AYSMPTOTIC INVARIANTS OF GROUPS: NOTES FROM MARK SAPIR'S MSRI MINICOURSE GEOMETRIC GROUP THEORY PROGRAM FALL 2007

Contents

1. Lecture 1, October 22

Topics to be covered:

(1) Asymptotic Cones
(2) Dehn Functions
(3) Divergence

1.1. Dehn Diagrams.

Let $G = \langle X \mid R\rangle$. We assume R is symmetric.

If $g =_G 1$ then there are $h_i \in G$ and $r_i \in R$ such that $g = \prod h_i r_i h_i^{-1}$. We think of this as a *lollipop diagram* for g, as in Figure 1.

This is a planar, simply connected graph with boundary label equal to g in the free group.

On the other hand, if $g =_G 1$ then we can view g as the boundary of a *disc diagram* or *van Kampen Diagram*. Every cell in the diagram has boundary labeled by a word from R. A disc diagram is *reduced* if it does not contain a subdiagram with two cells and boundary label equal to 1 in the free group.

We can always assume that the boundary label is a reduced word, that is, it does not contain subwords of the form aa^{-1} or $a^{-1}a$. If it is not reduced we can reduce it by the process of *folding*.

Remark. If in the process of folding, we have to fold a pair of edges with the same initial and the same terminal vertex, we have to remove the whole open subdisc bounded by these edges (leaving only the edge obtained by folding) .

Lemma 1.1 (van Kampen)**.** $g =_G 1 \iff$ *g is the boundary of a disc diagram over R.*

Remark. Cells may touch themselves, as in Figure 4.

Thanks to Sam Kim and Tullia Dymarz for help with the note taking.

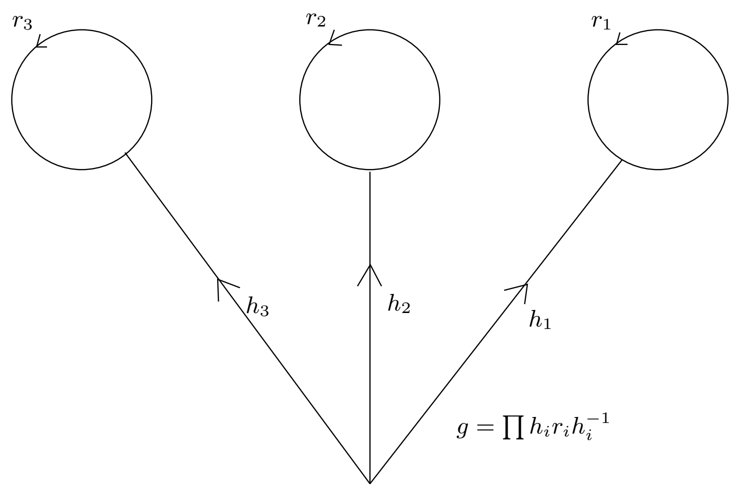

FIGURE 1. A lollipop diagram

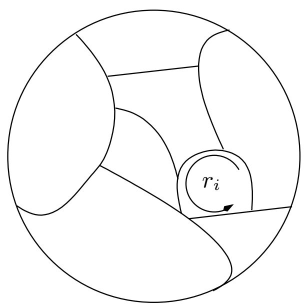

FIGURE 2. A disc diagram

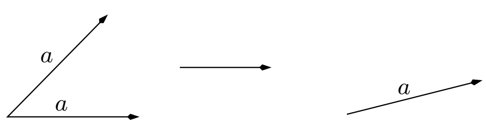

FIGURE 3. Folding

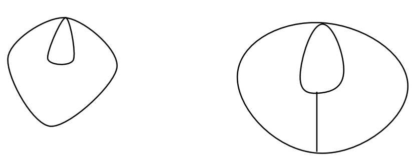

FIGURE 4. Bad cells

This is bad, but usually it is possible to get rid of these bad configurations. [This issue is addressed further in Lecture 2.]

