MIDTERM MATH 320

There are 4 problems and two pages on this exam. You may use the book and your course notes, but no other resources. In order to receive credit, you must show all your work and give complete proofs of all your assertions. GOOD LUCK!

Problem 1. (16 points) Find all solutions of the following system of equations.

$$\begin{aligned} 2x + 2y + 2z - w &= 2 \\ 3x + 3y + z + 2w &= 4 \\ 12x + 12y + 8z + w &= 14 \end{aligned}$$

Problem 2. (20 points) Let $h : \mathbb{R}^4 \to \mathbb{R}^3$ be the homomorphism determined by the following matrix

$$\begin{pmatrix} 3 & 2 & -2 & 1 \\ 0 & 3 & -1 & -1 \\ 6 & 3 & 3 & 3 \end{pmatrix}$$

(1) Find a basis for the kernel of h.
(2) Find a basis for the image of h.
(3) Calculate the rank of h.
(4) Calculate the nullity of h.

Problem 3. (24 points) Determine with proof whether the following subsets S of P_3, the space of polynomials $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ of degree at most 3 with real coefficients, are subspaces. If S is a subspace, find a basis and determine the dimension of S.

(1)
$$S = \{p(x) \in P_3 \mid p(1) = 3p(0)\}.$$

(2)
$$S = \{p(x) \in P_3 \mid \frac{d^2p(x)}{dx^2} + \frac{dp(x)}{dx} = 0\}.$$

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Problem 4. (40 points) For each of the following four statements, determine whether the statement is TRUE or FALSE. If the statement is true, give a proof. If the statement is false, give a counterexample. You will receive NO credit if you do not give a proof or a counterexample.

(1) A matrix M is invertible if and only if its transpose M^T is invertible.

(2) Let M be an $n \times n$ diagonal matrix. Then the dimension of the space of solutions of a homogeneous linear system of equations with coefficient matrix M is equal to the number of diagonal entries of M that are equal to zero.

(3) Let N be a $k \times n$ matrix of rank r. Let M be an $m \times k$ matrix. The matrix MN has rank r if and only if $\text{NullSpace}(M) \cap \text{ColumnSpace}(N) = \{0\}$.

(4) A sequence

$$0 \longrightarrow V_1 \xrightarrow{h_1} V_2 \xrightarrow{h_2} V_3 \longrightarrow 0$$

is called a *short exact sequence* of finite dimensional vector spaces if V_i are finite dimensional vector spaces and h_i are homomorphisms such that $h_1 : V_1 \to V_2$ is one-to-one (injective), $h_2 : V_2 \to V_3$ is onto (surjective) and the kernel of h_2 equals the image of h_1. For a short exact sequence of finite dimensional vector spaces,

$$\dim(V_1) - \dim(V_2) + \dim(V_3) = 0.$$