Facts we can use

$$\lim_{x \to \infty} \frac{1}{x} = 0, \ \lim_{x \to -\infty} \frac{1}{x} = 0.$$

By limit laws we then have

$$\lim_{x \to \infty} \frac{c}{x^n} = 0$$

for any constant $c$ and for any positive exponent $n$ (whether it is an integer or a fraction). This is very handy!

## 3.1 Examples

**Example 3.1** *Evaluate the following limits:*

1. $\lim_{x \to \infty}(3 + \frac{10}{x^2})$

2. $\lim_{x \to \infty} \frac{3 + 2x + 4x^2}{x^2}$

3. $\lim_{x \to \infty} \frac{\cos(x^5)}{\sqrt{x}}$

4. $\lim_{x \to -\infty}(5 + \frac{100}{x} + \frac{\sin^4(x^5)}{x^2})$

5. $\lim_{x \to -\infty}(3x^7 + x^2)$

1.

$$
\begin{aligned}
\lim_{x \to \infty}\left(3 + \frac{10}{x^2}\right) &= \lim_{x \to \infty} 3 + \lim_{x \to \infty} \frac{10}{x^2} && \text{(Sum Law)} \\
&= 3 + 0 && \left(\text{as } x \to \infty, \frac{1}{x} \to 0\right) \\
&= 3
\end{aligned}
$$

2.

$$
\begin{aligned}
\lim_{x \to \infty} \frac{3 + 2x + 4x^2}{x^2} &= \lim_{x \to \infty} \frac{3}{x^2} + \lim_{x \to \infty} \frac{2\cancel{x}}{x^{\cancel{2}}} + \lim_{x \to \infty} \frac{4\cancel{x^2}}{\cancel{x^2}} && \text{(Sum Law)} \\
&= \lim_{x \to \infty} \frac{3}{x^2} + \lim_{x \to \infty} \frac{2}{x} + \lim_{x \to \infty} \frac{4}{1} && \text{(Cancelling common factors)} \\
&= 0 + 0 + 4 \\
&= 4
\end{aligned}
$$

3. We know that

$$-1 \le \cos(x) \le 1,$$

no matter what we take the cosine of. So it is also true if we replace $x$ with $x^5$.

$$-1 \le \cos(x^5) \le 1$$

If we divide by $\sqrt{x}$ we get

$$\frac{-1}{\sqrt{x}} \le \frac{\cos(x^5)}{\sqrt{x}} \le \frac{1}{\sqrt{x}}.$$

(The direction of the inequality does not change since $\sqrt{x} \ge 0$, and we are taking the limit as $x \to \infty$.) We already know that the limits of both upper and lower bounding functions as $x \to \infty$ is 0, so by the squeeze theorem,

$$\lim_{x \to \infty} \frac{\cos(x^5)}{\sqrt{x}} = 0.$$

4.

$$\lim_{x \to -\infty} \left( 5 + \frac{100}{x} + \frac{\sin^4(x^5)}{x^2} \right) = \lim_{x \to -\infty} 5 + \lim_{x \to -\infty} \frac{100}{x} + \lim_{x \to -\infty} \frac{\sin^4(x^5)}{x^2} \qquad \text{(Sum law)}$$

$$= 5 + 0 + 0 \qquad\qquad ( \lim_{x \to -\infty} \frac{\sin^4(x^5)}{x^2} = 0 \text{ by the squeeze theorem})$$

$$= 5$$

5. When we take the limit at infinity of a polynomial, we have a theorem that says it is equal to the limit of just the term with the highest power. So

$$\lim_{x \to -\infty} (3x^7 + x^2) = \lim_{x \to -\infty} 3x^7 = -\infty,$$

negative because as $x \to -\infty$, $x$ takes only negative values, and raising it to an odd power preserves the sign, so $3x^7$ is still negative.

**Example 3.2** *Find horizontal asymptotes or slant asymptotes if they exist.*

1. $f(x) = \frac{4x^2 - 7}{8x^2 + 5x + 2}$

2. $f(x) = \frac{40x^5 + x^2}{16x^4 - 2x}$

3. $f(x) = \frac{3x^2 - 2x + 7}{2x - 5}$

For the horizontal asymptotes are $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. So these must be evaluated.

We have theorem 2.7 which tells us how to handle the limits at infinity of a rational function (a polynomial divided by another polynomial)

1. Because the exponent of the leading terms of the numerator and denominator are both 2 (they are the same exponent), the limits at infinity are the ratio of the coefficients, $\frac{4}{8} = \frac{1}{2}$. So $y = \frac{1}{2}$ is the horizontal asymptote.