Example 1.2.

$$G = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$$

Consider the word $g = a^2ba^{-2}b^{-1} =_G 1$. This has a disc diagram:

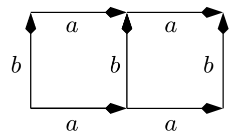

FIGURE 5. $g = a^2ba^{-2}b^{-1}$

We can make it a lollipop diagram by two unfolding steps:

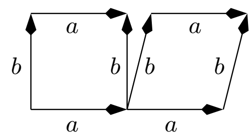

FIGURE 6. $g = a^2ba^{-1}b^{-1}ba^{-1}b^{-1}$

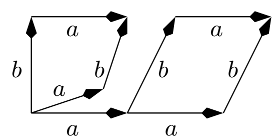

FIGURE 7. $g = a(aba^{-1}b^{-1})a^{-1} \cdot aba^{-1}b^{-1}$

1.2. **Dehn Functions.** Define Area(g) = minimum number of cells in a van Kampen diagram with boundary g.

Define the *Dehn function* of G, $d\colon \mathbb{N} \to \mathbb{N}$ by:

$$d(n) = \max_{g =_G 1, |g| \leq n} \text{Area}(g)$$

Every function d' such that $d'(n) > d(n)$ is an isoperimetric function of the presentation.

Example 1.3. For $\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1 \rangle$, $d(n) \sim n^2$.

Consider a trivial word g in $\mathbb{Z}^2$ of length $\leq n$. There is a disc diagram for g, and the disc is tiled by squares whose boundary label is given by the relator in the presentation, $aba^{-1}b^{-1}$. Pick such a square and look at, say, one of the sides labeled b. If this side is not on the boundary of the disc diagram, then there is another square next to it sharing a b-side. Keep extending this way to find a *b-band* in the disc diagram.

In particular, consider the word $a^{\frac{n}{4}}b^{\frac{n}{4}}a^{-\frac{n}{4}}b^{-\frac{n}{4}}$. This has an obvious disc diagram as a sqaaure of side length $n/4$ consisting of $\frac{n^2}{16}$ cells. Can it have a smaller van Kampen diagram?

Need to show:

(1) no a (or b) -annuli, ie. no a-band closes up on itself
(2) any a-band and any b-band intersect at most once

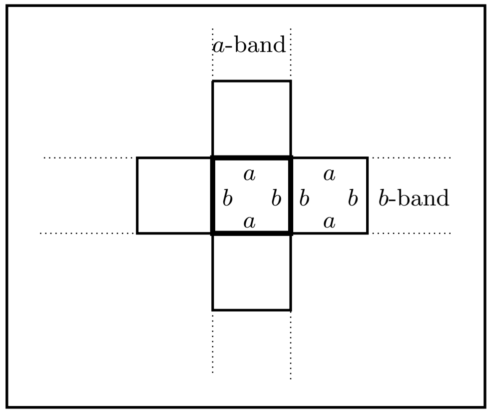

FIGURE 8. Bands in a dics diagram

Then there are at least $\frac{n}{4}$ a-bands and $\frac{n}{4}$ b-bands and at least $\frac{n^2}{16}$ distinct cells which are intersections of a and b-bands, so $d(n) \succeq n^2$.

On the other hand, if g is any trivial word of length $\leq n$ then a disc diagram for g contains at most $\frac{n}{2}$ a-bands and at most $\frac{n}{2}$ b-bands, so has area at most $\frac{n^2}{4}$. Therefore, $d(n) \sim n^2$.

Alternate Proof: $\mathbb{Z}^2$ acts properly, cocompactly on $\mathbb{R}^2$, which has isoperimetric function n^2.

Proposition 1.4. *The Dehn function of a group having a synchronous combing is* $\preceq n^2$*.*

Proof. Let G be a group with a synchronous combing and $g =_G 1$. Given a disc diagram for g, we can replace it with a digram made up of $|g|$ geodesic triangles. These triangle have two sides of length bounded in terms of $|g|$, and one side of length 1. Using synchronicity, the number of cells in such a triangle is linear in $|g|$, so the total area is $\preceq n^2$.

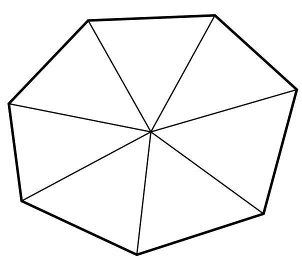

FIGURE 9. Replacing a disc by triangles

We can bound the number of cells in such a triangle, so the area is linear in the length of the boundary. □

Later in the course we will see that a group is hyperbolic if and only if it has linear Dehn function.

