Math320 Final Exam Spring 2008 Solutions

1. $A = \begin{pmatrix} 1 & -1 & 2 \\ -3 & -1 & 6 \\ -1 & -1 & 4 \end{pmatrix}$ $\quad |A - xI| = \begin{vmatrix} 1-x & -1 & 2 \\ -3 & -1-x & 6 \\ -1 & -1 & 4-x \end{vmatrix}$

$= (1-x)(-1-x)(4-x) + 6 + 6 + 2\cdot(1-x) + 6\cdot(1-x) - 3\cdot(4-x)$

$= -4 + x + 4x^2 - x^3 + 12 + 2 - 2x + 6 - 6x - 12 + 3x$

$= -x^3 + 4x^2 - 4x = (-x)(x-2)(x-2)$

E-values $\lambda = 0, 2$

$\boxed{\lambda=0}$ find nullspace of $A = A - 0$

$\begin{bmatrix} 1 & -1 & 2 \\ -3 & -1 & 6 \\ -1 & -1 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{nullspace} = \left\{ z \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \mid z \in \mathbb{R} \right\}}$

$\boxed{\lambda=2}$ nullspace of $A - 2I$ $\quad \begin{bmatrix} -1 & -1 & 2 \\ -3 & -3 & 6 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{nullspace} = \left\{ \begin{pmatrix} -y+2z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\}}$

$\boxed{X = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$

Want $A = XDX^{-1}$ or equivalently, $AX = XD$

Check $AX = \begin{pmatrix} 1 & -1 & 2 \\ -3 & -1 & 6 \\ -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -2 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \cancel{1} & -1 & 2 \\ \cancel{3} & 1 & 0 \\ \cancel{1} & 0 & 1 \end{pmatrix}$

$XD = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(see calculations also on next page)

2. $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & c & 0 \\ 5 & 1 & 1 \end{pmatrix}$ $\quad |A| = 3c \quad$ So,

Consider $|A - xI| = \begin{vmatrix} 3-x & 0 & 0 \\ 0 & c-x & 0 \\ 5 & 1 & 1-x \end{vmatrix} = (3-x)(c-x)(1-x)$

Independent of the value of c; $x = 3, 1$ will be eigenvalues. If c is 3 or 1 these will be double roots.

Consider $x=3$ $\quad A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c-3 & 0 \\ 5 & 1 & -2 \end{pmatrix}$

If $c=3$, this nullspace has dimension two (Two free variables.)

If $c \neq 3$, the null space is dimension one.

Consider $x=1$ $\quad A - 1 \cdot I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & c-1 & 0 \\ 5 & 1 & 0 \end{pmatrix}$

Whether $c=1$ or $c \neq 1$ this nullspace has dimension one. $\{(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}) z \mid z \in \mathbb{R}\}$

Calculations

$x=3, c=3 \Rightarrow A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 1 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1}{5} & \frac{-2}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

nullspace $= \left\{ \begin{pmatrix} \frac{-1}{5}y + \frac{2}{5}z \\ y \\ z \end{pmatrix} = y \cdot \begin{pmatrix} \frac{-1}{5} \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} \frac{2}{5} \\ 0 \\ 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\}$

$x=3, c \neq 3 \Rightarrow A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c-3 & 0 \\ 5 & 1 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & \frac{-2}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

nullspace $= \left\{ \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix} \cdot z \mid z \in \mathbb{R} \right\}$

$x=1$, c = anything $\Rightarrow A - 1 \cdot I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & c-1 & 0 \\ 5 & 1 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

nullspace $= \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z \mid z \in \mathbb{R} \right\}$

Consider $x=c$, with $c \neq 1$ or 3 $\quad A - cI = \begin{pmatrix} 3-c & 0 & 0 \\ 0 & 0 & 0 \\ 5 & 1 & 1-c \end{pmatrix}$

$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1-c \\ 0 & 0 & 0 \end{pmatrix}$ nullspace $= \left\{ \begin{pmatrix} 0 \\ 1-c \\ 1 \end{pmatrix} \cdot z \mid z \in \mathbb{R} \right\}$

$\boxed{\text{Summary}}$ If $c=3$, A is diagonalizable.
(e-values $\lambda=3$ with e-vectors $\begin{pmatrix} \frac{-1}{5} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$; e-value $\lambda=1$ e-vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$)

If $c \neq 1$ or 3, A is diagonalizable
(e-values $\lambda=3$, e-vector $\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$; $\lambda=1$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; $\lambda=c$, $\begin{pmatrix} 0 \\ 1-c \\ 1 \end{pmatrix}$)