2. Because the exponent of the leading term of the numerator is greater than the largest exponent in the denominator, then neither limits (at $\infty$ or $-\infty$) exist. There is no horizontal asymptote. BUT because the highest power in the numerator is 1 more than the denominator, there is a slant asymptote. We find that by doing polynomial division (or synthetic division).

|  |  | $5x$ |  |  |  |  |  |
|---|---|---|---|---|---|---|---|
| $16x^4 - 2x$ | $40x^5$ | $0x^4$ | $0x^3$ | $1x^2$ | $0x$ | 0 |  |
|  | $-40x^5$ | $0x^4$ | $0x^3$ | $+10x^2$ | $0x$ |  |  |
|  | 0 | 0 | 0 | $11x^2$ | $0x$ | 0 |  |

So

$$\frac{40x^5 + x^2}{16x^4 - 2x} = 5x + \frac{11x^2}{16x^4 - 2x}.$$

As $x \to \infty$, the fraction portion goes to zero and the behavior is dominated by $5x$ - this is the slant asymptote: $y = 5x$ (it is the same as $x \to -\infty$.

3. Again, we use division to determine the slant asymptote.

$$
\begin{array}{r|lll}
 & \frac{3}{2}x & \frac{11}{4} & \\
\hline
2x - 5 & 3x^2 & -2x & +7 \\
 & -3x^2 & +\frac{15}{2}x & \\
\hline
 & & \frac{11}{2}x & 7 \\
 & & -\frac{11}{2}x & \frac{55}{4} \\
\hline
 & & & \frac{88}{4}
\end{array}
$$

So

$$\frac{3x^2 - 2x + 7}{2x - 5} = \frac{3}{2}x + \frac{11}{4} + \frac{88/4}{2x - 5}.$$

As before, the behavior as $x \to \infty$ or $-\infty$ has the last term go to zero, so our slant asymptote is $y = \frac{3}{2}x + \frac{11}{4}$.

**Example 3.3** *True or false:*

  1. *A graph of a function never crosses one of its horizontal asymptotes.*

  2. *A graph of a function can have at most 2 horizontal asymptotes.*

1. FALSE. The definition of a horizontal asymptote is only the long term behavior of a function as $x \to \infty$ or when $x \to -\infty$. It says nothing about crossing over this "dotted line". In fact, $f(x) = \frac{\sin(x)}{x}$ is a good example of a function that has a horizontal asymptote of $y = 0$ that is crossed infinitely many times.

2. TRUE. When you think about the definition it is obvious - there can possibly be one asymptote as $x$ goes to $\infty$ and a second as $x$ goes to $-\infty$ and that's it.

**Example 3.4** *Find vertical and horizontal asymptotes (and the left-hand/right-hand limits for any vertical asymptotes)*

  1. $f(x) = \sqrt{|x|} - \sqrt{|x - 1|}$

  2. $f(x) = \frac{|1 - x^2|}{x(x+1)}$

1. First of all, we get these vertical asymptotes when division by zero occurs (and a couple other situations) but that doesn't happen here. So there are no vertical asymptotes. for the limit as $x \to \infty$ you may apply some intuitive reasoning that for larger and larger values of $x$, the difference between these terms is going to be smaller and smaller until it is zero. this is correct, but not very analytical. We can use the method of conjugates to shine some light on the math.

$$\lim_{x \to \infty} \sqrt{|x|} - \sqrt{|x-1|} = \lim_{x \to \infty} \frac{\sqrt{|x|} - \sqrt{|x-1|}}{1} \left( \frac{\sqrt{|x|} + \sqrt{|x-1|}}{\sqrt{|x|} + \sqrt{|x-1|}} \right)$$

$$= \lim_{x \to \infty} \frac{|x| - |x-1|}{\sqrt{|x|} + \sqrt{|x-1|}}$$

$$= \lim_{x \to \infty} \frac{x - (x-1)}{\sqrt{x} + \sqrt{x-1}} \qquad \text{(because } x \text{ is positive and large)}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x} + \sqrt{x-1}} \left( \frac{1/\sqrt{x}}{1/\sqrt{x}} \right)$$

$$= \lim_{x \to \infty} \frac{1/\sqrt{x}}{1 + \sqrt{1 - 1/x}}$$

$$= \frac{0}{1\sqrt{1-0}}$$

$$= 0$$

Now it is a lot clearer that the numerator approaches 0 while the denominator approaches 2. If we instead take the limit as $x \to -\infty$ you will see that we get