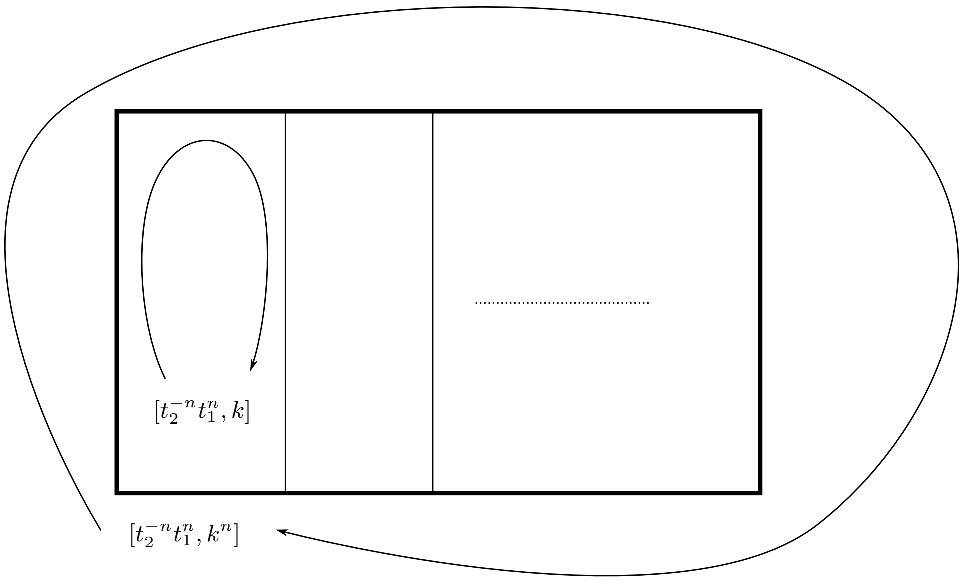

FIGURE 10. A disc diagram for $[t_2^{-n}t_1^n, k^n]$.

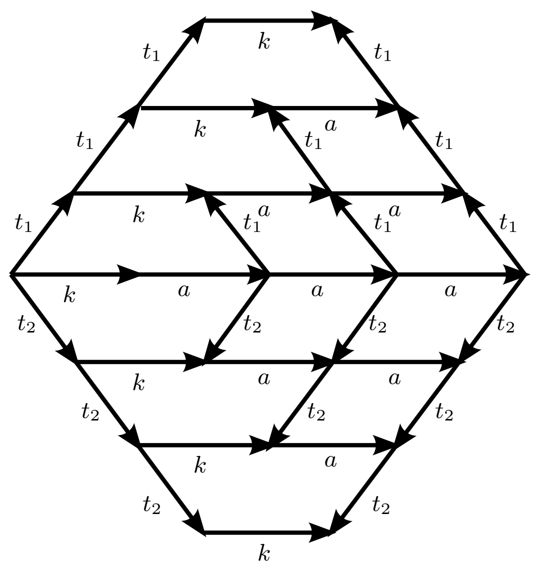

FIGURE 11. A disc diagram for $[t_2^{-3}t_1^3, k]$

Example 1.5.

$$G = \langle a, k, t_1, t_2 \mid t_i a = a t_i,\ t_i k t_i^{-1} = ka \rangle$$

$\langle a, k \rangle \lhd G$, G is free-by-free.

Claim $d_G(n) \sim n^3$.

Consider the word $g_n = [t_2^{-n}t_1^n, k^n]$. This has a disc diagram given by n copies of $[t_2^{-n}t_1^n, k]$, see Figure 10 and Figure 11.

$\text{Area}(g_n) \preceq n^3$.

To show it has the same lower bound consider t_i-bands and k-bands. Show there are no annuli or bigons.

