Exam 1 Review With Solutions

STAT 381, Spr15

Instructor: Brian Powers

Chapter 2

1. In how many ways can 5 different trees be planted in a row?
   ${}_5P_5 = 5! = 120$

2. How many subsets of $S = \{1, 2, 3, \ldots, 100\}$ contain 2 elements?
   $\binom{100}{2} = {}_{100}C_2 = \frac{100!}{2!98!} = \frac{100 \cdot 99}{2} = 4950$

3. In how many ways can 100 students be assigned 40 into dorm A, 35 into dorm B and 25 into dorm C?
   $\binom{100}{40}\binom{60}{35}\binom{25}{25} = \binom{100}{40,35,25} = \frac{100!}{40!60!}\frac{60!}{25!35!}\frac{25!}{25!} = \frac{100!}{40!35!25!}$

4. How many ways can a president, a vice president and a 3 person advisory committee be assigned from 30 people?
   $30 \cdot 29 \cdot \binom{28}{3} = 30 \cdot 29 \cdot \frac{28 \cdot 27 \cdot 26}{3 \cdot 2 \cdot 1} = 2850120$

Chapter 3

1. If A and B are independent events with $P(A) = 0.6$ and $P(B) = 0.3$, find the following:
   (a) $P(A \cup B)$ (b) $P(A' \cap B)$ (c) $P(A' \cup B')$ (d) $P(A|B)$ (e) $P(B'|A')$
   (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .3 - (.6)(.3) = .72$
   (b) $P(A' \cap B) = P(A')P(B) = (1 - P(A))P(B) = (.4)(.3) = .12$
   (c) $P(A' \cup B') = P(A') + P(B') - P(A' \cap B') = .4 + .7 - (.4)(.7) = .82$
   or $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - .18 = .82.$
   (d) $P(A|B) = P(A) = .6$
   (e) $P(B'|A') = P(B') = .7$

2. If A and B are mutually exclusive events with $P(A) = 0.6$ and $P(B) = 0.3$, find the following:
   (a) $P(A \cup B)$ (b) $P(A' \cap B)$ (c) $P(A' \cup B')$ (d) $P(A|B)$ (e) $P(B'|A')$
   (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .6 + .3 - 0 = .9$
   (b) $P(A' \cap B) = P(B) = .3$
   (c) $P(A' \cup B') = P(A') + P(B') - P(A' \cap B') = .4 + .7 - .1 = 1.0$
   (d) $P(A|B) = P(A \cap B)/P(B) = 0$
   (e) $P(B'|A') = P(B' \cap A')/P(A') = .1/.4 = .25$

3. You have to pass through an obstacle course. The probabilities that you make a mistake on each of the 4 obstacles is (respectively) 0.2, 0.3, 0.25 and 0.5. You pass the course if you make no more than 2 mistakes. What is the probability that you pass the course?
   Let A,B,C and D represent the events "make a mistake on obstacle A" and so forth.
   $P(\text{pass}) = P(\text{0,1, or 2 mistakes}) = 1 - P(\text{3 or 4 mistakes}).$

$P(4 \text{ mistakes}) = P(A \cap B \cap C \cap D) = (.2)(.3)(.25)(.5) = .0075$

$$\begin{aligned} P(3 \text{ mistakes}) &= P(A' \cap B \cap C \cap D) + P(A \cap B' \cap C \cap D) \\ &\quad + P(A \cap B \cap C' \cap D) + P(A \cap B \cap C \cap D') \\ &= (.8)(.3)(.25)(.5) + (.2)(.7)(.25)(.5) + (.2)(.3)(.75)(.5) + (.2)(.3)(.25)(.5) \\ &= .0775 \end{aligned}$$

So $P(\text{pass}) = 1 - (.0075 + .0775) = .915$

4. A random person has a probability of 0.36 of being a descendant of Ghengis Khan. A company advertises a blood test which can tell you if you are a descendant, and it is correct 99% of the time. If you take the test and it comes back negative, what is the probability you actually ARE descended from Ghengis?
Let G be the event of being related to Ghengis Khan, Y be the event that the test comes back as a "yes", and N be a "no". We are told the following probabilities:

$$P(G) = .36, P(Y|G) = .99, P(N|G') = .99$$

which implies that $P(N|G) = .01$ and $P(Y|G') = .01$. We are asked to find $P(G|N)$. This can be found by the Bayes formula

$$P(G|N) = \frac{P(G \cap N)}{P(N)} = \frac{P(G)P(N|G)}{P(G)P(N|G) + P(G')P(N|G')} = \frac{(.36)(.01)}{(.36)(.01) + (.64)(.99)}$$

This comes out to be about .00565 or .565%. Not very likely.