If $c=1$, A is not diagonalizable.
(e-values $\lambda=3$, e-vector $\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$; $\lambda=1$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ no other e-values or independent e-vectors)

$\boxed{\text{Answer: } c \neq 1, A \text{ is diagonalizable}}$

3. Mostly not covered in Spring 2009 Math 320.
For those who are curious: $A = XDX^{-1}$ with $X = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

a) $\det A = \det(XDX^{-1}) = (\det X)(\det D)(\det X^{-1})$
$= (-1) \cdot (-1) \cdot (\frac{1}{-1}) = -1 = |A|$

$\det(A^6) = (\det A)^6 = (-1)^6 = 1 = |A^6|$

Uses $\det(MN) = \det(M)\det(N)$

b) Note $A^3 = (XDX^{-1}) \cdot (XDX^{-1}) \cdot (XDX^{-1})$
$= X \cdot D^3 \cdot X^{-1}$
$= XDX^{-1} = A!$

$D^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}!$

So $B = A$ has $A^3 = A$

c) $e^A = Xe^DX^{-1} = X\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}X^{-1}$

Spring 2008 continued

4. Find basis for S: $\begin{bmatrix} 1 & -2 & 2 & 5 \\ 1 & -2 & 0 & 1 \\ 1 & -2 & 1 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a) basis $= \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix} \right\}$ $\dim S = 2$

b) orthogonalize: $k_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $k_2 = \begin{pmatrix}2\\0\\1\end{pmatrix} - \mathrm{Proj}_{\begin{pmatrix}1\\1\\1\end{pmatrix}} \begin{pmatrix}2\\0\\1\end{pmatrix}$
$= \begin{pmatrix}2\\0\\1\end{pmatrix} - \left(\frac{3}{3}\right)\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\rangle$

$b = \begin{pmatrix}4\\2\\3\end{pmatrix} = c_1 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\quad c_1 = \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot \vec{b}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}} = \frac{9}{3} = 3$

$= 3 \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} + 1 \cdot \begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\quad c_2 = \frac{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot \vec{b}}{\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}} = \frac{2}{2} = 1$

5. $V = \mathrm{Span}\{\cos x, \sin x\}$

a) $D(\cos x) = -\sin x$
$D(\sin x) = \cos x$

So $D(a\cos x + b \sin x) = -a \sin x + b \cos x$

So $D\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b\\-a\end{pmatrix}$

or $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} = \mathrm{Rep}_{E,E}(D)$ b) $E = \langle \cos x, \sin x \rangle$

or check D preserves linear combinations

i) $D\left(r \cdot \begin{pmatrix}x\\y\end{pmatrix} + s \cdot \begin{pmatrix}z\\w\end{pmatrix}\right)$
$= D\begin{pmatrix}rx+sz\\ry+sw\end{pmatrix} = \begin{pmatrix}ry+sw\\-rx-sz\end{pmatrix}$

ii) $D\left(r \cdot \begin{pmatrix}x\\y\end{pmatrix}\right) = D\begin{pmatrix}rx\\ry\end{pmatrix} = \begin{pmatrix}ry\\-rx\end{pmatrix} = r \cdot \begin{pmatrix}y\\-x\end{pmatrix}$ $\Big\}$ $r\cdot\begin{pmatrix}y\\-x\end{pmatrix} + s \cdot \begin{pmatrix}w\\-z\end{pmatrix}$

$D\left(s \cdot \begin{pmatrix}z\\w\end{pmatrix}\right) = s \cdot \begin{pmatrix}w\\-z\end{pmatrix}$

6. a) One way to see this: is to see from the rref that the rank is 3. Since $L: \mathbb{R}^3 \to \mathbb{R}^4$ the domain has dimension 3 so nullity must be 0 (so rank + nullity = dim (domain)). $\ker(L) = \{\vec{0}\}$. Or: rref(A) has no free variables.

b) range(L) = column space (L) = Span $\left\{ \begin{pmatrix}3\\2\\1\\6\end{pmatrix}, \begin{pmatrix}1\\5\\-1\\7\end{pmatrix}, \begin{pmatrix}2\\7\\1\\8\end{pmatrix} \right\}$ ($\neq$ col space of rref(A)!)

c) rank(L) = 3, nullity = 0 (see a, b)

d) L is not an isomorphism since $L: \mathbb{R}^3 \to \mathbb{R}^4$. but target = co-domain has dim = 4. Rank = dim of range = 3. So L is not onto. (L' is an isomorphism $L': \mathbb{R}^3 \to \mathrm{Range}(L)$.)

