Solutions for Take-Home Quiz due 2/9

1. Show that the set of 2 by 2 matrices forms a vector space.

Let $a, b, c, d, k, a', b', c', d'$ be field elements. The equation
$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} + k'\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} ka + k'a' & kb + k'b' \\ kc + k'c' & kd + k'd' \end{pmatrix}$
shows that the set of 2 by 2 matrices is closed under linear combinations.

This is not enough to show it is a vector space unless we consider it as a subspace of some set which we already know is a vector space. To do this one can consider 2 by 2 matrices as a subset of $\mathbb{R}^4$ (take $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the vector $(a\ b\ c\ d)$.)

Another approach would be to check all of the ten axioms directly for 2 by 2 matrices. This is actually pretty simple and not that long.

2. Does the following set of vectors span $\mathbb{R}^3$? $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right\}$

Combine the vectors into a matrix:
$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 0 & 2 & 4 \\ 3 & -1 & 2 & 6 \end{pmatrix}$
Row reduce it:
$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
The reduced form has exactly two "leading variables". Thus the set spans a 2-dimensional subspace of $\mathbb{R}^3$. Since $\mathbb{R}^3$ is three-dimensional, the answer is "no".

3.(See 2.3.1.24) Find a basis in P_2 (polynomials of degree less than or equal to two) for the subspace of polynomials such that $p(3) = 0$.

One possibility: $\{(x-3)^2, (x-3)\}$