For a general word $g =_G 1$ with $|g| \leq n$ there are at most $\frac{n}{2}$ t-bands and $\frac{n}{2}$ k-bands, so at most $\frac{n^2}{4}$ (t,k)-cells. There are at most $\frac{n^2}{4} + \frac{n}{2}$ a-bands, and the length of an a-band is at most the number of t-bands, so there are at most $\sim n^3$ (a,t)-cells. Thus $\mathrm{Area}(g) \preceq n^3$.

1.3. Asymptotic Cones.

Let ω be an ultrafilter on $\mathbb{N}$. $\omega\colon \mathcal{P}(\mathbb{N}) \to \{0,1\}$ with finite sets going to 0.

If $|b_n| < b$ then

$$\lim_{n\to\infty}^{\omega} b_n = a \iff \forall \epsilon > 0,\, \omega(\{n \mid |a - b_n| \leq \epsilon\}) = 1$$

Remark. Every sequence of numbers from a finite set is constant (up to ω-measure 0).

Let X be a metric space, $o \in X$ the *observation point*, and $d_n \to \infty$.

$$\prod\nolimits_b X = \{(a_n) \in \prod X \mid \frac{d_X(a_n, o)}{d_n} \leq C \text{ for some } C\}$$

Define

$$d((a_n),(b_n)) = \lim_{n\to\infty}^{\omega} \frac{d_X(a_n,b_n)}{d_n}$$

Define an equivalence relation $\sim$ on $\prod_b X$ by $(a_n) \sim (b_n) \iff d((a_n),(b_n)) = 0$.

Definition 1.6. The *asymptotic cone*, $\mathrm{Con}(X, \omega, o, (d_n))$ of X with respect to ω, o, and (d_n) is:

$$\prod\nolimits_b X/\sim$$

2. Lecture 2, October 29

2.1. Applications of Dehn Diagrams.

Recall from Lecture 1 that there are some issues with folding if you have two edges with the same label sharing both their initial and terminal vertices, or when you have a cell that touches itself. Olshanski introduced an alternative to folding to deal with these issues.

We introduce two new types of cells, called *0-cells*:

(1) Small 0-cells: polygons with all edges labeled 1.
(2) Large 0-cells: polygons with a pair of edges sharing a label but with opposite orientation, and all other edges labeled 1.

When computing lengths of word do not count edges labeled 1. When computing areas of trivial words do not count the 0-cells.

Introducing 0-cells is the first step towards a graded group presentation, where each of the relators is assigned a grade. A disc diagram with respect to a graded group presentation becomes "chicken soup", that is, in the disc we see some cells, the "chunks", corresponding to relators of a certain grade, floating around in a "broth" of cells of the other grades.

Let S be a closed, orientable surface and let F be a fixed free group. Consider a homomorphism $\phi\colon \pi_1(S) \to F$. We can use Dehn diagrams to give some information about ϕ.

Here is an explicit example which will illustrate the theorems:

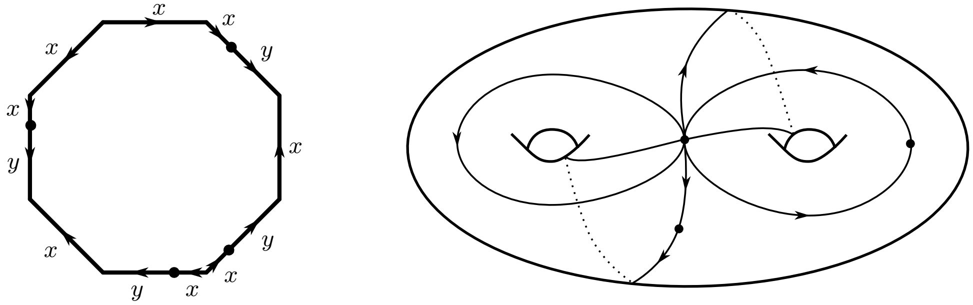

FIGURE 12. The octagon and genus 2 surface.

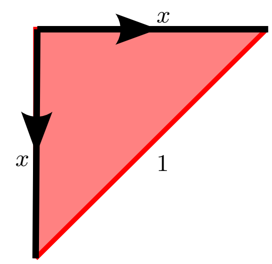

FIGURE 13. A large 0-cell.

Example 2.1. Let $\pi_1(S_2) = \langle a, b, c, d \mid [a, b][c, d] = 1\rangle$.