5. Your new neighbors have 2 children and you know at least one of them is a boy. You see one of them playing in the backyard and he is a boy - what is the probability the other child is a boy too? (Assume boys and girls are born with equal probability).
It's important to realize that you don't know whether you are looking at the younger or older child. The children could have been born in the following ways: BB, BG, GB. We are excluding the possibility that the children were born GG, since we know that one of the kids is a boy.
$P(2 \text{ boys}|\text{at least 1 boy}) = P(BB)/P(BB \cup BG \cup GB) = 1/3$

6. My fear of animals depends on how many legs they have, given in the table below, along with the probability that a random encounter will have the given number of legs:

# Legs	0	2	4	6	8	>8
Probability of encounter	0.05	0.70	0.10	0.05	0.09	0.01
Probability of fear	0.5	0.1	0.3	0.6	0.8	0.9

If I encounter an animal and I am afraid of it, what is the probability that it had no legs?
Let A be the event that I am afraid, and L the number of legs. You are asked for

$$P(L = 0|A)$$

We want to use Bayes rule for this

$$P(L = 0|A) = \frac{P(L = 0)P(A|L = 0)}{\sum_l P(L = l)P(A|L = l)}$$

The numerator is $(.05)(.5) = .025$, the denominator is

$$(.05)(.5) + (.7)(.1) + (.1)(.3) + (.05)(.6) + (.09)(.8) + .(01)(.9) = .236$$

So our answer is $P(L = 0|A) = .025/.236 \approx .1059$.

Chapter 4

1. A jar has 5 red and 10 blue marbles. I pick a handful of 4 marbles out. Let X be the number of red marbles in my hand. Find the pmf of X, find its expected value and its variance.
   The probability that you draw x red marbles is

$$P(X = x) = \frac{({}_5C_x)({}_{10}C_{4-x})}{{}_{15}C_4}$$

   So the probability mass function is given by

x	$f(x)$
0	$(1)(210)/1365 = 2/13$
1	$(5)(120)/1365 = 40/91$
2	$(10)(45)/1365 = 30/91$
3	$(10)(10)/1365 = 20/273$
4	$(5)(1)/1365 = 1/273$

   The expected value is

$$E(X) = 0\left(\frac{2}{13}\right) + 1\left(\frac{40}{91}\right) + 2\left(\frac{30}{91}\right) + 3\left(\frac{20}{273}\right) + 4\left(\frac{1}{273}\right) = \frac{4}{3}$$

   Variance is calculated by

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= 0^2\left(\frac{2}{13}\right) + 1^2\left(\frac{40}{91}\right) + 2^2\left(\frac{30}{91}\right) + 3^2\left(\frac{20}{273}\right) + 4^2\left(\frac{1}{273}\right) - \left(\frac{4}{3}\right)^2 \\ &\approx .6984 \end{aligned}$$

2. A 65 year old couple are considering a joint life insurance policy. The man has a probability of .90 of living at least 5 more years, .95 for the woman (and assume the event of either person dying is independent of the other). The insurance policy pays \$100,000 if one of them dies and \$150,000 if both die during this time. What is a fair cost for this policy?
   Since their deaths are assumed to be independent,
   $P(\text{neither dies}) = (.9)(.95) = .855$
   $P(\text{one dies}) = P(\text{only husband dies}) + P(\text{only wife dies}) = (.1)(.95) + (.9)(.05) = .14$
   $P(\text{both die}) = (.1)(.05) = .095$
   The payout is $0, 100000, 150000$ respectively in the three cases. The payout is the value the random variable takes. So its expected value is

$$E(X) = 0(.855) + 100000(.14) + 150000(.095) = 28250$$

   So a fair price for the policy would be exactly \$28,250, since that is the expected payout.

3. **The St. Petersburg Paradox** The game is as follows: You pay \$1000 to play. The pot starts at \$1. I flip a coin. If it is a tails you take the pot and the game is over. If it is heads then I double the pot and flip again. I keep flipping and doubling the pot until the coin comes up tails. What is the (net) expected value of the game? What if it costs \$1,000,000 to play?

