Math 320

Feb. 20 Midterm Solutions

1. (20 pts)

Show that the set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ **spans** $\mathbb{R}^3$**.**

Given an arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ we can write

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a-b)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (b-c)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

2. (20 pts)

(a) Find a basis for the row space of $A = \begin{pmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$.

Basis $= \{ \begin{pmatrix} 1 & 0 & -3 & 0 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 4 & 0 & -6 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 & 7 \end{pmatrix} \}$

(b) Find the nullspace of A**.**

$$\text{Nullspace} = \left\{ \begin{pmatrix} 3a-5b \\ -4a+6b \\ a \\ -7b \\ b \end{pmatrix} \mid a, b \in \mathbb{R} \right\} \subset \mathbb{R}^5$$

(c) Find a basis and the dimension of the nullspace of A**.**

$$\text{Basis} = \left\{ \begin{pmatrix} 3 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 6 \\ 0 \\ -7 \\ 1 \end{pmatrix} \right\}, \text{ Dimension} = 2.$$

3. (30 pts)

(a) Find a basis for the subspace of 2 by 2 matrices given by $W = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a + c = 0 \}$**.**

Rewrite W as $\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \}$. There are only two free variables, a and b, so the basis will have two elements. Plug 0 and 1 in for a and b, respectively, to get one basis element, and the reverse to get the other.

Basis $= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$.

(b) Prove that this is a basis for W**.**

The set spans W, since

$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
and is linearly independent, since
$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$
implies $a = 0$ and $b = 0$.

4. (32 pts) Answer each of the following TRUE or FALSE. If TRUE, explain why; if FALSE give a counter example.
(a) If $\{v, w, x\}$ is a linearly dependent set, then $\{v, w\}$ is also linearly dependent.

FALSE. In $\mathbb{R}^2$, set $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) If $\{v, w, x\}$ spans V, then $\{v, w, x, y\}$ also spans V, where $y \in V$.

TRUE. Given any three scalars a, b, c, we have
$av + bw + cx = av + bw + cx + 0y$,
so the span of $\{v, w, x, y\}$ contains V. Since $y \in V$ there exist scalars i, j, k such that $y = iv+jw+kx$.
Given any four scalars a, b, c, d we have
$av + bw + cx + dy = (a + di)v + (b + dj)w + (c + dk)x$,
so V contains the span of $\{v, w, x, y\}$.

(c) A linear system of three equations and five unknowns always has at least one solution.

FALSE. The system
$x_1 + x_2 + x_3 + x_4 + x_5 = 1$
$x_1 + x_2 + x_3 + x_4 + x_5 = 2$
$x_1 + x_2 + x_3 + x_4 + x_5 = 3$
has no solution.

5. (30 pts)
(a) Is $W = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ a vector space? (Assume the usual $\cdot$ and $+$ operations from $\mathbb{R}^3$.)

No, W is not a vector space.

(b) If yes, find a basis. If no, show it is not a vector space.

Assume toward contradiction that W is a vector space. Then it is a subspace of $\mathbb{R}^3$ and therefore contains $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Writing this in the form given in the definition, we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

which implies

$0 = 1 - 2a - 3b$

$0 = 2 + a$

$0 = 3 + b.$

The last two equations in particular imply $a = -2$ and $b = -3$, which when plugged into the first equation gives the falsehood

$0 = 1 - 2(-2) - 3(-3) = 14.$