Let $F = \langle x, y\rangle$ be the free group on two generators.

Define a homomorphism $\phi\colon \pi_1(S_2) \to F$ by:

$$\phi(a) = x$$
$$\phi(b) = xy$$
$$\phi(c) = xy$$
$$\phi(d) = x$$

We know a disc digram for the word $[a, b][c, d]$ in $\pi_1(S_2)$, it is just a single 2-cell, an octagon. Use ϕ to push this digram into F. The edges of the octagon are now labeled by their images in F, as in Figure 12.

Notice in Figure 12 that the top side and the top left side share a vertex and are both labeled x leaving the vertex. Introduce a large 0-cell by connecting the endpoints of these two sides by an edge labeled 1, as in Figure 13. For the remainder of this example edges that appear in color are considered to have label 1; edges that are black have the label from Figure 12.

Add this 0-cell to the disc diagram. We can also pull this back to S_2, as in Figure 14

The 0-cell that we added touches itself; it is a bad cell. To eliminate this inconvenience add eight small 0-cells around the vertex. In Figure 15 these are the eight yellow quadrilaterals.

In the bottom right corner of the disc there is a small 0-cell connecting two edges which leave vertices of the small 0-cell with label x. We attach a large 0-cell as in Figure 16.

We can continue in this way and add more large 0-cells, as in Figure 17.

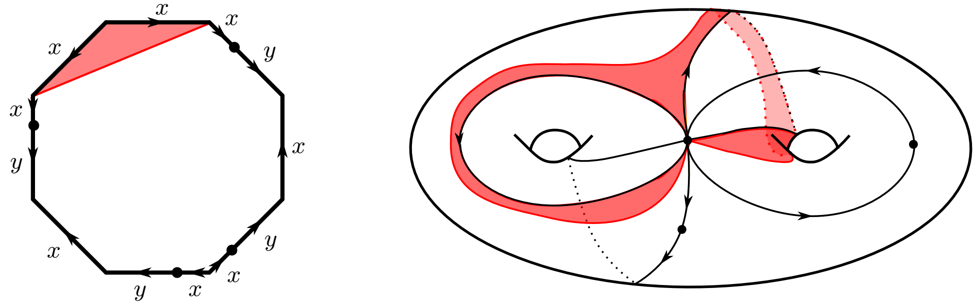

FIGURE 14. Add a large 0-cell.

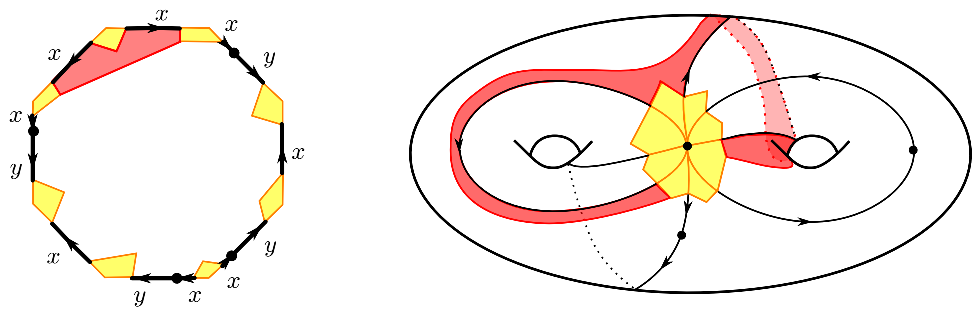

FIGURE 15. Add small 0-cells.