$$\lim_{x \to -\infty} \sqrt{|x|} - \sqrt{|x-1|} = \lim_{x \to -\infty} \frac{|x| - |x-1|}{\sqrt{|x|} + \sqrt{|x-1|}}$$

$$= \lim_{x \to -\infty} \frac{-x - (-x+1)}{\sqrt{-x} + \sqrt{-x+1}} \qquad \text{(because } x \text{ is negative and large)}$$

$$= \lim_{x \to -\infty} \frac{-1}{\sqrt{-x} + \sqrt{-x+1}} \left( \frac{1/\sqrt{-x}}{1/\sqrt{-x}} \right)$$

$$= \lim_{x \to -\infty} \frac{1/\sqrt{-x}}{1 + \sqrt{1 + 1/x}}$$

$$= \frac{0}{1\sqrt{1-0}}$$

$$= 0$$

The same limit. $y = 0$ is the horizontal asymptote.

2. This function has division by zero when $x = 0$ and $x = -1$.

$$\lim_{x \to 0^+} \frac{|1 - x^2|}{x(x+1)} = \lim_{x \to 0^+} \frac{1 - x^2}{x(x+1)} = \infty$$

Because both factors in the denominator will be positive, and the numerator is close to 1.

$$\lim_{x \to 0^-} \frac{|1 - x^2|}{x(x+1)} = \lim_{x \to 0^-} \frac{1 - x^2}{x(x+1)} = -\infty$$

Because the numerator will be close to 1, while the denominator has a negative factor and a positive one.

Now turning to the behavior near $x = -1$:

$$\lim_{x \to -1^+} \frac{|1 - x^2|}{x(x+1)} = \lim_{x \to -1^+} \frac{1 - x^2}{x(x+1)} \qquad (\text{because } 1 - x^2 > 0 \text{ for } x > -1 \text{ but close})$$

$$= \lim_{x \to -1^+} \frac{(1 - x)\cancel{(1+x)}}{x\cancel{(x+1)}} \qquad (\text{by factoring})$$

$$= \lim_{x \to -1^+} \frac{1 - x}{x} \qquad (\text{by cancelling})$$

$$= -2$$

$$\lim_{x \to -1^-} \frac{|1 - x^2|}{x(x+1)} = \lim_{x \to -1^-} \frac{x^2 - 1}{x(x+1)} \qquad (\text{because } 1 - x^2 < 0 \text{ for } x < -1)$$

$$= \lim_{x \to -1^-} \frac{(x - 1)\cancel{(x+1)}}{x\cancel{(x+1)}} \qquad (\text{by factoring})$$

$$= \lim_{x \to -1^-} \frac{x - 1}{x} \qquad (\text{by cancelling})$$

$$= -2$$

So we have a vertical asymptote at $x = 0$ but a removable discontinuity at $x = -1$. Now turning our attention to the horizontal asymptotes.

$$\lim_{x \to \infty} \frac{|1 - x^2|}{x(x+1)} = \lim_{x \to -1^+} \frac{x^2 - 1}{x(x+1)} \qquad (\text{because } 1 - x^2 < 0 \text{ for large } x)$$

$$= \lim_{x \to \infty} \frac{(x - 1)\cancel{(x+1)}}{x\cancel{(x+1)}} \qquad (\text{by factoring})$$

$$= \lim_{x \to \infty} \frac{x - 1}{x} \qquad (\text{by cancelling})$$

$$= 1 \qquad (\text{ratio of leading coefficients})$$

And we will find that this is the same limit when $x \to -\infty$.

**Example 3.5** *The hyperbolic sine function is*

$$\sinh(x) = \frac{e^x - e^{-x}}{2}.$$

*Find its limit as $x \to \infty$ and as $x \to -\infty$.*

This example is now rather straight forward after all of the previous ones. We should first realize that

$$\lim_{x \to \infty} e^x = \infty, \ \lim_{x \to -\infty} e^x = 0, \ \lim_{x \to \infty} e^{-x} = 0, \ \text{and} \ \lim_{x \to -\infty} e^{-x} = \infty.$$

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{2} = \infty$$

since the $e^{-x}$ term has a finite limit of zero while the other is infinite.

$$\lim_{x \to -\infty} \frac{e^x - e^{-x}}{2} = -\infty$$

since the $e^x$ term has a finite limit of zero while the other is infinite, and it is being subtracted.