SOLUTIONS To Fall 2006 FINAL Exam

1. a) $|A - xI| = \begin{vmatrix} 2-x & 0 & 0 \\ 1 & 1-x & -1 \\ 1 & -1 & 1-x \end{vmatrix} = (2-x)(1-x)(1-x) - (2-x)$
$= (2-x)[1 - 2x + x^2 - 1] = (-x)(x-2)^2$

$\underline{x=2}$: $A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ rref$(A-2I) = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ nullspace = $\lambda = 2$ e-space $= \left\{ \begin{pmatrix} y+z \\ y \\ z \end{pmatrix} \middle| y, z \in \mathbb{R} \right\}$

Yes, A is diagonalizable. ($\lambda = 2$, $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ are lin. indep. e-vectors, know $\lambda = 0$ also has 1 e-vector $\begin{pmatrix}0\\1\\1\end{pmatrix}$)

b) $|A - xI| = \begin{vmatrix} 2-x & 3 \\ -3 & 8-x \end{vmatrix} = (2-x)(8-x) + 9 = x^2 - 10x + 25 = (x-5)^2$

$A - 5I = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ nullspace = e-space for $\lambda = 5$ $\left\{ \begin{pmatrix} y \\ y \end{pmatrix} \middle| y \in \mathbb{R} \right\}$ only $\underline{1 \text{ dimensional}}$

No, A is not diagonalizable

2a) With given e-vectors, e-values $A = XDX^{-1}$ with $X = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(multiply out see A has a typo on the exam)

$A = \begin{pmatrix} -7 & 14 & 10 \\ 1 & -2 & -1 \\ -8 & 16 & 11 \end{pmatrix}$!

(b), c) Not covered Spring 2009
(b) $B = X D' X^{-1}$
$D' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$)

d) One can show L is linear because of rules of matrix multiplication.
$A(v + w) = Av + Aw$ $\quad A(r \cdot v) = r \cdot Av$ for r a scalar.

e) $\mathrm{Rep}_{X,X}(L) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $X = \{x_1, x_2, x_3\}$ This is the point of e-vectors!

f) $\det \begin{pmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix} = 1$, so this is a basis. (Or row reduce...) $(\;)^{-1}$

g) Too computationally involved to be a good exam question. Better just to ask for formula: $\mathrm{Rep}_{Y,Y} L = \mathrm{Rep}_{E,Y}(id)\, \mathrm{Rep}_{E,E}(L)\, \mathrm{Rep}_{Y,E}(id)$
$= \begin{pmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ -4 & -3 & 3 \end{pmatrix} \begin{pmatrix} -7 & 14 & 10 \\ 1 & -2 & -1 \\ -8 & 16 & 11 \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 \\ 2 & 1 & 2 \\ 6 & 1 & 5 \end{pmatrix}$

3a) $\left(\begin{array}{cccc|c} 4 & 4 & -4 & 0 & 3 \\ 2 & 0 & 2 & -1 & 2 \\ 1 & 3 & -5 & 1 & 6 \\ 2 & 3 & -4 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 3/4 \\ 0 & -2 & 4 & -1 & 1/2 \\ 0 & 2 & -4 & 1 & 21/4 \\ 0 & 1 & -2 & 1 & -1/2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 3/4 \\ 0 & 1 & -2 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & \bullet \end{array}\right)$ NON-ZERO So $b \notin S$ So $S \neq \mathbb{R}^4$

b) $\left\langle \begin{pmatrix}4\\2\\1\\3\end{pmatrix}, \begin{pmatrix}4\\0\\3\\3\end{pmatrix} \right\rangle$ is a basis, $\dim S = 2$.

c) $k_1 = \begin{pmatrix}4\\2\\1\\2\end{pmatrix}$ $\quad k_2 = \begin{pmatrix}4\\0\\3\\3\end{pmatrix} - (\;-\;) \cdot \begin{pmatrix}4\\2\\1\\2\end{pmatrix} = \begin{pmatrix}0\\-2\\2\\1\end{pmatrix}$ check $k_1 \cdot k_2 = 0$
$-4 + 2 + 2 = 0$

d) closest is $\mathrm{Proj}_{k_1}(b) + \mathrm{Proj}_{k_2}(b) = \frac{24}{25}\begin{pmatrix}4\\2\\1\\2\end{pmatrix} + \frac{1}{9} k_2$ = very close to $\begin{pmatrix}4\\2\\1\\2\end{pmatrix}$!