PRACTICE PROBLEMS FOR THE MATH 330 FINAL

Here is a list of sample problems that you should be comfortable solving quickly and accurately. I have not included problems involving material from Chapters 18, 20, 21, 22 since we have not covered that material yet.

(1) Find the lcm and the gcd of the integers 1240 and 820.

(2) Find all the subgraoups of the group $\mathbb{Z}/240\mathbb{Z}$ and describe their orders.

(3) Find the number of elements of order 20 in $\mathbb{Z}/60\mathbb{Z} \oplus \mathbb{Z}/80\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z}$.

(4) Find the number of subgroups of order 9 in $\mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. (Caution: They need not all be cyclic.)

(5) Let p be a prime number. Calculate the inner automorphism groups of S_{2p}, A_{2p} and D_{2p} up to isomorphism.

(6) Calculate the automorphism group of $\mathbb{Z}/1800\mathbb{Z}$ up to isomorphism. Is this group cyclic? Express the group as a direct product of cyclic groups.

(7) List the conjugacy classes of the symmetric group S_6. Determine how many elements each of the conjugacy class has. Do the same problem for the alternating group A_5.

(8) Let G be an abelian group of order 225.
   (a) Suppose G has 24 elements of order 45. Determine the isomorphism class of G.
   (b) Suppose G has no elements of order 45, but has exactly 4 elements of order 5. Determine the isomorphism class of G.
   (c) Suppose G has more than 24 elements of order 45. Determine the isomorphism class of G.

(9) Find all the group homomorphisms $f : \mathbb{Z}/160\mathbb{Z} \to \mathbb{Z}/100\mathbb{Z}$ and determine their kernels.

(10) Consider the group homomorphism $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ with the property that $f((1,0)) = (6,12)$ and $f((0,1)) = (2,4)$. Determine the image of f. Determine the kernel of f. Consider the factor group $\mathbb{Z} \oplus \mathbb{Z}/f(\mathbb{Z} \oplus \mathbb{Z})$. Does this group contain any elements of order 5? Does it contain any elements of order 2? Does it contain any elements of infinite order? Is it cyclic?

(11) Classify all abelian groups of order 1080 up to isomorphism.

(12) Let G be a group of order 12 whose center contains an element of order 4. Prove that G must be abelian. Does G have to be cyclic?

(13) Prove that the intersections of two normal subgroups of a group G is a normal subgroup of G.

(14) Consider the ring $R = \mathbb{Z}/30\mathbb{Z}$. Find all the units in R. Find all the zero divisors in R. Find all the maximal ideals in R. Find all the prime ideals in R.

(15) Find all ring homomorphisms $f : \mathbb{Z}/60\mathbb{Z} \to \mathbb{Z}/48\mathbb{Z}$. Describe their kernels.

(16) Let $\mathbb{C}$ denote the field of complex numbers. Describe all the ideals of $\mathbb{C}$. Describe all the prime ideals of $\mathbb{C}[x]$. Describe all the maximal ideals of $\mathbb{C}[x]$

(17) Let F be a field. Let $F[x, y]$ denote the ring of polynomials in two variables with coefficients in F. Is $F[x, y]$ a principal ideal domain?

(18) Consider the polynomial $x^2 + x - 6$. Find all the solutions of this polynomial in $\mathbb{Z}/p\mathbb{Z}$ where p is a prime. How does your answer change if you instead find the solutions in $\mathbb{Z}/12\mathbb{Z}$?

(19) Are the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{7}]$ isomorphic rings? Prove or disprove.

(20) Determine the number of elements in the ring $\mathbb{Z}[i]/ < 5 + i >$. Determine the characteristic of this ring.

(21) Prove that the ring $\mathbb{Q}(\sqrt{2})$ is isomorphic to $\mathbb{Q}[x]/ < x^2 - 2 >$.

(22) Calculate 109! mod 113.

(23) Prove that $x^2 + 2$ is a prime ideal in $\mathbb{Z}[x]$. Is it maximal?

(24) Is $\mathbb{R}[x]/ < x^3 + x + 1 >$ a field? Is it an integral domain? How about $\mathbb{R}[x]/ < x^2 + 3 >$?

(25) Determine all the ring automorphisms of $\mathbb{Q}(i)$.

(26) Are the rings $\mathbb{R}[x, y]/ < x - y >$ and $\mathbb{R}[x, y]/ < x^2 - y^2 >$ isomorphic? Prove your answer.

(27) Prove that the number of elements in a finite field F has to be p^n where p is a prime number and n is a positive integer. Express the additive group $(F, +)$ as a direct sum of cyclic groups.

In addition to these problems there are many good problems in the book. You can practice by solving the supplementary exercises on pages 275-277 and 339-340 and any other problem in the book that you have not solved during the semester.