Solutions for Take-Home Quiz due 1/28

1. Prove the following statement: Two matrices have the same reduced echelon form if and only if they are row equivalent. (Hint: This was proved quickly in class. Write down the complete argument using equivalence relation properties and quoting statements from the book.)

$\Leftarrow$: Suppose two matrices A and B have the same reduced echelon form, C. Then $A \sim C$ and $B \sim C$, where $\sim$ denotes row equivalence. Row equivalence is symmetric and transitive (Lemma 1.5), so $A \sim B$.

$\Rightarrow$: Suppose two matrices A and B are row equivalent, that is A B. Both A and B have unique reduced echelon forms (Theorem 2.7); call them A' and B' respectively. Then $A \sim A'$ and $B \sim B'$. Again using symmetry and transitivity (Lemma 1.5) we have $A' \sim B'$. Thus A' and B' are both reduced echelon forms of A. Hence by uniqueness, $A' = B'$.

2. Is the vector $\begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix}$ in the set of vectors given by $\begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} y$ with x and y real numbers? (Hint: This can be rewritten as a linear system of equations. Then use Gauss-Jordan reduction to solve the system.

We want to solve $\begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} y.$

Combine the constant terms to get $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} y = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}.$

In matrix form this is $\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 3 & 1 & 7 \\ 4 & 2 & 10 \end{array}\right)$

which row-reduces to $\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$

giving values $x = 2$ and $y = 1$. Plugging into the original equation,

$\begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot 2 + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot 1$

indeed. The answer is "Yes".