^3}$
SOLUTION: Quotient rule with $u = x \cos x, u' = -x \sin x + \cos x$ (by product rule), $v = 1 + x^3, v' = 3x^2$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{u'v - v'u}{v^2} \\ &= \frac{(-x \sin x + \cos x)(1 + x^3) - (3x^2)(x \cos x)}{(1 + x^3)^2} \end{aligned}$$

3. Evaluate the following limit or state it does not exist.

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx}, \text{ where } a \text{ and } b \text{ are constants with } b \neq 0$$

SOLUTION:

$$\begin{aligned} \lim_{x\to 0} \frac{\sin ax}{\sin bx} &= \lim_{x\to 0} \frac{\sin ax}{\sin bx}\left(\frac{abx}{abx}\right) && \text{multiply by 1} \\ &= \lim_{x\to 0} \frac{a \sin ax}{ax} \lim_{x\to 0} \frac{bx}{b \sin bx} && \text{limit product rule} \\ &= \left(a \lim_{u\to 0} \frac{\sin u}{u}\right)\left(\frac{1}{b} \lim_{v\to 0} \frac{v}{b \sin v}\right) && \text{let } u = ax, v = bx \\ &= \frac{a}{b}(1)\left(\lim_{v\to 0} \frac{b \sin v}{v}\right)^{-1} && = \text{evaluate limit and use limit exponent rule} \\ &= \frac{a}{b}(1)^{-1} && \text{evaluate limit} \end{aligned}$$

4. Find the following derivatives using the product rule

$$\frac{d}{dx}(\sin^2 x) \quad \frac{d}{dx}(\sin^3 x) \quad \frac{d}{dx}(\sin^4 x)$$

Make a conjecture about $\frac{d}{dx}(\sin^n x)$. See if you can prove it by induction!
SOLUTION:

$$\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x \sin x) = (\sin x \cos x) + (\cos x \sin x) = 2\cos x \sin x$$

$$\frac{d}{dx}(\sin^3 x) = \frac{d}{dx}(\sin x \sin^2 x) = (\sin x(2\cos x \sin x)) + (\cos x \sin^2 x) = 3\cos x \sin^2 x$$

$$\frac{d}{dx}(\sin^4 x) = \frac{d}{dx}(\sin x \sin^3 x) = (\sin x(3\cos x \sin^2 x)) + (\cos x \sin^3 x) = 4\cos x \sin^3 x$$

We may conjecture that

$$\frac{d}{dx}(\sin^n x) = n\cos x \sin^{n-1} x$$

Proof by induction:
We have already shown the base case for $n = 2, 3, 4$. Assume for $n \geq 2$ that $\frac{d}{dx}(\sin^n x) = n\cos x \sin^{n-1} x$. We wish to show that $\frac{d}{dx}(\sin^{n+1} x) = (n+1)\cos x \sin^{(n+1)-1}$.

$$\begin{aligned}
\frac{d}{dx}(\sin^{n+1} x) &= \frac{d}{dx}(\sin x \sin^n x) && \text{factor out a } \sin x \\
&= (\cos x \sin^n x) + (\sin x \frac{d}{dx}(\sin^n x)) && \text{product rule} \\
&= \cos x \sin^n x + \sin x(n\cos x \sin^{n-1} x) && \text{invoke inductive hypothesis} \\
&= \cos x \sin^n x + n\cos x \sin^n x \\
&= (n+1)\cos x \sin^{(n+1)-1} x
\end{aligned}$$

Done! It's worth noting that this is simply a special case of the **Chain Rule**, a very powerful rule for derivatives.

5. Use the fact that $\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h)$ to prove that $\frac{d}{dx}\cos x = -\sin x$ using the limit definition.
SOLUTION:

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h} \\
&= \cos x \left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) - \sin x \left(\lim_{h\to 0} \frac{\sin h}{h}\right) \\
&= \cos x(0) - \sin x(1) \\
&= -\sin x
\end{aligned}$$