

*You may use results from the textbook, the online notes, and notes you took in class. You may not simply refer to or copy proofs from these sources.*

1. Show that any finite group can be realized (up to isomorphism) as a Galois group in the following steps. (24pts)

(a) Let  $G$  be a finite group. Show that there is a natural number  $n$ , a subgroup  $H$  of the symmetric group  $S_n$  and an isomorphism  $G \cong H$ . (You may use the book and notes. Or do it yourself using group actions.)

(b) Now suppose  $G = S_n$ . Let  $F$  be a field of characteristic 0 and  $E = F(T_1, \dots, T_n)$  be the rational function field in  $n$  variables over  $F$ . Show that  $G$  acts naturally on  $E$  by field automorphisms, by permuting the variables.

(c) Still with  $G = S_n$ , let  $K = E^G = \{\alpha \in E \mid \sigma(\alpha) = \alpha\}$  be the fixed field of the  $G$ -action. Show that  $E/K$  is the splitting field of the polynomial  $\prod_{i=1}^n (X - T_i)$  over  $K$ . (That is, you need to show that this polynomial has coefficients in  $K$ , and that  $E$  is generated over  $K$  by its roots.)

(d) Use the main theorem of Galois theory to conclude that  $\text{Gal}(E/K) \cong S_n$ . (Hint: Suppose  $E/K$  is any normal separable extension. Let  $G$  be any group of automorphisms of  $E$  fixing the elements of  $K$ . Then the main theorem implies that  $G = \text{Gal}(E/K)$ .)

(e) Now combine part (a) and the main theorem of Galois theory to show that for any finite group  $G$  there exists a normal separable field extension with Galois group isomorphic to  $G$ .

2. Let  $\alpha$  and  $\beta$  be algebraic numbers over  $\mathbb{Q}$ . Let  $d_\alpha = [\mathbb{Q}(\alpha) : \mathbb{Q}]$  and  $d_\beta = [\mathbb{Q}(\beta) : \mathbb{Q}]$ . Recall that the product  $\alpha\beta$  is also algebraic over  $\mathbb{Q}$ . Give an upper bound for  $[\mathbb{Q}(\alpha\beta) : \mathbb{Q}]$  and examples for  $\alpha$  and  $\beta$  such that the upper bound is attained. (13pts)

3. Let  $F$  be a field and  $E/F$  an extension of degree two. Suppose that the characteristic of  $F$  is *not* two. Show that  $E/F$  is a Galois (that is, separable and normal) extension. (The "separable" part means you have to show that no irreducible polynomial over  $F$  that has a root in  $E$  can have a multiple root. You may, but don't have to, use the main theorem of Galois theory for the "normal" part.) (13pts)