PROBLEM SET 9

This homework is due Wednesday November 2 in the beginning of class. No late homework will be accepted. You may collaborate on the homework. However, the final write-up must be yours and should reflect your own understanding of the problem. Please be sure to properly cite any help you get. The following problems refer to problems in the book.

Do problems 1.6 ac, 1.8, 1.11, 1.15, 1.17 on page 278.
Do problems 2.10, 2.11, 2.12, 2.15, 2.21 on page 287.
Do problems 3.10a, 3.12, 3.15, 3.26 on page 291.

Problem 1. Consider the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,2,3,0)$, $(2,-1,3,1)$ and $(1,1,2,2)$. Find an orthonormal basis for this subspace with respect to the usual inner product. Find the projection of $(5,3,12,13)$ to this subspace.

Problem 2. In this problem you will learn about orthogonal polynomials. Show that

$$\langle p,q\rangle = \int_{-1}^{1} p(t)q(t)dt$$

defines an inner product on the vector space of real valued polynomials. Starting with $1, x, x^2, x^3$ construct an orthonormal basis Q_0, Q_1, Q_2, Q_3 for the space of polynomials of degree at most three with respect to this inner product. The resulting polynomials (or rather a slight modification of them) are known as Legendre polynomials. Define the degree n Legendre polynomial

$$P_n = \sqrt{\frac{2}{2n+1}} Q_n.$$

For the polynomials you found, check that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

Check also that

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

In fact, these properties hold more generally and can be used to calculate the Legendre polynomials of any degree to obtain an orthogonal basis for the space of polynomials with respect to this inner product.

Problem 3. In this problem you will learn about Fourier series. Let P be the space of 2π periodic, real-valued, continuous functions, i.e. the space of continuous functions such that $f(x) = f(x+2\pi)$. Check that

$$\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$$

defines an inner product on P. Prove that

$$\frac{1}{2}, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \ldots$$

are an orthonormal set of vectors in P. The expansion of $f(x) \in P$ in terms of these vectors

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

is called the Fourier expansion of f. Calculate the Fourier expansion of the zigzag function

$$f(x) = \begin{cases} \frac{x}{\pi} - 2k & \text{if } 2k\pi \le x \le (2k+1)\pi \\ 2k - \frac{x}{\pi} & \text{if } (2k-1)\pi \le x \le 2k\pi \end{cases}$$