DAVID TALL AND SHLOMO VINNER

CONCEPT IMAGE AND CONCEPT DEFINITION IN MATHEMATICS WITH PARTICULAR REFERENCE TO LIMITS AND CONTINUITY

ABSTRACT. The concept image consists of all the cognitive structure in the individual's mind that is associated with a given concept. This may not be globally coherent and may have aspects which are quite different from the formal concept definition.

The development of limits and continuity, as taught in secondary school and university, are considered. Various investigations are reported which demonstrate individual concept images differing from the formal theory and containing factors which cause cognitive conflict.

Compared with other fields of human endeavour, mathematics is usually regarded as a subject of great precision in which concepts can be defined accurately to provide a firm foundation for the mathematical theory. The psychological realities are somewhat different. Many concepts we meet in mathematics have been encountered in some form or other before they are formally defined and a complex cognitive structure exists in the mind of every individual, yielding a variety of personal mental images when a concept is evoked. In this paper we formulate a number of general ideas intended to be helpful in analysing these phenomena and apply them to the specific concepts of continuity and limits.

1. CONCEPT IMAGE AND CONCEPT DEFINITION

The human brain is not a purely logical entity. The complex manner in which it functions is often at variance with the logic of mathematics. It is not always pure logic which gives us insight, nor is it chance that causes us to make mistakes. To understand how these processes occur, both successfully and erroneously, we must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived.

Many concepts which we use happily are not formally defined at all, we learn to recognise them by experience and usage in appropriate contexts. Later these concepts may be refined in their meaning and interpreted with increasing subtlety with or without the luxury of a precise definition. Usually in this process the concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colours the meaning of the concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or

otherwise. During the mental processes of recalling and manipulating a concept, many associated processes are brought into play, consciously and unconsciously affecting the meaning and usage.

We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

For instance the concept of subtraction is usually first met as a process involving positive whole numbers. At this stage children may observe that subtraction of a number always *reduces* the answer. For such a child this observation is part of his concept image and may cause problems later on should subtraction of negative numbers be encountered. For this reason all mental attributes associated with a concept, whether they be conscious or unconscious, should be included in the concept image; they may contain the seeds of future conflict.

As the concept image develops it need not be coherent at all times. The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole.

We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. Children doing mathematics often use different processes according to the context, making different errors depending on the specific problem under consideration. For instance adding $\frac{1}{2} + \frac{1}{4}$ may be performed correctly but when confronted by $\frac{1}{2} + \frac{1}{3}$, an erroneous method may be used. Such a child need see no conflict in the different methods, he simply utilises the method he considers appropriate on each occasion.

The definition of a concept (if it has one) is quite a different matter. We shall regard the *concept definition* to be a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his (evoked) concept image. Whether the concept definition is given to him or constructed by himself, he may vary it from time to time. In this way a *personal* concept definition can differ from a *formal* concept definition, the latter being a concept definition which is accepted by the mathematical community at large.

For each individual a concept definition generates its *own* concept image (which might, in a flight of fancy be called the "concept definition image"). This is, of course, part of the concept image. In some individuals it may be empty, or virtually non-existent. In others it may, or may not be coherently related to other parts of the concept image. For instance the concept definition of a mathematical function might be taken to be "a relation between two sets A and B in which each element of A is related to precisely one element in B". But individuals who have studied functions may or may not remember the concept definition and the concept image may include many other aspects, such as the idea that a function is given by a rule or a formula, or perhaps that several different formulae may be used on different parts of the domain A. There may be other notions, for instance the function may be thought of as an *action* which maps a in A to $f(a)$ in B, or as a *graph*, or a *table of values*. All or none of these aspects may be in an individual's concept image. But a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure. Initially the student in this position can operate quite happily with his restricted notion adequate in its restricted context. He may even have been taught to respond with the correct formal definition whilst having an inappropriate concept image. Later, when he meets functions defined in a broader context he may be unable to cope. Yet the teaching programme itself has been responsible for this unhappy situation.

As we shall see shortly the concept images of limit and continuity are quite likely to contain factors which conflict with the formal concept definition.

Some of these are subtle and may not even be consciously noted by the individual but they can cause confusion in dealing with the formal theory. The latter is concerned only with that part of the concept definition image which is generally mutually acknowledged by mathematicians at large. For instance, the verbal definition of a limit "$s_n \to s$" which says "we can make s_n as close to s as we please, provided that we take n sufficiently large" induces in many individuals the notion that s_n cannot be equal to s (see Schwarzenberger and Tall, 1978). In such an individual this notion is part of his concept definition image, but not acknowledged by mathematicians as part of the formal theory.

We shall call a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition, a *potential conflict factor*. Such factors need never be evoked in circumstances which cause actual cognitive conflict but if they are so evoked the factors

concerned will then be called *cognitive conflict factors*. For instance the definition of a complex number $x + iy$ as an ordered pair of real numbers (x, y) and the identification of $x + i0 = (x, 0)$ as the real number x is a potential conflict factor in the concept of complex number. This is because it includes a potential conflict with the set-theoretic notion that the element x is *distinct* from the ordered pair $(x, 0)$. Students in a questionnaire (Tall, 1977b) often regarded a real number such as $\sqrt{2}$ as not being a complex number and yet several of these defined real numbers as "complex numbers with imaginary part zero". Thus $\sqrt{2}$ was regarded as real and $\sqrt{2} + i0$ as complex. These were conveniently considered as being distinct entities or the same, depending on the circumstances, without causing any cognitive conflict. They only become *cognitive* conflict factors when evoked simultaneously.

In certain circumstances cognitive conflict factors may be evoked subconsciously with the conflict only manifesting itself by a vague sense of unease. We suggest that this is the underlying cause for such feelings in problem solving or research when the individual senses something wrong somewhere; it may be a considerable time later (if at all) that the reason for the conflict is consciously understood.

A more serious type of potential conflict factor is one in the concept image which is at variance not with another part of the concept image but with the formal concept definition itself. Such factors can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors unless the formal concept definition develops a concept image which can then yield a cognitive conflict. Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous.

The notions so far described are all clearly manifested in the various concepts of limit and continuity. In the remainder of the article we describe a few of the problems caused by a concept image which does not coherently relate to the concept definition and the resulting potential conflicts.

2. PRACTICAL CURRICULUM PROBLEMS

There are several practical problems imposed in the teaching of the concepts of limits and continuity. If we confine ourselves to the three notions set out:

(i) limit of a sequence $\lim\limits_{n \to \infty} s_n$

(ii) limit of a function $\lim\limits_{x \to a} f(x)$

(iii) continuity of a function $f: D \to \mathbb{R}$