PRACTICE PROBLEMS FOR THE FINAL EXAMINATION MATH 18.100B, ANALYSIS I

You may freely use Rudin's book Principles of Mathematical Analysis, your problem sets and your class notes. However, you may not use any other materials. In order to receive full credit on the problems you must prove any assertion that is not proved in Rudin or in the class notes. You have three hours for the exam. (This is a set of practice problems including final exam problems from previous years and Rudin problems. They are considerably harder than the final exam problems.)

Problem 1. Prove that the following subsets of the complex numbers $\mathbb{C}$ considered as a metric space are connected.

(1) $\{ z \in \mathbb{C} \ : \ z = \exp(it^3) \ \text{ for some } \ t \in \mathbb{R} \}$.
(2) $\{ z \in \mathbb{C} \ : \ 1 < |z| < 2 \}$

Problem 2. Suppose $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a continuous, open map. If U is a connected subset of $\mathbb{R}^1$, then does $f^{-1}(U)$ have to be connected?

Problem 3. Prove that the function

$$f(x) = \exp(\frac{x^2 - 3}{x^2 + 2})$$

is a differentiable function on $\mathbb{R}^1$. Explain why f achieves a minimum on the interval $[0, 1]$.

Problem 4. Suppose $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a twice differentiable function. Assume that 0 is a local minimum of f. Prove that $f''(0) \geq 0$.

Problem 5. Suppose that $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a differentiable function. Suppose that $|f'(x)|$ is bounded on $\mathbb{R}^1$. Prove that f is uniformly continuous. Give an example to show that if $|f'(x)|$ is not bounded, then f does not have to be uniformly continuous.

Problem 6. Prove that the formula

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$$

defines a continuously differentiable real valued function on $\mathbb{R}^1$.

Problem 7. Prove that for any monotonically increasing function $\alpha : [0, 1] \to \mathbb{R}^1$, the function defined by $f(0) = 1$ and $\frac{\sin(x)}{x}$ if $x \neq 0$ is Riemann-Stieltjes integrable.

Problem 8. Let $F : [0, 1]^2 \to \mathbb{R}^1$ be a continuous function satisfying

$$\sup_{[0,1]^2} |F(x, y)| \leq \frac{1}{2}.$$

Prove the following statements:

(1) If $f \in C([0, 1])$, then the function

$$g(x) = \int_0^1 F(x, y) f(y) dy$$

is also in $C([0, 1])$.

(2) $||g|| \leq \frac{1}{2}||f||$, where $||r(x)||$ denotes $\sup_{[0,1]} |r(x)|$.

(3) Let $h(x)$ be a fixed, real valued function on $[0, 1]$. Then there exists a unique function $r(x) \in C([0, 1])$ such that

$$r(x) = h(x) + \int_0^1 F(x, y)r(y)dy.$$

(Hint: Think about contractions and problem 5 on the second midterm.)

Problem 9. For the following series answer the following questions: What is the radius of convergence of the series? What is the boundary behavior of the series? In what intervals does the series converge uniformly? On what intervals is the sum differentiable? What is the derivative?

(1)
$$\sum_{n=0}^{\infty} \frac{(\sqrt{n+1} - \sqrt{n})\ z^n}{n}$$

(2)
$$\sum_{n=2}^{\infty} \frac{z^n}{n\ \log(n)}$$

(3)
$$\sum_{n=1}^{\infty} \frac{n!\ z^n}{n^n}$$

Problem 10. Do problem 5 on page 166 of Rudin.

Problem 11. Do problem 12 on page 198 of Rudin.

Problem 12. Do problem 22 on page 201 of Rudin.

Problem 13. Show that there does not exist a one-to-one continuous function from the interval $[0, 2\pi]$ onto the unit circle in $\mathbb{R}^2$. Does there exist a continuous, one-to-one function from the interval $[0, 2\pi)$ onto the unit circle in $\mathbb{R}^2$? Does there exist a continuous one-to-one function from $\mathbb{R}^1$ onto $(0, 1)$?

Problem 14. Show that if $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a twice differentiable, convex function, then $f''(x) \geq 0$ for all $x \in \mathbb{R}^1$.

Problem 15. Compute the surface area and the volume of the unit sphere in $\mathbb{R}^n$. Recall that the unit sphere is given by $\sum_{i=1}^n x_i^2 = 1$.

Problem 16. Let $f(x)$ be the function defined on Problem 18 on page 100 of Rudin. Show that $f(x)$ is Riemann integrable on any interval $[0, y]$. Compute the function

$$F(y) = \int_0^y f(x)dx.$$

Show that $F(y)$ is differentiable and compute its derivative. Why doesn't this contradict the Fundamental Theorem of Calculus?

Throughout the course I suggested many additional problems from Rudin's book. Working through some of those problems would be great practice for the final exam. In particular,

Problem 17. Solve Problem 23 on page 169 of Rudin.

Problem 18. Solve Problem 24 on page 170 of Rudin.

Problem 19. Solve Problem 25 on page 170 of Rudin.

Problem 20 Solve Problem 25 on page 118 of Rudin.