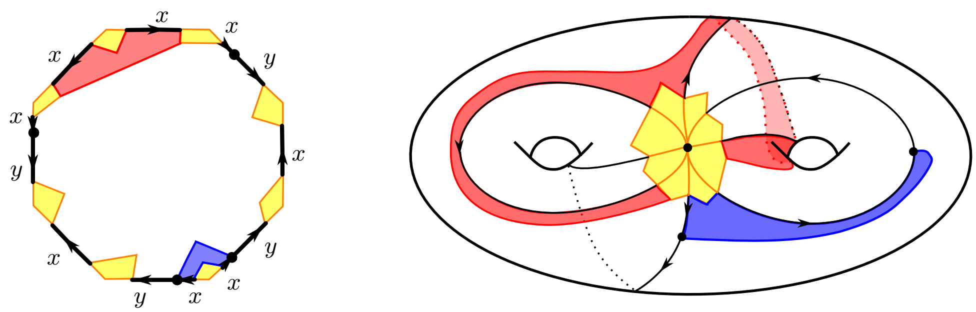

FIGURE 16. The first 2 large 0-cells

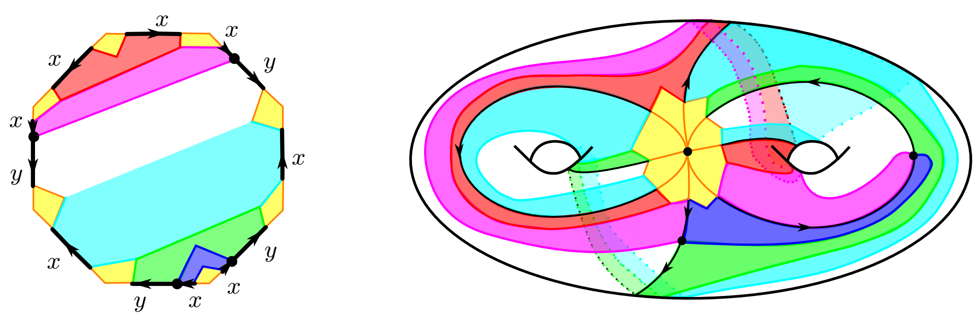

FIGURE 17. Six large 0-cells and 8 small 0-cells.

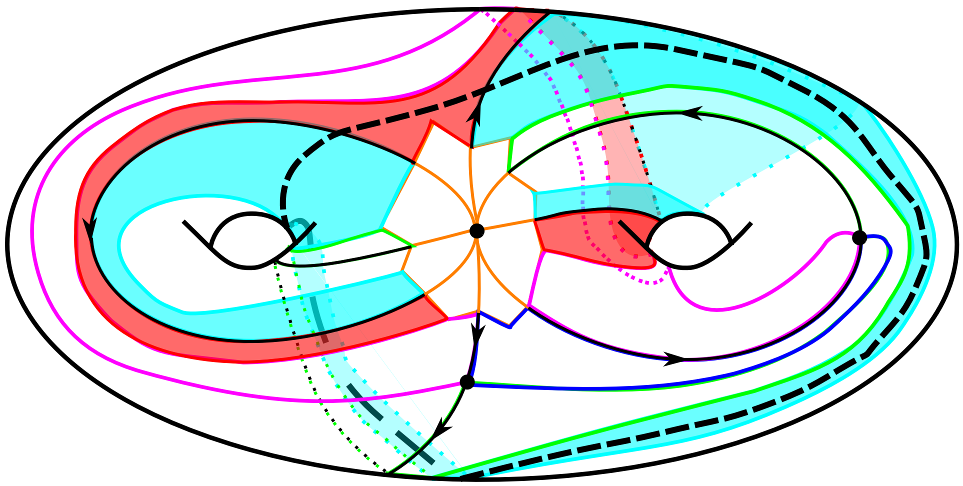

FIGURE 18. An x-annulus with a curve in the kernal of ϕ.

Pulling the 0-cells back to the surface gives us a tesselation of the surface by 0-cells. It is possible to recover the homomorphism ϕ from this tesselation. For any element of the fundamental group, realize it as a curve on the surface. Push the curve to lie in the 1-skeleton of the tesselation, and read off the edge labels. Choosing a different way to push the curve into the 1-skeleton does not change the element of the free group that you get, since the cells all have boundary labels which are trivial in the free group.

Theorem 2.2. *Given a closed surface S, a free group F, and a homomorphism $\phi\colon \pi_1(S) \to F$, there is a nontrivial element in $\ker \phi$ which is represented by a simple closed curve in S.*

Proof. Look for a band in the surface. If every 0-cell is small then ϕ is the trivial homomorphism, so every simple closed curve is in the kernal. If there is a large 0-cell then complete it to a band. The surface has no boundary, so bands close up to give annuli. Each such annulus has boundary components labeled 1. The annulus is a neighborhood of some simple closed curve, and pushing this curve to either boundary component shows that the image under ϕ of its hopmotpy class is trivial.

