Math 547 **Homework 3** **Due September 28, 2007**

1. Let G be a topological group and $p : \tilde{G} \to G$ a covering space of G. (12 pts)

a) Show that there is a structure of a topological group on $\tilde{G}$ such that p is a homomorphism.

b) Prove that the topological group structure from part a) is essentially unique. That is, prove the following: given two topological group structures (m_1, i_1, e_1) and (m_2, i_2, e_2) on $\tilde{G}$ (where m_j is the continuous multiplication, i_j is the inverse map, and e_j is the neutral element) making p a homomorphism, then there is a (unique) deck transformation $D : \tilde{G} \to \tilde{G}$ with $D(e_1) = e_2$ and D is a homomorphism (meaning $D \circ m_1 = m_2 \circ (D \times D)$.)

c) Identify the kernel of the homomorphism p.

2. In this problem, we will study the example of the group $G = SO(n)$. (25 pts)

a) Show/ recall from linear algebra that $SO(2)$ is homeomorphic to a circle, and write down a path γ in $SO(2)$ generating its fundamental group $\pi_1(SO(2), I_2)$ based at the identity matrix.

b) Next we want to examine the case $n = 3$. First, show that $SU(2)$ is homeomorphic to the three sphere S^3 by proving that the natural action of $SU(2)$ on $S^3 = \{z \in \mathbb{C}^2 | \|z\| = 1\}$ is simply transitive.

c) Now, let $\mathfrak{su}(2)$ be (the Lie algebra of) 2×2 complex matrices A such that $A + A^* = 0$ and $tr(A) = 0$. Show that $\mathfrak{su}(2)$ is a three dimensional $\mathbb{R}$ vector space and the bilinear form $b(A, B) = -1/2tr(AB^*)$ is a positive definite symmetric form allowing us to identify $(\mathfrak{su}(2), b)$ with the standard Euclidean space $\mathbb{R}^3$.

d) Next, observe that $SU(2)$ acts on $\mathfrak{su}(2)$ continuously by conjugation (this is called the *adjoint action*). Show that this action preserves the form b and therefore gives a continuous homomorphism $p : SU(2) \to SO(3)$.

e) Prove that p is a covering map, making $SU(2)$ the universal covering space of $SO(3)$.

f) Determine the kernel of p. What is the fundamental group $\pi_1(SO(3), I_3)$?

Remark: The universal covering group of $SO(n)$ is usually called Spin(n) and plays an important role in mathematical physics.