   The payouts of the game can be given in the following table:

x	$f(x)$
$1 = 2^0$	$P(T) = 1/2$
$2 = 2^1$	$P(HT) = (1/2)^2$
$4 = 2^2$	$P(HHT) = (1/2)^3$
$\cdots$	$\cdots$

   So the expected payout is

   $$E(X) = 2^0\frac{1}{2^1} + 2^1\frac{1}{2^2} + 2^2\frac{1}{2^3} + \cdots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

   So even with a buy-in of \$1000, or even \$1,000,000, the net expected value is infinite. But realistically, at a certain point (say 1 quadrillion dollars, 1 followed by 15 zeroes) the pot can't grow any larger since this is how much money is on earth. The pot grows $\times 1000$ after roughly 10 flips of the coin ($2^{10} = 1024$). So it takes 50 flips for this to happen. If the game ends after 50 flips, then the expected payout is actually just \$25, so it's not worth the buy-in.

4. Consider a random variable X which can take any positive integer value (i.e. $1, 2, 3, \ldots$). Its pmf is

   $$f(x) = \frac{c}{4^x}.$$

   Find the value c, find its cdf, and calculate $P(X < 4)$. Try to find its expected value (tricky).

   To find the value of c, we solve the summation

   $$1 = \sum_{x=1}^{\infty} \frac{c}{4^x} = c\sum_{x=1}^{\infty}\left(\frac{1}{4}\right)^x$$

   The power series converges to $\frac{1}{1-1/4} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$, so $c = 3$.

   $P(X < 4) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} = \frac{63}{64}$.

   The expected value is

   $$E(X) = \sum_{x=1}^{\infty} \frac{3x}{4^x}$$

   Recall that

   $$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots) = \frac{1}{(1-x)^2}$$

   if $-1 < x < 1$. In our case, $x = \frac{1}{4}$, and the exponents are the variables. We want to re-express our expected value so it is in this form.

   $$E(X) = \sum_{x=1}^{\infty} \frac{3x}{4^x} = \frac{3}{4}\left(1 + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{4}\right)^3 + \cdots\right) = \frac{3}{4}\left(\frac{1}{(1-1/4)^2}\right)$$

   This evaluates to $\frac{4}{3}$.

5. Determine k so that the following is a valid pdf for X:

$$f(x) = k/\sqrt[3]{x}, \quad 0 < x < 4$$

Then find $E(X)$, $P(X < 2)$, and $Var(X)$.
We set the definite integral over the support equal to 1 to find the value of k:

$$1 = \int_0^4 kx^{-1/3}dx = k\frac{3}{2}x^{2/3}\Big|_0^4 = \frac{k3\sqrt[3]{16}}{2} = k3\sqrt[3]{2}$$

So $k = 1/3\sqrt[3]{2}$.

$$E(X) = \int_0^4 \frac{1}{3\sqrt[3]{2}}x^{2/3}dx = \frac{1}{3\sqrt[3]{2}}\frac{3}{5}x^{5/3}\Big|_0^4 = \frac{1}{5\sqrt[3]{2}}\sqrt[3]{4^5} = \frac{\sqrt[3]{512}}{5} = \frac{8}{5} = 1.6$$

$$P(X < 2) = \int_0^2 \frac{1}{3\sqrt[3]{2}}x^{-1/3}dx = \frac{1}{3\sqrt[3]{2}}\frac{3}{2}x^{2/3}\Big|_0^2 = \frac{1}{\sqrt[3]{4}}$$

$$E(X^2) = \int_0^4 \frac{1}{3\sqrt[3]{2}}x^{5/3}dx = \frac{1}{3\sqrt[3]{2}}\frac{3}{8}x^{8/3}\Big|_0^4 = \frac{1}{8\sqrt[3]{2}}\sqrt[3]{4^8} = \frac{\sqrt[3]{32768}}{8} = \frac{32}{8} = 4$$

So $Var(X) = E(X^2) - E(X)^2 = 4 - 1.6^2 = 1.44$

6. The cdf of Y is given by

$$F(y) = \begin{cases} 0 & y < 1 \\ \ln(y) & 1 \le y \le e \\ 1 & x > e \end{cases}$$

Find $f(y)$, $E(Y)$ and $Var(Y)$. Find $P(Y > 2)$.