If every annulus is homotopically trivial in the surface then again ϕ is just the trivial map, so every simple closed curve is in the kernal. □

In Theorem 2.1 there are two x-annuli and one y-annulus. The red and cyan large 0-cells match up to make an x-annulus, as do the magenta and blue. The green and white large 0-cells form the y-annulus.

Figure 18 illustrates the red and cyan x-annulus, along with a curve in the kernal of ϕ.

In fact, we can get a stronger result by repeating this argument.

Theorem 2.3. *Given a closed surface S, a free group F, and a homomorphism $\phi\colon \pi_1(S) \to F$, there is a pants decomposition of S such that ϕ is trivial on the homotopy classes of the pants curves.*

Proof. Find an annulus giving an essential simple closed curve in the kernal, as above. We can insure that the anulus does not touch itself by adding small 0-cells, if necessary.

The anulus is a neighborhood of a simple closed curve in the kernal of ϕ; this curve will be one of the curves in the pants decomposition.

Remove the interior of the annulus. If the curve was separating then we get two surfaces with one boundary component each. If the curve was non-separating then we get a new surface with two boundary components. In either case, the genus of each new surface is smaller, and each boundary component is labeled by all 1's. Glue small 0-cells to each boundary component to get a collection of closed surfaces. Now apply Theorem 2.2 to each of these new surfaces.

Proceed by induction until we have a collection of spheres. Look at subcomplexes of the original surface correspoding to each of these spheres. ϕ is trivial on each such subcomplex, so choose simple closed curves to complete the pants decomposition. □

Corollary 2.4. *Let $g > 1$, S_g the orientable surface of genus g, and F_g the free group of rank g. There is a unique homomorphism from $\pi_1(S_g)$ onto F_g, up to precomposition by elements of the mapping class group of the surface and postcomposition by free group automorphisms.*

The following is a conjecture of Stallings which is closely related to the Poincaré Conjecture.

Conjecture 2.5. *Let $g > 1$, S_g the orientable surface of genus g, and F_g the free group of rank g. Consider a surjective homomorphism $\phi\colon \pi_1(S_g) \to F_g \times F_g$. There is a nontrivial element of $\ker\phi$ which is represented by a simple closed curve in S_g.*

2.2. **Asymptotic Cones.** Recall from last time, $\omega\colon \mathcal{P}(\mathbb{N}) \to \{0,1\}$ is an ultrafilter,

(1) Every sequence of numbers with a bounded number of values is ω-constant.
(2) If b_n are real numbers in some fixed interval, $\lim^\omega(b_n)$ exists and is unique.
(3) Given a sequence of metric spaces with observation points, (Y_i, o_i),

$$\lim{}^\omega(Y_i) = \prod\nolimits_b Y_i/\sim$$

where

$$(y_i) \sim (y_i') \iff 0 = d((y_i),(y_i')) = \lim{}^\omega(d_i(y_i, y_i'))$$

$\lim^\omega(Y_i)$ is the connected component of $\prod_b Y_i/\sim$ containing (o_i).

(4) $\mathrm{Con}^\omega(X,(d_i),o) = \lim^\omega(X/d_i, o)$

Proposition 2.6. *Suppose X is a homogenous geodesic metric space.*

(1) *The construction does not depend on o.*
(2) $\mathrm{Con}^\omega(X,(d_i))$ *is a complete geodesic metric space.*

Remark. The homogeneity condition can be weakened: X need only be almost homogeneous.

Exercise 2.7. Prove the triangle inequality is satisfied.

Exercise 2.8. Prove Con^ω is complete.

Proof of homogeneity. Let $A = (a_i)$, $B = (b_i) \in \mathrm{Con}^\omega$. Suppose $\gamma_i(a_i) = c_i$ with $d(c_i, b_i) < C$. Then consider the map γ which corresponds to γ_i on the i-th coordinate. γ is an isometry which takes A to B.
Proof that the cone is a geodesic space. Suppose we have a sequence of geodesics $(p_i\colon [0, l_i] \to X)$. Let $Y_i = p_i([0, l_i]) \subset X/d_i$.

