Math310: Final Exam
Fall 2006

Problem 1.(50 pts) Find the eigenvalues and corresponding eigenspaces for the matrix. Decide whether the matrix is diagonalizable or not. (Explain!)

$$a) \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \qquad b) \begin{pmatrix} 2 & 3 \\ -3 & 8 \end{pmatrix}.$$

Problem 2. Let $A = \begin{pmatrix} -10 & 14 & 10 \\ 1 & -2 & -1 \\ -8 & 16 & 11 \end{pmatrix}$.

It is given that A has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = -1$ with corresponding eigenvectors $x_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

a) (15 pts) Write down a factorization $A = XDX^{-1}$, where D is diagonal.

b) (10 pts) Find B such that $B^2 = A$.

c) (15 pts) Compute e^A.

d) (10 pts) If map $L : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $L(x) = Ax$ for any $x \in \mathbb{R}^3$, show that L is a linear transformation.

e) (15 pts) What is the matrix representation of L with respect to the basis $\{x_1, x_2, x_3\}$?

f) (10 pts) Let $y_1 = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$, $y_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $y_3 = \begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$. Show that $\{y_1, y_2, y_3\}$ is a basis of $\mathbb{R}^3$.

g) (25 pts) What is the matrix representation of L with respect to $\{y_1, y_2, y_3\}$?

Problem 3. Let

$$S = \text{Span}\left\{ \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -5 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 2 \\ 6 \\ 1 \end{pmatrix}.$$

a) (10 pts) Does b belong to S? Can S be equal to $\mathbb{R}^4$?

b) (15 pts) Find a basis of S. What is the dimension of S?

c) (25 pts) Use the Gram-Schmidt process and your answer for part b) to find an orthonormal basis of S.

d) (Extra credit: 15 pts) Find an element of S which is closest to b.