Part 1. Group actions.

1. Let G be a finite group and $H < G$ a subgroup of index 2. Show that H is normal in G.

2. Let $G \times X \to X$ be an action of a group on a set.

a) Define what it means for the action to be *faithful*.

b) Define the *orbit* of a point $x \in X$.

c) Define what it means for the action to be *transitive*.

d) Give a definition of a *fixed point* of the action.

e) Define the *stabilizer* of a point $x \in X$.

3. Let p be a prime number and let $G \times X \to X$ be an action of a p-group on a finite set. Suppose p does not divide $|X|$. Show that the action has a fixed point.

Part 2. Solvable and simple groups. Sylow theorems.

4. Let p and q be different primes and let G be a group with pq elements. Show that G is solvable.

5. Let $p < q$ be primes and G a group of order pq^n for some $n \geq 0$. Show that G is solvable.

6. Suppose G is a finite group that has no non-trivial subgroup. Show that the order of G is a prime.

7. Using that A_5 is simple, prove that the symmetric group S_n is not solvable for $n \geq 5$.

Part 3. Field extensions.

8. Let $F \subseteq E$ be a finite field extension. Show that any intermediate extension $K \subseteq E$ is finite over F. Bonus question: Is the corresponding statement also true for *finitely generated* extensions?

9. Let $f(X)$ be a polynomial with coefficients in the field F. Show that the splitting field E of F is a finite extension of F.

10. Suppose F is an algebraically closed field. Show that F has no non-trivial finite field extension.

Part 4. Separable, inseparable and normal extensions. Galois theory.

11. Give the three equivalent characterizations of what it means for an extension $F \subseteq E$ to be normal.

12. Let $F \subseteq K \subseteq E$ be finite separable extensions. Suppose E/F is normal of degree 15 and $[K : F] = 3$. Show that K/F is normal. (*Hint:* Use the main theorem of Galois theory.)

13. Let F be the splitting field of the polynomial $X^5 - 1$ over $\mathbb{Q}$. Compute $\mathrm{Gal}(F/\mathbb{Q})$.

14. Same as 13., but now over the field $GF(5)$ with 5 elements.

15. Give an example of a purely inseparable field extension. What is its Galois group?

16. Give an example of a normal extension of $\mathbb{Q}$ (automatically separable because...?) whose Galois group is isomorphic to $\mathbb{Z}/3$.

17. Give an example of a normal extension of $\mathbb{Q}$ with Galois group isomorphic to S_3.

18. *Bonus question:* Let F be a an infinite field of characteristic p and u and v two transcendentals. Let $K = F(u, v)$ and $E = F(u^p, v^p)$. Show that the extension $E \subseteq K$ has infinitely many intermediate extensions.