$\lim^\omega Y_i \subset \mathrm{Con}^\omega(X)$ exists and is a geodesic.

To see this consider the possiilities:

(1) If $(p_i(0))$ and $(p_i(l_i))$ are both in the cone then the limit of the Y_i's is a finite geodesic segment.
(2) If some points of the limit are in the cone, but one or both of $(p_i(0))$ and $(p_i(l_i))$ are not, then the limit is either a geodesic ray or a biinfinite geodesic.
(3) Otherwise, the limit is empty.

Let $x_i = p_i(t_i)$, and let $t = \lim^\omega(t_i)$. Define $p(t) = (x_i)$. If p_i is a (λ, c)-quasi-geodesic then $\lim^\omega p_i$ is a $(\lambda, 0)$-quasi-geodesic.

Proposition 2.9. *If X and Y are (λ, c)-quasi-isometric then $\mathrm{Con}^\omega(X)$ and $\mathrm{Con}^\omega(Y)$ are $(\lambda, 0)$-quasi-isometric.*

There is a dictionary to translate between coarse properties of a space and its asymptotic cone. For example:

Theorem 2.10 (Gromov)**.** *X is δ-hyperbolic if and only if all asymptotic cones are $\mathbb{R}$-trees.*

Proof. Pick three points in the cone and look at sequences of trinagles in the $X_i = X/d_i$.

[picture here]

Claim:

$$\lim^\omega([a_i, b_i]) = [A, B]$$
$$\lim^\omega([b_i, c_i]) = [B, C]$$
$$\lim^\omega([c_i, a_i]) = [C, A]$$

a_i, b_i, and c_i form a δ/d_i-thin geodesic triangle in X_i, so A, B, and C form a 0-thin triangle. □

Unfortunately, this proof is not correct. We do not know that the geodesics joining A, B, C must be limits of geodesics in X. To get around this problem we approximate ΔABC by k-gons in the X_i.

Lemma 2.11. *$\forall \epsilon > 0\ \exists k$ and a sequence of k-gons $P_i \subset X$ such that $d(\lim^\omega P_i, \Delta ABC) < \epsilon$. In fact, we can specify particular points of ΔABC to be vertices of the P_i.*

Exercise 2.12. Prove the lemma.

Hint. Suppose X is not hyperbolic but every cone is a tree. $\forall M\ \exists$ and M-thick triangle $\Delta a_m b_m c_m$, ie $d([a_m, b_m], [b_m, c_m] \cup [c_m, a_m]) > M$. Let x_m be the point on $[a_m, b_m]$ at maximum distance, and let $d_m = d(x_m, [b_m, c_m] \cup [c_m, a_m])$. By assumption, the cone defined using (d_i) is a tree, but $d((x_i), [(b_i), c(i)] \cup [(c_i), (a_i)]) = 1$, a contradiction.

2.3. **Applications of Asymptotic Cones.**

Theorem 2.13 (Gromov)**.** *If every cone is simply connected, the Dehn funtion is polynomial.*

Here is another applicaiton due to Bestvina and Paulin:

Suppose Γ and G are groups with Γ finitely generated such that there are infinitely many pairwise non-conjugate $\phi_i \in \mathrm{Hom}(\Gamma, G)$. Each ϕ_i gives and action of Γ on G by $\gamma.g = \phi_i(\gamma)g$.

Let $\gamma_1, \ldots, \gamma_n$ be generators for Γ.

Let

$$d_i = \min_{x \in G} \max_{j=1..n} d(x, \phi_i(\gamma_j(x))).$$

Call this number the translation number of the Γ action coming from ϕ_i.

$d_i \to \infty$

Consider G/d_i.

Γ acts on $\mathrm{Con}^\omega(G, (d_i))$ with translation number 1.

If, in particular, G is δ-hyperbolic, then we get an action of Γ on an $\mathbb{R}$-tree, so Γ splits.

Theorem 2.14 (Paulin)**.** *If G has Property (T) then* $\mathrm{Out}(G)$ *is finite.*

Such groups with Property (T) do not act on $\mathbb{R}$-trees.

3. Lecture 3, November 19

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