Math 547 Sec. **Homework 1** **Due September 7th, 2007**

1. This exercise asks you to give explicit formulas for various homotopies - because I had to do it, too. (2 pts each)

a) Let γ, δ and ϵ be loops in the pointed space $(X,\ x_0)$ (that is, maps from the pair $(I,\ \partial I)$ to the pair $(X,\ x_0)$.) Give an explicit homotopy *rel* ∂I making $\gamma * (\delta * \epsilon)$ homotopic to $(\gamma * \delta) * \epsilon$.

b) Give an explicit (pointed!) homotopy between the identity map id of the pointed space $(\mathbb{R},\ 0)$ and the constant map $c : (\mathbb{R},\ 0) \to (\mathbb{R},\ 0)$ with value 0.

c) Use b) to prove that $\pi_1(\mathbb{R},\ 0) = \{e\}$ is the trivial group.

2. Assume there exists a pointed space $(X,\ x_0)$ such that $\pi_1(X,\ x_0) \neq \{e\}$. Show that $\pi_1(S^1,\ *) \neq \{e\}$. Here the circle $(S^1,\ *)$ is represented as the quotient space of the interval I obtained by identifying the endpoints 0 and 1, with basepoint the image of 0 and 1. Tip: given the assumption, you can give an explicit loop that is non-zero in the fundamental group of the circle. (6 pts)

3. This problem is about categories and functors. Recall that a category $\mathbf{C}$ has a class ob$\mathbf{C}$ of objects, and for any two objects x and y, a set $\mathrm{Mor}_{\mathbf{C}}(x, y)$ of morphisms; there is an associative composition law; and for each object x there is an identity morphism $id_x \in \mathrm{Mor}_{\mathbf{C}}(x, x)$ that is left- and right-neutral with respect to composition.
Also recall that a functor $F : \mathbf{C} \to \mathbf{D}$ is an assignment that assigns to each object $x \in \mathrm{ob}\mathbf{C}$ an object $F(x) \in \mathrm{ob}\mathbf{D}$ and to each pair of objects (x, y) a map of sets $F = F_{x,y} : \mathrm{Mor}_{\mathbf{C}}(x, y) \to \mathrm{Mor}_{\mathbf{D}}(F(x), F(y))$ such that for any composable pair (f, g) of morphisms in $\mathbf{C}$, we have $F(fg) = F(f)F(g)$, and for any object x we have $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$. (2 pts each)

a) Explain how a group G can be considered as a category with one object.

b) Let G and H be groups, considered as categories with one object. What is a functor from G to H, in terms of the language of groups?

c) Give two examples for a functor from the category $\mathbf{Vect}_k$ of vector spaces over a field k to the category **Ens** of sets.

d) A morphism $f : x \to y$ in a category is called an isomorphism (in that category) provided there is a morphism $g : y \to x$ such that $fg = \mathrm{id}_y$ and $gf = \mathrm{id}_x$. What is an isomorphism in the category of topological spaces with continuous maps as morphisms?

e) A category with the property that all morphisms are isomorphisms is called a *groupoid*. Show that a group, considered as category with one object, is a groupoid.

4. Let X be a topological space. We can make X into a category ΠX in the following way: the objects are the points $x \in X$; and given $x, y \in X$, a morphism $x \to y$ is a homotopy class *rel* ∂I of paths $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. You can draw pictures of the homotopies that may be needed in this problem. (3 pts each)

a) Give the composition law in this category. Explain why it is associative.

b) Identify the identity morphism of some object $x \in \mathrm{ob}\Pi X$. Explain why it is left- and right-neutral for the composition law.

c) Given $x_0 \in X$, what is $\mathrm{Mor}_{\Pi X}(x_0, x_0)$ equal to?

d) Show that ΠX is a groupoid; it is called the *fundamental groupoid* of X and encodes a lot of information about the space X.