$$f(y) = \frac{d}{dy}F(y) = \begin{cases} \frac{1}{y} & 1 \le y \le e \\ 0 & \text{otherwise} \end{cases}$$

$$E(Y) = \int_1^e yf(y)dy = \int_1^e 1dy = y|_1^e = e - 1$$

$$E(Y^2) = \int_1^e y^2 f(y)dy = \int_1^e ydy = \frac{1}{2}y^2\Big|_1^e = \frac{e^2 - 1}{2}$$

So $Var(Y) = E(Y^2) - E(Y)^2 = \frac{e^2-1}{2} - (e^2 - 2e + 1) = \frac{-e^2+4e-3}{2} \approx .242.$

$$P(Y > 2) = 1 - P(Y < 2) = 1 - F(2) = 1 - \ln(2)$$

7. If X and Y are independent, with $\mu_X = 10, \sigma_X^2 = 8, \mu_Y = -3, \sigma_Y = 2$, find the mean and variance of:
(a) $W = 3X - 8Y$ (b) $T = X + Y$
Note: I meant to write $\sigma_Y^2 = 2$, but we'll go with the typo. σ_Y is the standard deviation of Y, so $Var(Y) = 4$.
$E(W) = E(3X - 8Y) = 3E(X) - 8E(Y) = 3(10) - 8(-3) = 30 + 24 = 52$
$Var(W) = Var(3X-8Y) = 3^2Var(X)+(-8)^2Var(Y) = 9(8)+64(4)=72+256=328$
$E(T) = E(X + Y) = E(X) + E(Y) = 10 - 3 = 7$
$Var(T) = Var(X + Y) = Var(X) + Var(Y) = 8 + 4 = 12$

8. Let X be the number of heads out of 4 flips of a fair coin, let Y_i be the ith roll of a 6-sided die. Find the mean and variance of:
(a) $X+Y_1$ (b)Y_1-Y_2 (c) $2X+3Y_2+1.52$

We need the pmf of X first.

x	$f(x)$
0	$\frac{1}{2^4}=\frac{1}{16}$
1	$\binom{4}{1}\frac{1}{16}=\frac{4}{16}$
2	$\binom{4}{2}\frac{1}{16}=\frac{6}{16}$
3	$\binom{4}{3}\frac{1}{16}=\frac{4}{16}$
4	$\binom{1}{16}$

$E(X)=0(\frac{1}{16})+1(\frac{4}{16})+2(\frac{6}{16})+3(\frac{4}{16})+4(\frac{1}{16})=\frac{4+12+12+4}{16}=2$
$E(X^2)=0^2(\frac{1}{16})+1^2(\frac{4}{16})+2^2(\frac{6}{16})+3^2(\frac{4}{16})+4^2(\frac{1}{16})=\frac{4+24+36+16}{16}=5$
$Var(X)=E(X^2)-E(X)^2=5-2^2=1$

$E(Y_i)=1(\frac{1}{6})+2(\frac{1}{6})+3(\frac{1}{6})+4(\frac{1}{6})+5(\frac{1}{6})+6(\frac{1}{6})=\frac{21}{6}=3.5$
$E(Y_i^2)1^2(\frac{1}{6})+2^2(\frac{1}{6})+3^2(\frac{1}{6})+4^2(\frac{1}{6})+5^2(\frac{1}{6})+6^2(\frac{1}{6})=\frac{1+4+9+16+25+36}{6}=\frac{91}{6}$
$Var(Y_i)=E(Y_i^2)-E(Y)^2=\frac{91}{6}-\frac{21^2}{6^2}=\frac{108}{36}=3$
(a) $E(X+Y_1)=E(X)+E(Y_1)=2+3.5=5.5,$

$Var(X+Y_1)=Var(X)+Var(Y_1)=5+3=8.$

(b) $E(Y_1-Y_2)=E(Y_1)-E(Y_2)=3.5-3.5=0,$
$Var(Y_1-Y_2)=Var(Y_1)+(-1)^2Var(Y_2)=3+3=6$

(c) $E(2X+3Y_2+1.52)=2E(X)+3E(Y_2)+1.52=2(2)+3(3.5)+1.52=16.02,$
$Var(2X+3Y_2+1.52)=2^2Var(X)+3^2Var(Y)=4(1)+9(3)4+18=22$